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Метод усереднення для автономної випадкової коливної системи третього порядку

У роботі вивчається асимптотична поведінка автономної коливної системи, яка описується диференціальним рівнянням третього порядку з малими нелінійними зовнішніми збуреннями типу багатовимірного "білого" та Пауссонівського шумів.

Ключові слова: метод усереднення, автономна коливна система, стохастичне диференціальне рівняння.

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Averaging method for autonomous third order random oscillating system

The asymptotic behavior of autonomous oscillating system describing by differential equation of third order with small non-linear external perturbations of multidimensional "white noise" and "Poisson noise" types is studied.

Key Words: averaging method, autonomous oscillating system, stochastic differential equation.

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1 Introduction

The overview of papers devoted to the averaging method, proposed by N.M.Krylov, N.N.Bogolyubov [1], and its applications to random oscillatory systems of different types is presented in O.D.Borysenko, O.V.Borysenko [2] with corresponding references.

This paper deals with investigation of the behaviour, as $\varepsilon \to 0$, of the general type third order autonomous oscillating system described by stochastic differential equation

$$x'''(t) + ax''(t) + b^2x'(t) + ab^2x(t) = \\ = \varepsilon^{k_0} f_0(x(t), x'(t), x''(t)) + \\ + f_\varepsilon(x(t), x'(t), x''(t)) \quad (1)$$

with non-random initial conditions $x(0) = x_0, x'(0) = x'_0, x''(0) = x''_0$, where $\varepsilon > 0$ is a small parameter, $f_\varepsilon(x, x', x'')$ is a random function such that

$$\int_0^t f_\varepsilon(x(s), x'(s), x''(s))\, ds = \\ = \sum_{i=1}^m \varepsilon^{k_i} \int_0^t f_i(x(s), x'(s), x''(s))\, dw_i(s) + \\ + \varepsilon^{k_{m+1}} \int_0^t \int_{\mathbb{R}} f_{m+1}(x(s), x'(s), x''(s), z)\, \nu(ds, dz),$$

$k_i > 0, i = \overline{0, m+1}$; $f_i, i = \overline{0, m+1}$ are non-random functions; $w_i(t), i = \overline{1, m}$ are independent one-dimensional Wiener processes; $\tilde{\nu}(dt, dy) = \nu(dt, dy) - \Pi(dy)dt$, $E\nu(dt, dy) = \Pi(dy)dt$, $\nu(dt, dy)$ is the Poisson measure independent on $w_i(t), i = \overline{1, m}$; $\Pi(A)$ is a finite measure on Borel sets $A \in \mathbb{R}$, $a > 0, b > 0$.

We will consider the equation (1) as the system of stochastic differential equations

$$\begin{aligned} dx(t) &= x'(t)dt, \\ dx'(t) &= x''(t)dt, \\ dx''(t) &= [-ax''(t) - b^2x'(t) - ab^2x(t) + \\ &+ \varepsilon^{k_0} f_0(x(t), x'(t), x''(t)) + \\ &+ \varepsilon^{k_{m+1}} \int_{\mathbb{R}} f_{m+1}(x(t), x'(t), x''(t), z)\Pi(dz)]dt + \\ &+ \sum_{i=1}^m \varepsilon^{k_i} f_i(x(t), x'(t), x''(t)) dw_i(t) + \\ &+ \varepsilon^{k_{m+1}} \int_{\mathbb{R}} f_{m+1}(x(t), x'(t), x''(t), z)\tilde{\nu}(dt, dz), \\ x(0) &= x_0,\ x'(0) = x'_0,\ x''(0) = x''_0. \end{aligned} \quad (2)$$

In what follows we will use the constant $K > 0$ for the notation of different constants, which are not depend on ε.

