УДК 519.21

О.І. Пономаренко[1], доцент

Дискретні зображення випадкових функцій в нормованих просторах

Розглядаються зображення випадкових функцій із значеннями в нормованих просторах у вигляді функціональних рядів з випадковими коефіцієнтами при широких припущеннях відносно їхніх аргументів та властивостей

Ключові слова: узагальнена випадкова функція другого порядку в нормованому просторі, зображення типу Карунена-Лоева, зображення базисного типу.

[1]Київський національний університет імені Тараса Шевченка, 01033, Київ, вул. Володимирська, 64

O. I. Ponomarenko[1], Associate Professor

Discrete representations of random functions in normed spaces

The representations of random functions with values in normed spaces in the form of functional series with random coefficients are considered under wide assumptions with respect of their arguments and properties

Key words: generalized random function of second order in normed space, Karunen-Loéve type representation, basis type representation.

[1]National Taras Shevchenko University of Kuiv, 01033, Kyiv, 64, Volodymyrska str.

E-mail: [1]*dmponoma@mts.com.ua*

Communicated by Prof. Kozachenko Yu.V.

1. Introduction

In the theory of stochastic processes and random fields of second order the important role plays different discrete representations of such objects in the form of finite and infinite sums of functional members with random coefficients.

The most known such representations of Karhunen-Loéve type for scalar processes on segments of real line ([1]) and spectral expansions of scalar homogeneous random fields on compact homogeneous spaces ([2]).

The aim of this paper is investigation of discrete representations of random functions in more general assumptions about their domains and values. As sets of arguments of considered random functions we use different compact topological spaces, measurable spaces with positive finite measures and arbitrary sets. We also consider generalized random functions with values in arbitrary normed space X. The case when X is Hilbert space was studied in [3].

Now we give some facts and notations which are necessary in further.

Denote by $L_2(\Omega)$ Hilbert space of all complex second order random variables defined on some probability space (Ω, F, P) endowed by strong topology. Let X be a complex normed space and X^* be its topological dual space endowed by strong topology.

The generalized random element Ξ on Ω in X is continuous linear operator $\Xi: X^* \to L_2(\Omega)$. Every such element is generated by some usual random element ξ on Ω with values in some extension of space X (see[4]). The space of such random elements in X with denoted by $L(X^*, L_2(\Omega))$.

Let $\overline{L}(X^*, X^{**})$ be a space of all antilinear (or conjugate linear) continuous operators from X^* into X^{**} (second dual space for X). Define the expectation $\mathrm{E}\Xi \in X^{**}$ for $\Xi \in L(X^*, L_2(\Omega))$ by the equality $(\mathrm{E}\Xi)(x^*) = \mathrm{E}(\Xi x^*), x^* \in X^*$ and covariance operator $[\Xi, \Psi] \in \overline{L}(X^*, X^{**})$ of elements $\Xi, \Psi \in L(X^*, L_2(\Omega))$ by the equality

$$\mathrm{E}(\Xi x^*)(\Psi y^*) = ([\Xi, \Psi] y^*)(x^*), x^*, y^* \in X^*$$

The generalized random function of second order Ξ_t on set T is a mapping of T into $L(X^*, L_2(\Omega))$. Denote by m_t its mean function $m_t = \mathrm{E}\Xi_t, t \in T$ and by $R(t,s) = [\Xi_t, \Xi_s], t, s \in T$ its covariance function. If function Ξ_t is continuous on topological space T in strong topology of

$L(X^*, L_2(\Omega))$ the $R(t,s)$ is continuous on $T \times T$ in weak topology of $\overline{L}(X^*, X^{**})$ and vise versa.

Note that the class of covariance functions of the set of all second order random functions in X coinsides with the set of all positive definite $\overline{L}(X^*, X^{**})$ — valued operator kernel $R(t,s)$ on T (i.e. for all $n \in N, x_i^* \in X^*, t_i \in T, i = 1,...,n$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(R(t_i,t_j)x_j^*)(x_i^*) \geq 0).$$

2. Representations of Karhunen-Loéve type

Let T be a compact topological space, B be a σ-algebra of Borel sets in T and μ be a σ-additive finite measure on B such that $\mu(V) > 0$ for every open set $V \in B$. Denote by $k(t,s)$ continuous complex-valued positive definite kernel on T and by $\{\varphi_j(t), j \in J\}$ complete orthonormal system of eigenfunctions corresponding to system of eigenvalues $\{\lambda_j, j \in J\}$, $\lambda_j > 0$ of integral operator K,

$$K\varphi(t) = \int_T k(t,s)\varphi(s)\mu(ds), \varphi \in L_2(T,B,\mu), \tag{1}$$

$$K\varphi_j(t) = \lambda_j \varphi_j(t), j \in J,$$

$$\int_T \varphi_j(t)\overline{\varphi_i(t)}\mu(dt) = \delta_{ji} = \begin{cases} 0, & j \neq i, \\ 1, & j = i. \end{cases}$$

