УДК 512.53

Є.А. Кочубінська[1], к. ф.-м. н., асистент

Про комбінаторику часткового вінцевого добутку скінченних симетричних інверсних напівгруп.

У статті зібрано деякі комбінаторні результати (наприклад, потужність напівгрупи, комбінаторика відношень Гріна та нільпотентних елементів), що стосуються вінцевого добутку скінченних інверсних напівгруп. Через рекурсивне визначення вінцевого добутку виникають рекурентні формули, в які входять різноманітні відомі комбінаторні об'єкти.

Ключові слова: інверсна симетрична напівгрупа, відношення Гріна, вінцевий добуток.

[1]Київський національний університет імені Тараса Шевченка, механіко-математичний факультет, 01601, Київ, вул. Володимирська 64, e-mail: eugenia@univ.kiev.ua

E.A. Kochubinska[1], Ph.D., Assistant Prof.

On combinatorics of wreath product of finite symmetric inverse semigroups

Some combinatorial results (e.g. cardinality of semigroup, combinatorics of Green's relations, combinatorics of nilpotents) concerning wreath product of finite inverse symmetric semigroup are presented. Because of the recursive definition of the wreath product, a number of recurrent formulae, containing different types of known combinatorial objects, arises.

Keywords: symmetric inverse semigroup, Green's relations, wreath product.

[1]Taras Shevchenko National University of Kyiv, Department of Mechanics and Mathematics, 01601, Kyiv, Volodymyrska str., 64, e-mail: eugenia@univ.kiev.ua

Статтю представив доктор фізико-математичних наук, професор В. В. Кириченко

1 Basic definitions

For a set X, let $\mathcal{I}(X)$ denote the set of all partial bijections on X. Clearly, it is a semigroup under natural composition law. This semigroup is called the *full symmetric inverse semigroup* on X. If $X = \{1, \dots, n\}$, then semigroup $\mathcal{I}(X)$ is called the full symmetric inverse semigroup of rank n and is denoted $\mathcal{I}_n$.

Let S be a semigroup, (P, X) be a semigroup of partial transformations of the set X. Define the set S^{PX} as a set of partial functions from X to semigroup S:

$$S^{PX} = \{f : A \to S | \operatorname{dom}(f) = A, A \subseteq X\}.$$

Given $f, g \in S^{PX}$, the product fg is defined in a following way:

$$\operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g),$$
$$(fg)(x) = f(x)g(x) \text{ for all } x \in \operatorname{dom}(fg).$$

For $a \in P, f \in S^{PX}$, define f^a as:

$$(f^a)(x) = f(xa),$$
$$\operatorname{dom}(f^a) = \{x \in \operatorname{dom}(a); xa \in \operatorname{dom}(f)\}.$$

Wreath product of semigroup S with semigroup (P, X) of partial transformations of the set X is a set

$$\{(f, a) \in S^{PX} \times (P, X) \,|\, \operatorname{dom}(f) = \operatorname{dom}(a)\}$$

with composition defined by

$$(f, a) \cdot (g, b) = (fg^a, ab).$$

We will denote the wreath product of semigroups S and (P, X) by $S \wr_p P$.

Wreath product of inverse semigroups is an inverse semigroup. We can recursively define the wreath product of any finite number of inverse semigroups. Let T be a k-level n-regular rooted tree. By a partial automorphism we mean a root-preserving tree homomorphism defined on a connected subtree of T. The set $\operatorname{PAut} T$ of all partial automorphisms of T forms an inverse semigroup under partial automorphisms composition.

Theorem 1. [3] *Let T be a rooted k-level n-regular tree. Then*

$$\operatorname{PAut} T \cong \underbrace{\mathcal{I}_n \wr_p \mathcal{I}_n \wr_p \cdots \wr_p \mathcal{I}_n}_{k} = (\wr_p \mathcal{I}_n)^k.$$

2 Basic combinatorics

This sections contains brief summary of results from [3].

For an arbitrary function F we denote

$$F^k(x) = \underbrace{F(F \ldots (F}_{k}(x)) \ldots).$$

Proposition 1. *$|(\wr_p \mathcal{I}_n)^k| = S^k(1)$, where $S(x) = \sum_{i=1}^n \binom{n}{i}^2 i! x^i$*

Let $E(\mathcal{I}_n)$ be the set of idempotents of semigroup $\mathcal{I}_n$.

Proposition 2. *An element $(f, a) \in S \wr_p \mathcal{I}_n$ is an idempotent if and only if $a \in E(\mathcal{I}_n)$ and*

$$f(\mathrm{dom}(a)) \subseteq E(\mathcal{I}_n).$$

In terms of partial tree automorphisms, it means that idempotent is a partial automorphism, which acts identically on its domain.

Proposition 3. *Let $E((\wr_p \mathcal{I}_n)^k)$ be the set of idempotents of semigroup $(\wr_p \mathcal{I}_n)^k$. Then*

$$\left|E((\underset{p}{\wr} \mathcal{I}_n)^k)\right| = F^k(1) = \underbrace{(((1+1)^n + 1)^n \ldots + 1)^n}_{k},$$

where $F(x) = (x+1)^n$.