2 Auxiliary result

From Borysenko O. and Malyshev I. [3], using the obvious modifications we obtain following results

Lemma 1. *Let for each $x \in \mathbb{R}^d$ there exists*

$$\lim_{T\to\infty} \frac{1}{T} \int_A^{T+A} f(t,x)\,dt = \bar{f}(x)$$

uniformly with respect to A, the function $\bar{f}(x)$ is bounded, continuous, function $f(t,x)$ is bounded and continuous in x uniformly with respect to (t,x) in any region $t \in [0,\infty), |x| \leqslant K$, and stochastic processes $\xi(t) \in \mathbb{R}^d$, $\eta(t) \in \mathbb{R}$ are continuous, then

$$\lim_{\varepsilon\to 0} \int_0^t f\left(\frac{s}{\varepsilon} + \eta(s), \xi(s)\right) ds = \int_0^t \bar{f}(\xi(s))\,ds$$

almost surely for all arbitrary $t \in [0,T]$.

Remark 1. *Let $f(t,x,z)$ is bounded and uniformly continuous in x with respect to $t \in [0,\infty)$ and $z \in \mathbb{R}$ in every compact set $|x| \leqslant K, x \in \mathbb{R}^d$. Let $\Pi(\cdot)$ be a finite measure on the σ-algebra of Borel sets in $\mathbb{R}$ and let*

$$\lim_{T\to\infty} \frac{1}{T} \int_A^{T+A} f(t,x,z)\,dt = \bar{f}(x,z),$$

uniformly with respect to A for each $x \in \mathbb{R}^d, z \in \mathbb{R}$, where $\bar{f}(x,z)$ is bounded, uniformly continuous in x with respect to $z \in \mathbb{R}$ in every compact set $|x| \leqslant K$. Then for any continuous processes $\xi(t) \in \mathbb{R}^d$ and $\eta(t) \in \mathbb{R}$ we have

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}} f\left(\tfrac{s}{\varepsilon} + \eta(s), \xi(s), z\right) \Pi(dz)ds = \\ = \int_0^t \int_{\mathbb{R}} \bar{f}(\xi(s),z)\,\Pi(dz)ds.$$

3 Main result

Let us consider the following representation of processes $x(t), x'(t), x''(t)$:

$$x(t) = C(t)e^{-at} + A_1(t)\cos(bt) + A_2(t)\sin(bt),$$

$$x'(t) = -aC(t)e^{-at} - bA_1(t)\sin(bt) + bA_2(t)\cos(bt),$$

$$x''(t) = a^2C(t)e^{-at} - b^2A_1(t)\cos(bt) - b^2A_2(t)\sin(bt),$$

$$N(t) = C(t)\exp\{-at\}.$$

Then

$$\begin{aligned} N(t) &= \frac{b^2x(t) + x''(t)}{a^2+b^2}, \\ A_1(t) &= \cos\alpha\cos(bt+\alpha)x(t) - \frac{\sin bt}{b}x'(t) - \\ &\quad -\frac{\sin\alpha\sin(bt+\alpha)}{b^2}x''(t), \\ A_2(t) &= \cos\alpha\sin(bt+\alpha)x(t) + \frac{\cos bt}{b}x'(t) + \\ &\quad +\frac{\sin\alpha\cos(bt+\alpha)}{b^2}x''(t), \end{aligned}$$

where $\alpha = \operatorname{arctg}(b/a)$. We can apply Ito formula [4] to stochastic process $\xi(t) = (N(t), A_1(t), A_2(t))$ and obtain for the process $\xi(t)$ the system of stochastic differential equations

$$\begin{aligned} dN(t) &= -aN(t)dt + \frac{1}{a^2+b^2}dH(t) \\ dA_1(t) &= -\frac{\sin\alpha\sin(bt+\alpha)}{b^2}dH(t) \qquad (3), \\ dA_2(t) &= \frac{\sin\alpha\cos(bt+\alpha)}{b^2}dH(t) \end{aligned}$$