Consider the generalized random function $\Xi_t, t \in T$ in X with covariance function of the form

$$R(t,s) = k(t,s)A, t,s \in T,$$

where A is positive operator from $\overline{L}(X^*, X^{**}), (Ax^*)(x^*) > 0$ for all $x^* \neq 0$.

Theorem 1. *Under forgoing assumptions the random function* Ξ_t *is continuous in uniform topology of* $L(X^*, L_2(\Omega))$ *and admits representation*

$$\Xi_t = \sum_{j \in J}\sqrt{\lambda_j}\varphi_j(t)\Phi_j, t \in T, \tag{2}$$

where Φ_j *are random elements from* $L(X^*, L_2(\Omega))$ *such that* $\left[\Phi_j, \Phi_i\right] = \delta_{ji}A, j,i \in J$. *The sery (2) is uniformly convergent on* T *in uniform topology of* $L(X^*, L_2(\Omega))$. *The covariance function of* Ξ_t *admits the representation*

$$R(t,s) = \sum_{j \in J}\lambda_j \varphi_j(t)\overline{\varphi_i(s)}A, \tag{3}$$

where the sery (3) is uniformly convergent on $T \times T$ *in uniform topology of* $\overline{L}(X^*, X^{**})$.

For the proof note that the representation (2) follows from theorem 2.1 in [3]. Now we apply to the representation (2) the theorem about representations of generalized random functions in vector topological spaces (theorem 2, [5]). Then we have the representation (2). The indicated convergence of the sery (2) in consequence of type of convergence of sery (3).

Example 1. In the case, when

$$R(t,s) = e^{-\alpha|t-s|}A, \alpha > 0, t,s \in [-\ell, \ell], \ell > 0,$$

Ξ_t is generalized Orstein-Uhlenbeck process in X and

$$\Xi_t = \sum_{j=0}^{\infty}(B_j \cos a_j\, t\Phi_j^{(1)} + C_j \sin b_j\, t\Phi_j^{(2)}),$$

where B_j and C_j are some normalization constants, a_j and $b_j, j = 1,2,3...$ are roots of equations $a_j(\ell a_j) = \alpha$, $b_j(\ell b_j) = \alpha$ and elements $\Phi_j^{(i)} \in L(X^*, L_2(\Omega))$ are such that $\left[\Phi_j^{(p)}, \Phi_i^{(q)}\right] = \delta_{pq}\delta_{ji}A$.

In more general situation Ξ_t is generalized weakly stationary process in X with continuous time and covariance function of the form

$$R(t,s) = k(t-s)A,$$

where k is real continuous positive definite function. Then in accordance with theorem 1 process Ξ_t, $t \in [-\pi, \pi]$ admits random Fourier expansion with random orthogonal coefficients $\Phi_0, \Phi_k^r, r = 1,2$ of the form

$$\Xi_t = \sqrt{\frac{a_0}{2}}\Phi_0 + \sum_{n=1}^{\infty}\sqrt{a_n}(\Phi_n^1 \cos nt + \Phi_n^2 \sin nt)$$

because operator K with kernel $k(t-s)$ has eigenvalues $\lambda_n = \pi a_n$, where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} k(t)\cos nt dt\,, n = 0,1,2,...$$

and two eigenfunctions $\cos nt$ and $\sin nt$ correspond for the case $n \ge 1$ to every λ_n and eigenfunction $\varphi_0(t) \equiv 1$ corresponds to λ_0.

In particular, when $k(t) = \cos^2 t$ the operator K has nonzero eigenvalues $\lambda_0 = \pi$, $\lambda_2 = \frac{\pi}{2}$ with corresponding eigenfunctions $\varphi_0(t) \equiv 1$, $\varphi_2^{(1)}(t) = \cos 2t$, $\varphi_2^{(2)}(t) = \sin 2t$ and process Ξ_t is finite sum with three members.