$$\begin{aligned} N(0) &= \frac{b^2x_0 + x_0''}{a^2+b^2}, \\ A_1(0) &= \frac{a^2x_0 - x_0''}{a^2+b^2}, \\ A_2(0) &= \frac{ax_0'' + (a^2+b^2)x_0' + ab^2x_0}{b(a^2+b^2)}, \end{aligned}$$

where

$$\begin{aligned} dH(t) = {} & [\varepsilon^{k_0}\tilde{f}_0(t,N(t),A_1(t),A_2(t)) + \\ & +\varepsilon^{k_{m+1}}\int_{\mathbb{R}} \tilde{f}_{m+1}(t,N(t),A_1(t),A_2(t),z)\Pi(dz)]dt + \\ & +\sum_{i=1}^m \varepsilon^{k_i}\tilde{f}_i(t,N(t),A_1(t),A_2(t))dw_i(t) + \\ & +\varepsilon^{k_m+1}\int_{\mathbb{R}} \tilde{f}_{m+1}(t,N(t),A_1(t),A_2(t),z)\tilde{\nu}(dt,dz), \end{aligned}$$

$\tilde{f}_i(t,N,A_1,A_2) = f_i(N + A_1\cos bt + A_2\sin bt, -aN - bA_1\sin bt + bA_2\cos bt, a^2N - b^2A_1\cos bt - b^2A_2\sin bt)$, $i = \overline{0,m}$, $\tilde{f}_{m+1}(t,N,A_1,A_2,z) = f_{m+1}(N + A_1\cos bt + A_2\sin bt, -aN - bA_1\sin bt + bA_2\cos bt, a^2N - b^2A_1\cos bt - b^2A_2\sin bt, z)$.

Theorem 3.1. *Let $\Pi(\mathbb{R}) < \infty$, $t \in [0,t_0]$, $k = \min(k_0, 2k_1, \ldots, 2k_m, k_{m+1})$. Let us suppose, that functions $f_i, i = \overline{0,m+1}$ bounded and satisfy Lipschitz condition on x, x', x''. If given below matrix $\bar{\sigma}^2(A_1,A_2)$ is non-negative definite, then*

1. For $k_0 = 2k_i = k_m + 1$, $i = \overline{1,m}$ the stochastic process $\xi_\varepsilon(t) = \xi(t/\varepsilon^k)$ weakly converges, as $\varepsilon \to 0$, to the stochastic process $\bar{\xi}(t) = (0, \bar{A}_1(t), \bar{A}_2(t))$, where $\bar{A}(t) =$

$(\bar{A}_1(t), \bar{A}_2(t))$ is the solution to the system of stochastic differential equations

$$\begin{aligned} d\bar{A}(t) &= \bar{\alpha}(\bar{A}(t))dt + \bar{\sigma}(\bar{A}(t))d\bar{w}(t), \\ \bar{A}(0) &= (A_1(0), A_2(0)), \end{aligned} \tag{4}$$

where $\bar{\alpha}(\bar{A}) = (\bar{\alpha}^{(1)}(A_1, A_2), \bar{\alpha}^{(2)}(A_1, A_2))$,

$$\bar{\alpha}^{(1)}(A_1, A_2) = -\frac{1}{2\pi b(a^2+b^2)}\times$$
$$\times \int_0^{2\pi} \hat{f}_{(1)}(\psi, A_1, A_2)(a\sin\psi + b\cos\psi)\, d\psi,$$

$$\bar{\alpha}^{(2)}(A_1, A_2) = \frac{1}{2\pi b(a^2+b^2)}\times$$
$$\times \int_0^{2\pi} \hat{f}_{(1)}(\psi, A_1, A_2)(a\cos\psi - b\sin\psi)\, d\psi,$$

$$\bar{\sigma}(A_1, A_2) = \left\{\bar{B}(A_1, A_2)\right\}^{\frac{1}{2}} =$$
$$= \left\{\frac{1}{2\pi b^2(a^2+b^2)^2}\int_0^{2\pi} \hat{f}_{(2)}(\psi, A_1, A_2)B(\psi)\, d\psi\right\}^{\frac{1}{2}},$$

$$B(\psi) = (B_{ij}(\psi), i, j = 1, 2),$$
$$B_{11}(\psi) = (a\sin\psi + b\cos\psi)^2,$$
$$B_{12}(\psi) = B_{21}(\psi) = -(a\sin\psi + b\cos\psi)(a\cos\psi - b\sin\psi),$$
$$B_{22}(\psi) = (a\cos\psi - b\sin\psi)^2,$$
$$\hat{f}_{(1)}(\psi, A_1, A_2) = \tilde{f}_0(\psi, 0, A_1, A_2) + \int_{\mathbb{R}} \tilde{f}_{m+1}(\psi, 0, A_1, A_2, z)\Pi(dz),$$

$$\hat{f}_{(2)}(\psi, A_1, A_2) = \sum_{i=1}^{m} \tilde{f}_i^2(\psi, 0, A_1, A_2),$$

$\bar{w}(t) = (\bar{w}_i(t), i = 1, 2)$, $\bar{w}_i(t), i = 1, 2$ – independent one-dimensional Wiener processes.

2. If $k < k_0$ then in the averaging equation (4) we must put $\tilde{f}_0 \equiv 0$; if $k < 2k_j$ for some $1 \leqslant j \leqslant m$, then in the averaging equation (4) we must put $\tilde{f}_j \equiv 0$ for all such j; if $k < k_{m+1}$ then in the averaging equation (4) we must put $\tilde{f}_{m+1} \equiv 0$.

Proof. Let us make a change of variable $t \to t/\varepsilon^k$ in equation (3) and obtain for the process $\xi_\varepsilon(t) = (N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t)) = (N(t/\varepsilon^k), A_1(t/\varepsilon^k), A_2(t/\varepsilon^k))$ the system of stochastic differential equations

$$\begin{aligned} dN_\varepsilon(t) &= -\frac{a}{\varepsilon^k}N_\varepsilon(t)dt + \frac{1}{a^2+b^2}dH_\varepsilon(t) \\ dA_1^\varepsilon(t) &= -\frac{\sin\alpha\sin(bt/\varepsilon^k + \alpha)}{b^2}dH_\varepsilon(t) \\ dA_2^\varepsilon(t) &= \frac{\sin\alpha\cos(bt/\varepsilon^k + \alpha)}{b^2}dH_\varepsilon(t), \end{aligned} \tag{5}$$

where

$$\begin{aligned} dH_\varepsilon(t) &= [\varepsilon^{k_0-k}\tilde{f}_0(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t)) + \\ &+ \varepsilon^{k_{m+1}-k}\int_{\mathbb{R}} \tilde{f}_{m+1}(\frac{t}{\varepsilon^k}, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), z)\Pi(dz)]dt \\ &+ \sum_{i=1}^{m} \varepsilon^{k_i-k/2}\tilde{f}_i(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t))dw_i^\varepsilon(t) + \\ &+ \varepsilon^{k_{m+1}}\int_{\mathbb{R}} \tilde{f}_{m+1}(t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz), \end{aligned}$$

$w_i^\varepsilon(t) = \varepsilon^{k/2}w_i(t/\varepsilon^k)$, $\tilde{\nu}_\varepsilon(t, A) = \nu(t/\varepsilon^k, A) - \Pi(A)t/\varepsilon^k$, here A is Borel set in $\mathbb{R}$. For any $\varepsilon > 0$ the processes $w_i^\varepsilon(t), i = \overline{1, m}$ are the independent Wiener processes and $\tilde{\nu}_\varepsilon(t, A)$ is the centered Poisson measure independent on $w_i^\varepsilon(t), i = \overline{1, m}$.