Example 2. Let $\Xi_t, t \in M_\ell^n = \times_{i=1}^n [0, \ell_i]$, $\ell = (\ell_1, ..., \ell_n), n \ge 1$ be a random function in X with covariance

$$R(t,s) = \left(\prod_{i=1}^n min(t_i, s_i)\right) A,\ t = (t_1,...,t_n),$$
$$s = (s_1, ..., s_n).$$

This function is generalized Chentsov-Wiener field on $M_\ell^n, n > 1$ with orthogonal n-dimensional increments and in case $n = 1$, when Ξ_t is Gaussian, is generalized Brownian motion in space X. Ξ_t has the representation

$$\Xi_t = \frac{\left(2\sqrt{2}\right)^n \sqrt{\ell_1 ... \ell_n}}{\pi^n} \times$$
$$\times \sum_{j \in Z_+^n} \left(\prod_{r=1}^n \frac{\sin(2j_r+1)\pi t_r / 2\ell_r}{(2j_r+1)}\right)\Phi_j,$$
$$t \in M_\ell^n, \Phi_j \in L(X^*, L_2(\Omega)), \left[\Phi_j, \Phi_i\right] = \delta_{ij} A.$$

Example 3. Let $\Xi_t, t \in M_\ell^n$ be random function in X with covariance

$$R(t,s) = \left(\prod_{r=1}^n \ell_r^{-1}[\ell_r \min(t_r, s_r) - t_r s_r]\right) A.$$

Then Ξ_t admits the representation

$$\Xi_t = \frac{2^{\frac{n}{2}}\sqrt{\ell_1 ... \ell_n}}{\pi^n}\sum_{j \in N^n}\left(\prod_{r=1}^n \frac{1}{j_r}\sin\frac{j_r \pi t_r}{\ell \partial_r}\right)\Phi_j,$$

$t \in M_l^n, \Phi_j \in L(X^*, L_2(\Omega)), \left[\Phi_j, \Phi_i\right] = \delta_{ij} A,$
$\mathrm{N} = \{1, 2, ...\}.$

When $n = 1, \Xi_t$ is "stochastic bridge" on $[0, \ell]$ and Brownian bridge if Ξ_t is Gaussian, when $n = 2$ Ξ_t is "stochastic sheet" on $[0, \ell]^2$ and Brownian sheet in X if Ξ_t is Gaussian.

3. Discrete representations of basic type

The discrete representation of basis type for generalized random function $\Xi_t, t \in T$ (T is nonempty set) in space X is expansion of the form

$$\Xi_t = \sum_{j \in J} \alpha_j(t)\Phi_j, t \in T,$$

where $\{\Phi_j, j \in J\}$ is system of elements from $L(X^*, L_2(\Omega))$ and $\{\alpha_j(t), j \in J\}$ is corresponding set of numerical functions, and expansion is unique (i.e. for given Φ_j functions $\alpha_j(t)$ are defined by unique way or for given $\alpha_j(t)$ elements Φ_j are defined by unique way).

Example 4. Let $\Xi_q, q \in Q$ be a continuous (in strong topology of $L(X^*, L_2(\Omega))$) homogeneous random function in X on compact homogeneous space Q with compact transformation group G, i.e. $\mathrm{E}\Xi_q = const$ and for all $g \in G, p, q \in Q$ $R(gp, gq) = R(p, q)$. Then Ξ_q admits expansion with respect to system of spherical functions $\{\psi_{ij}^{(\lambda)}(q), i = 1, ..., d_\lambda, j = 1, ..., r_\lambda, \lambda = 1, 2, ...\}$ on Q:

$$\Xi_q = \sum_\lambda \sum_{i=1}^{d_\lambda}\sum_{j=1}^{r_\lambda}\psi_{ij}^{(\lambda)}(q)\Phi_{ji}^{(\lambda)}, q \in Q, \qquad (4)$$

$$\Phi_{ij}^{(\lambda)} = \left(\int_Q |\psi_{ij}^{\lambda}(q)|^2 dq\right)^{-1} \int_Q \overline{\psi_{ij}^{(\lambda)}(q)} \Xi_q dq \in L(X^*, L_2(\Omega)),$$

where dq is G-invariant measure on Q and last integral is interpreted in strong sense. Representation (4) has many realization for different types of compact homogeneous spaces and compact groups (see [3], [6]).

In particular, when Q is sphere S_2 in R^3 with center in 0 with spherical coordinates of the points (θ,φ) and G in rotation group $SO(3)$ the homogeneous random field $\Xi_{\theta,\varphi}$ on S_2 has the expansion

$$\Xi_{\theta,\varphi} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(\theta,\varphi)\Phi_m^{\ell},$$

where $\{Y_m^{\ell}(\theta,\varphi), m=-\ell,...,\ell; \ell=0,1,...\}$ is the system of spherical harmonics on S_2 and

$$\Phi_m^{\ell} \in L(X, L_2(\Omega)),$$
$$[\Phi_m^{\ell}, \Phi_j^k] = \delta_{mj}\delta_{\ell k} F_m, F_m \in \overline{L}(X^*, X^{*}*),$$
$$(F_m(x^*))(x^*) \ge 0, x^* \in X^*.$$

In the case of homogeneous field Ξ_t on sphere S_{n-1} in R^n with center in 0 with respect to rotation group $SO(n)$ Ξ_t has representation in the form of sery with members which are products of hyperspherical harmonics

$$Y_{\ell,m_1,...,m_{n-3},\pm m_{n-2}}(\theta_1,...,\theta_{n-2},\varphi)$$

$$\ell = 0,1,2,...,0 \le m_{n-2} \le m_{n-3} \le ... \le m_1 \le \ell$$

and random elements

$$\Phi_{\ell,m_1,...,m_{n-3},\pm m_{n-2}} \in L(X, L_2(\Omega))$$

with covariance operators which are depends only on ℓ (there $(\theta_1,...,\theta_{n-2},\varphi)$ are spherical coordinates of point from S_{n-1}).

Now suppose that set T is endowed by σ-algebra of subsets A with finite measure μ on A. Let $\Xi_t, t \in T$ be a random function in X with covariance $R(t,s) = k(t,s)A, t,s \in T$, where positive definite kernel

$$k(t,s) \in L_2(T \times T, A \times A, \mu \times \mu). \tag{5}$$

If $\{\varphi_j(t), j \in J\}$ and $\{\lambda_j, j \in J\}, \lambda_j > 0$ are orthonormal system of eigenfunctions and system of corresponding eigenvalues for integral operator K of the form (1) in $L_2(T, A, \mu)$, then

$$k(t,s) = \sum_{j\in J} \lambda_j \varphi_j(t)\overline{\varphi_j(s)}, \tag{6}$$

where sery is convergent in $L_2(T \times T, A \times A, \mu \times \mu)$.

Suppose that elements of the set J is numerated in order of decreasing of λ_j with taking in account of corresponding multiplicity. Consider elements

$$\Phi_j = \int_T \overline{\varphi_j(s)} \Xi_s \mu(ds) \in L(X^*, L_2(\Omega)),$$

It is easy to see that

$$[\Phi_j, \Phi_i] = \delta_{ji}\lambda_j A, [\Xi_t, \Phi_j] = \lambda_j \varphi_j(t) A. \tag{7}$$

Now consider the approximation of Ξ_t by the sums $\Xi_t^n = \sum_{j=1}^n \varphi_j(t)\Phi_j$. For every $x^* \in X^*$ we have

$$\mathrm{E}|\Xi_t x^* - \Xi_t^n x^*|^2 =$$
$$= \left[k(t,t) - \sum_{j=1}^n \lambda_j |\varphi_j(t)|^2\right]\left(Ax^*\right)(x^*).$$

Then using (6) we obtain that

$$\int_T \mathrm{E}|\Xi_t x^* - \Xi_t^n x^*|^2 \mu(dt) \to 0 \text{ when } n \to \infty \tag{8}$$

(for comparison, see [3]). Hence the following result is true.

Theorem 2. *Let $\Xi_t, t \in T$ be a random function in X for which the assumption (5) take place. Then Ξ_t admits the representation*

$$\Xi_t = \sum_{j\in J} \varphi_j(t)\Phi_j, t \in T,$$

where $\Phi_j \in L(X^*, L_2(\Omega))$ satisfy the relations (7) and sery is convergent in the sense of relation (8).

Let T be a nonempty set and $k(t,s)$ be a complex-valued positive definite kernel on T.

Theorem 3. *Let* $\Xi_t, t \in T$ *be a random function in* X *with covariance* $R(t,s) = k(t,s)A$. *Then there exists such system of complex-valued* ω*-independent (in Krein's sense, see [7]) functions* $\{\alpha_j(t), j \in J\}$ *on* T, *which defined up to unitary transformation, that*

$$R(t,s) = \sum_{j \in J} \alpha_j(t)\overline{\alpha_j(s)}A, t, s \in T, \qquad (9)$$

$$\Xi_t = \sum_{j \in J} \alpha_j(t)\Phi_j, \Phi_j \in L(X^*, L_2(\Omega)), \quad (10)$$

where non more than countable summands are nonzero and $\left[\Phi_i, \Phi_j\right] = \delta_{ij}A$. *The sery (9) us convergent in uniform topology of* $\overline{L}(X^*, X^{**})$ *under every* $t, s \in T$ *and sery (10) is convergent in uniform topology of* $L(X^*, L_2(\Omega))$ *under every* $t \in T$.