Since we have relationship $N_\varepsilon(t) = \exp\{-at/\varepsilon^k\}C(t/\varepsilon^k)$ and process $C_\varepsilon(t) = C(t/\varepsilon^k)$ satisfies the stochastic equation

$$C_\varepsilon(t) = C(0) + \int_0^t \frac{\exp\{as/\varepsilon^k\}}{a^2+b^2}dH_\varepsilon(s),$$

where $C(0) = \frac{b^2x_0 + x_0''}{a^2+b^2}$, we can obtain estimate

$$\mathsf{E}|N_\varepsilon(t)|^2 \leqslant K[e^{-2at/\varepsilon^k} + \varepsilon^k(1 - e^{-2at/\varepsilon^k})\times$$
$$\times(t(\varepsilon^{2(k_0-k)} + \varepsilon^{2(k_{m+1}-k)}) + \sum_{i=1}^{m+1} \varepsilon^{2k_i-k})].$$

Therefore $\lim_{\varepsilon\to 0} \mathsf{E}|N_\varepsilon(t)|^2 = 0$ and it is sufficient to study the behaviour, as $\varepsilon \to 0$, of solution to the system of stochastic differential equations

$$\begin{aligned} dA_1^\varepsilon(t) &= -\frac{\sin\alpha\sin(bt/\varepsilon^k + \alpha)}{b^2}d\hat{H}_\varepsilon(t), \\ dA_2^\varepsilon(t) &= \frac{\sin\alpha\cos(bt/\varepsilon^k + \alpha)}{b^2}d\hat{H}_\varepsilon(t), \end{aligned} \tag{6}$$

with initial conditions $A_1^\varepsilon(0) = A_1(0)$, $A_2^\varepsilon(0) = A_2(0)$, where

$$\begin{aligned} d\hat{H}_\varepsilon(t) &= [\varepsilon^{k_0-k}\hat{f}_0(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t)) + \\ &+ \varepsilon^{k_{m+1}-k}\int_{\mathbb{R}} \hat{f}_{m+1}(\frac{t}{\varepsilon^k}, A_1^\varepsilon(t), A_2^\varepsilon(t), z)\Pi(dz)]dt \\ &+ \sum_{i=1}^{m} \varepsilon^{k_i-k/2}\hat{f}_i(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dw_i^\varepsilon(t) + \\ &+ \varepsilon^{k_{m+1}}\int_{\mathbb{R}} \hat{f}_{m+1}(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt, dz), \end{aligned}$$

$$\hat{f}_j(t, A_1, A_2) = \tilde{f}_j(t, 0, A_1, A_2),\ j = \overline{0, m}$$

$$\hat{f}_{m+1}(t, A_1, A_2, z) = \tilde{f}_{m+1}(t, 0, A_1, A_2, z),$$

Let us denote $A_\varepsilon(t) = (A_1^\varepsilon(t), A_2^\varepsilon(t))$. Using conditions on coefficients of equation (6) and properties of stochastic integrals we obtain estimates

$$\mathsf{E}||A_\varepsilon(t)||^2 \leqslant K[1+t^2(\varepsilon^{2(k_1-k)}+ +\varepsilon^{2(k_{m+1}-k)})+t\sum_{i=1}^{m+1}\varepsilon^{2k_i-k}],$$

$$\mathsf{E}||A_\varepsilon(t)-A_\varepsilon(s)||^2 \leqslant K[|t-s|^2(\varepsilon^{2(k_0-k)}+ +\varepsilon^{2(k_{m+1}-k)})+|t-s|\sum_{i=1}^{m+1}\varepsilon^{2k_i-k}].$$

Similarly for the process $\zeta_\varepsilon(t) = (\zeta_1^\varepsilon(t),\zeta_2^\varepsilon(t))$, where

$$\zeta_1^\varepsilon(t) = -\int_0^t \frac{\sin\alpha\sin(bs/\varepsilon^k+\alpha)}{b^2}dM_\varepsilon(s),$$
$$\zeta_2^\varepsilon(t) = \int_0^t \frac{\sin\alpha\cos(bs/\varepsilon^k+\alpha)}{b^2}dM_\varepsilon(s),$$

where

$$dM_\varepsilon(t) = \sum_{i=1}^m \varepsilon^{k_i-k/2}\hat f_i(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dw_i^\varepsilon(t)+ +\varepsilon^{k_{m+1}}\int_{\mathbb{R}}\hat f_{m+1}(t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde\nu_\varepsilon(dt,dz),$$

we derive estimates

$$\mathsf{E}||\zeta_\varepsilon(t)||^2 \leqslant Kt\sum_{i=1}^{m+1}\varepsilon^{2k_i-k},$$

$$\mathsf{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^2 \leqslant K|t-s|\sum_{i=1}^{m+1}\varepsilon^{2k_i-k}.$$

Therefore for stochastic process $\eta_\varepsilon(t) = (A_\varepsilon(t),\zeta_\varepsilon(t))$ conditions of weak compactness [5] are fulfilled

$$\lim_{h\downarrow 0}\overline{\lim_{\varepsilon\to 0}}\sup_{|t-s|<h} \mathrm{P}\{|\eta_\varepsilon(t)-\eta_\varepsilon(s)|>\delta\}=0$$

for any $\delta>0,\ t,s\in[0,T]$,

$$\lim_{N\to\infty}\overline{\lim_{\varepsilon\to 0}}\sup_{t\in[0,T]} \mathrm{P}\{|\eta_\varepsilon(t)|>N\}=0,$$

and for any sequence $\varepsilon_n\to 0, n=1,2,\dots$ there exists a subsequence $\varepsilon_m=\varepsilon_{n(m)}\to 0, m=1,2,\dots$, probability space, stochastic processes $\bar A_{\varepsilon_m}(t) = (\bar A_1^{\varepsilon_m}(t),\bar A_2^{\varepsilon_m}(t))$, $\bar\zeta_{\varepsilon_m}(t)$, $\bar A(t) = (\bar A_1(t),\bar A_2(t))$, $\bar\zeta(t)$ defined on this space, such that $\bar A_{\varepsilon_m}(t)\to\bar A(t), \bar\zeta_{\varepsilon_m}(t)\to\bar\zeta(t)$ in probability, as $\varepsilon_m\to 0$, and finite-dimensional distributions of $\bar A_{\varepsilon_m}(t),\bar\zeta_{\varepsilon_m}(t)$ are coincide with finite-dimensional distributions of $A_{\varepsilon_m}(t),\zeta_{\varepsilon_m}(t)$. Since we interesting in limit behaviour of distributions, we can consider processes $A_{\varepsilon_m}(t)$, and $\zeta_{\varepsilon_m}(t)$ instead of $\bar A_{\varepsilon_m}(t),\bar\zeta_{\varepsilon_m}(t)$. From (6) we obtain equation

$$A_{\varepsilon_m}(t) = A(0)+\int_0^t \alpha_{\varepsilon_m}(s,A_{\varepsilon_m}(s))\,ds+\zeta_{\varepsilon_m}(t),$$
$$A_0 = (A_1(0),A_2(0)), \qquad (7)$$

where $\alpha_\varepsilon(t,A) = (\alpha_\varepsilon^{(1)}(t,A_1,A_2),\alpha_\varepsilon^{(2)}(t,A_1,A_2))$,

$$\alpha_\varepsilon^{(1)}(t,A_1,A_2) = = -\frac{\sin\alpha\sin(bt/\varepsilon^k+\alpha)}{b^2}\hat f_{(1)}^\varepsilon(t/\varepsilon^k,A_1,A_2),$$
$$\alpha_\varepsilon^{(2)}(t,A_1,A_2) = = \frac{\sin\alpha\cos(bt/\varepsilon^k+\alpha)}{b^2}\hat f_{(1)}^\varepsilon(t/\varepsilon^k,A_1,A_2),$$

where

$$\hat f_{(1)}^\varepsilon(t,A_1,A_2) = \varepsilon^{k_0-k}\hat f_0(t,A_1,A_2)+ +\varepsilon^{k_{m+1}-k}\int_{\mathbb{R}}\hat f_{m+1}(t,A_1,A_2,z)\Pi(dz).$$