For the proof we first of all note that expansion (9) is consequence of Krein expansion for kernel $k(t,s)$ (see [7],[3]). Then expansion (10) may be obtained by application of theorem 2 from [5] to expansion (9). Indicated convergence of sery (10) follows from type of convergence of sery (9).

Example 5. Let kernel $k(t,s)$ admits the Karhunen factorization

$$k(t,s) = \int_\Lambda f(t,\lambda), \overline{f(s,\lambda)}\mu(d\lambda), \qquad (11)$$

where $f(t,\cdot) \in L_2(\Lambda, A, \mu)$ (there μ is positive measure on measurable space (Λ, A)). Denote by $\{g_j(\lambda), j \in J\}$ an orthonormal base in $L_2(\Lambda, A, \mu)$. Then we have representations (9),(10) with the functions

$$\alpha_j(t) = \int_\Lambda f(t,\lambda)\overline{g_j(\lambda)}\mu(d\lambda). \qquad (12)$$

In particular, for random field $\Xi_t, t \in M^n = [0, 2\pi]^n$ in X of example 2, using the representation $k(t,s)$ with help of indicator function $1_t(\lambda), \lambda \in M^n$ of parallelepiped $\times_{r=1}^n [0, t_r]$, $t = (t_1, ..., t_n) \in M^n$ of the form $k(t,s) = (1_t, 1_s)_{L_2(M^n)}$, we have that

$$\Xi_t = (2\pi)^{-\frac{n}{2}} \sum_{j \in Z^n} \prod_{r=1}^n \frac{(1 - e^{ij_r t_r})}{ij_r} \Phi_j, t \in M^n.$$

Example 6. When in representation (11)

$$f(t,\lambda) = \left(\frac{2h\Gamma(\frac{3}{2} - h)}{\Gamma(h + \frac{1}{2})\Gamma(2 - 2h)} \right)^{\frac{1}{2}} \left[(\frac{1}{\lambda})^{h - \frac{1}{2}} (t - \lambda)^{h - \frac{1}{2}} - -(h - \frac{1}{2})\lambda^{\frac{1}{2} - h} \int_\lambda^t u^{h - \frac{3}{2}} (u - \lambda)^{h - \frac{1}{2}} du \right], h \in (0,1),$$

the mean $\mathrm{E}\Xi_t \equiv 0$ and

$$R(t,s) = \left(\int_0^{\min(t,s)} f(t,\lambda) f(s,\lambda) d\lambda \right) A,$$

then process Ξ_t is generalized fractional Brownian motion in X with Hurst index h if it is Gaussian. For process Ξ_t the representation (10) is true with

$$\alpha_j(t) = \int_0^t f(t,\lambda) g_j(\lambda) d\lambda, \ j \in J, t \in [0, \ell],$$

where $\{g_j(\lambda)\}$ is orthonormal base in $L_2[0, \ell]$.

Список використаних джерел

1. *Boulinski A.V. Shiriaev A.N. Theory of stochastic processes.* - M.: Fizmatgiz, 2003. — 399 p.
2. *Yaglom A.M.* Second-order homogeneous random fields. // Proc. Fourth Berkely Symp.on Math. Stat. and Probab., Vol.II, Univ. Calif. Press, 1961, p. 583-622.
3. *Ponomarenko O.I.* Discrete representations of random functions of second order I.// Theory Probab. and Math. Stat. 86, 2012, p.164-173.
4. *Ponomarenko O.I.* Random linear functionals of second order I. // Theory Probab. and Math. Stat., 54,1997, p. 145-158.
5. *Ponomarenko O.I.* Integralni zobrazhennia vypadkovyh funktsij zi znachenniamy v lokalno-opuklyh prostorah. // Теорія ймовір. та матем. статист. V.46, 1992, p.132-141.
6. *Ponomarenko O., Perun Y.* Multivariate random fields on some homogeneous spaces. Theory of Stoch. Processes, vol. 14(30), no.3-4, 2008, p.104-113.
7. *Krein M.G.* Ermitovo polozhitelnye yadra na odnorodnyh prostranstvah. I. // Ukr. Math, Journ.,V .1,4,1949, p.64-98.

Received 22.12.2012