It should be noted that process $\zeta_\varepsilon(t)$ is the vector-valued square integrable martingale with matrix characteristic

$$\langle\zeta_\varepsilon^{(l)},\zeta_\varepsilon^{(n)}\rangle(t) = \sum_{j=1}^m\int_0^t \sigma_\varepsilon^{(l,j)}(s,A_1^\varepsilon(s),A_2^\varepsilon(s))\times \times\sigma_\varepsilon^{(n,j)}(s,A_1^\varepsilon(s),A_2^\varepsilon(s))\,ds+ +\frac{1}{\varepsilon^k}\int_0^t\int_{\mathbb{R}}\gamma_\varepsilon^{(l)}(s,A_1^\varepsilon(s),A_2^\varepsilon(s),z)\times \times\gamma_\varepsilon^{(n)}(s,A_1^\varepsilon(s),A_2^\varepsilon(s),z)\,\Pi(dz)ds,\ l,n=1,2,$$

where

$$\sigma_\varepsilon^{(1,j)}(s,A_1,A_2) = = -\varepsilon^{k_j-k/2}\frac{\sin\alpha\sin(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat f_j(\tfrac{s}{\varepsilon^k},A_1,A_2),$$
$$\sigma_\varepsilon^{(2,j)}(s,A_1,A_2) = = \varepsilon^{k_j-k/2}\frac{\sin\alpha\cos(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat f_j(\tfrac{s}{\varepsilon^k},A_1,A_2),$$
$$\gamma_\varepsilon^{(1)}(s,A_1,A_2,z) = = -\varepsilon^{k_{m+1}}\frac{\sin\alpha\sin(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat f_{m+1}(\tfrac{s}{\varepsilon^k},A_1,A_2,z),$$
$$\gamma_\varepsilon^{(2)}(s,A_1,A_2,z) = = \varepsilon^{k_{m+1}}\frac{\sin\alpha\cos(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat f_{m+1}(\tfrac{s}{\varepsilon^k},A_1,A_2,z).$$

For processes $A_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ following estimates hold

$$\mathsf{E}||A_\varepsilon(t)-A_\varepsilon(s)||^4 \leqslant \leqslant K[(\varepsilon^{4(k_0-k)}+\varepsilon^{4(k_{m+1}-k)})|t-s|^4+ +\mathsf{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^4], \qquad (8)$$

$$\mathsf{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^4 \leqslant \leqslant K[\sum_{j=1}^{m+1}\varepsilon^{4k_j-2k}|t-s|^2+ \qquad (9)$$
$$+\varepsilon^{4k_{m+1}-3k/2}|t-s|^{3/2}+\varepsilon^{4k_{m+1}-k}|t-s|,$$

$$\mathsf{E}||A_\varepsilon(t)-A_\varepsilon(s)||^8 \leqslant K, \mathsf{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^8 \leqslant K. \qquad (10)$$

Since $A_{\varepsilon_m}(t) \to \bar{A}(t), \zeta_{\varepsilon_m}(t) \to \bar{\zeta}(t)$ in probability, as $\varepsilon_m \to 0$, then, using (10), from (8) and (9) we obtain estimates

$$\mathsf{E}||\bar{A}(t)-\bar{A}(s)||^4 \leqslant K(|t-s|^4+|t-s|^2),$$
$$\mathsf{E}||\bar{\zeta}(t)-\bar{\zeta}(s)||^4 \leqslant C|t-s|^2.$$

Therefore processes $\bar{A}(t)$ and $\bar{\zeta}(t)$ satisfy the Kolmogorov's continuity condition.

Let us consider the case $k_0 = 2k_j = k_{m+1},\ j=\overline{1,m}$. Under these conditions we have for $l,n=1,2$

$$\lim_{\varepsilon\to 0}\frac{1}{t}\int_0^t \alpha_\varepsilon^{(l)}(s,A_1,A_2)ds = \bar{\alpha}^{(l)}(A_1,A_2),$$

$$\lim_{\varepsilon\to 0}\frac{1}{t}\int_0^t[\sum_{j=1}^m \sigma_\varepsilon^{(l,j)}(s,A_1,A_2)\sigma_\varepsilon^{(n,j)}(s,A_1,A_2)+ +\frac{1}{\varepsilon^k}\int_R \gamma_\varepsilon^{(l)}(s,A_1,A_2,z)\times \times\gamma_\varepsilon^{(n)}(s,A_1,A_2,z)\Pi(dz)]ds = \bar{B}_{ln}(A_1,A_2), \qquad (11)$$

where functions $\bar{\alpha}^{(i)}(A_1,A_2)$ and $\bar{B}(A_1,A_2) = \{\bar{B}_{ij}(A_1,A_2),\ i,j=1,2\}$ are defined in the condition of theorem. Since processes $\bar{A}(t), \bar{\zeta}(t)$ are continuous, then from Lemma 1 and relationships (7), (11) it follows

$$\bar{A}(t) = A(0) + \int_0^t \bar{\alpha}(\bar{A}_1(s),\bar{A}_2(s))ds + \bar{\zeta}(t),$$
$$A(0) = (A_1(0),A_2(0)), \qquad (12)$$

where $\bar{\zeta}(t)$ is continuous vector-valued martingale with matrix characteristic

$$\langle\bar{\zeta}^{(i)},\bar{\zeta}^{(j)}\rangle(t) = \int_0^t \bar{B}_{ij}(\bar{A}_1(s),\bar{A}_2(s))ds, i,j=1,2.$$

Hence [6] there exists Wiener process $\bar{w}(t) = (w_i(t), i=1,2)$, such that

$$\bar{\zeta}(t) = \int_0^t \bar{\sigma}(\bar{A}_1(s),\bar{A}_2(s))\,d\bar{w}(s),$$
$$\bar{\sigma}(A_1,A_2) = \left\{\bar{B}(A_1,A_2)\right\}^{1/2}. \qquad (13)$$

Relationships (12), (13) mean that process $\bar{A}(t)$ satisfies equation (4). Under conditions of theorem the equation (4) has unique solution. Therefore process $\bar{A}(t)$ does not depend on choosing of sub-sequence $\varepsilon_m \to 0$, and finite-dimensional distributions of process $A_{\varepsilon_m}(t)$ converge to finite-dimensional distributions of process $\bar{A}(t)$. Since processes $A_{\varepsilon_m}(t)$ and $\bar{A}(t)$ are Markov processes then using the conditions for weak convergence of Markov processes we finish the proof of statement 1 of theorem.

Let us consider the case $k < k_0$ or $k < k_{m+1}$. Then the corresponding terms in the coefficients $\alpha_\varepsilon^{(i)}(t,A_1,A_2), i=1,2$ of equation (7) tend to zero, as $\varepsilon \to 0$.

In the case $k < 2k_j,\ j=\overline{1,m}$ in (11) we have

$$\sigma_\varepsilon^{(l,j)}(t,A_1,A_2)\sigma_\varepsilon^{(n,j)}(t,A_1,A_2) = O(\varepsilon^{2k_j-k}),$$
$$l,n=1,2.$$

Repeating with obvious modifications the proof of statement 1) of theorem we obtain proof of the statement 2). □

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