Recall that Green's $\mathcal{R}$-relation on inverse semigroup S is defined by

$$a \,\mathcal{R}\, b \Leftrightarrow aS^1 = bS^1,$$

similarly, Green's $\mathcal{L}$-relation is defined by

$$a \,\mathcal{L}\, b \Leftrightarrow S^1 a = S^1 b,$$

Green's $\mathcal{J}$-relation is defined by

$$a \,\mathcal{J}\, b \Leftrightarrow S^1 a S^1 = S^1 b S^1.$$

Green's $\mathcal{H}$- and $\mathcal{D}$-relations are derivative: $\mathcal{H} = \mathcal{R} \wedge \mathcal{L}$, $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.

Theorem 2. *Let $(f, a), (g, b) \in (\wr_p \mathcal{I}_n)^k$. Then*

1) $(f, a)\ \mathcal{L}\ (g, b)$ if and only if $\mathrm{ran}(a) = \mathrm{ran}(b)$ and $g^{b^{-1}}(z)\ \mathcal{L}\ f^{a^{-1}}(z)$ for all $z \in \mathrm{ran}(a)$, where a^{-1} is the inverse element for a;

2) $(f, a)\ \mathcal{R}\ (g, b)$ if and only if $\mathrm{dom}(a) = \mathrm{dom}(b)$ and $f(z) \,\mathcal{R}\, g(z)$ for all $z \in \mathrm{dom}(a)$;

3) $(f, a)\ \mathcal{H}\ (g, b)$ if and only if $\mathrm{ran}(a) = \mathrm{ran}(b)$ and $\mathrm{dom}(a) = \mathrm{dom}(b)$, $g^{b^{-1}}(z)\ \mathcal{L}\ f^{a^{-1}}(z)$ and $f(z) \,\mathcal{R}\, g(z)$ for $z \in \mathrm{dom}(a) \cap \mathrm{ran}(a)$;

4) $(f, a)\ \mathcal{D}\ (g, b)$ if and only if there exists a bijection map $x : \mathrm{dom}(b) \to \mathrm{dom}(a)$ such that $f(zx)\ \mathcal{D}\ g(z)$ for all $z \in \mathrm{dom}(x)$.

5) $\mathcal{D}{=}\mathcal{J}$.

It is well-known fact that in inverse semigroup each $\mathcal{R}$ ($\mathcal{L}$)-class contains exactly one idempotent, so the number of different $\mathcal{R}$ ($\mathcal{L}$)-classes in $(\wr_p \mathcal{I}_n)^k$ is equal to the number of idempotents of $(\wr_p \mathcal{I}_n)^k$.

Proposition 4. *The number of $\mathcal{D}$-classes of semigroup $(\wr_p \mathcal{I}_n)^k$ equals $P^k(1)$, where $P(x) = \binom{x+n}{n}$.*

3 Combinatorics of cross-sections

Now let ρ be an equivalence relation on a semigroup S. A subsemigroup $H \subset S$ is called *cross-section with respect to ρ* (or simply ρ-cross-section) provided that H contains exactly one element from every equivalence class. Correspondingly, cross-sections with respect to $\mathcal{R}$- ($\mathcal{L}$-) Green's relations are called $\mathcal{R}$- ($\mathcal{L}$-) cross-sections. Note that every $\mathcal{R}$- ($\mathcal{L}$-) equivalence class contains exactly one idempotent. Then the number of elements in every cross-section is $|E(S)|$, where $E(S)$ is the subsemigroup of all idempotents of S.

Observe that a subsemigroup H of semigroup $\mathcal{I}_n$ is an $\mathcal{R}$-cross-section if and only if for every subtree $\Gamma \subseteq T$ it contains exactly one element φ such that $\mathrm{dom}(\varphi) = \Gamma$.

Let now $\{1, \ldots, n\} = M_1 \sqcup M_2 \ldots \sqcup M_s$ be an arbitrary decomposition of $\{1, 2, \ldots, n\}$ into disjoint union of non-empty blocks, where the order of blocks is irrelevant. Assume that a linear order is fixed on the elements of every block: $M_i = \left\{m_1^i < m_2^i < \cdots < m_{|M_i|}^i\right\}$.

For each pair i, j $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant |M_i|$ denote by $a_{i,j}$ the element in $\mathcal{D}$-class D_{n-1} of rank $n-1$ of semigroup $\mathcal{I}_n$, containing *chain* $[m_1^i, m_2^i, \ldots, m_j^i]$, that acts as identity on the set

$$\{1, \ldots, n\} \setminus \{m_1^i, m_2^i, \ldots, m_j^i\}$$

and

$$a_{i,j}(m_k^i) = m_{k+1}^i,$$

$k = 1, \ldots, j-1$, $m_{j-1}^i \notin \mathrm{dom}(a_{i,j})$. Denote by $R = R(\overrightarrow{M_1}, \overrightarrow{M_2}, \ldots, \overrightarrow{M_k})$ the semigroup $\langle a_{i,j} |\ 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant |M_i| \rangle \sqcup \{e\}$. It is shown in [1, Chapter 12] that $R = R(\overrightarrow{M_1}, \overrightarrow{M_2}, \ldots, \overrightarrow{M_k})$ is an $\mathcal{R}$-cross-section of $\mathcal{I}_n$.

Define the map

$$\varphi_\mu : \prod_{i=1}^{k} (S \wr_p \mathcal{I}((M_i))) \to S \wr_p \mathcal{I}_n$$

in the following manner: φ_μ maps the product $\prod_{i=1}^{k}(f_i, a_i)$ to the element (f, a) such that $a\big|_{M_i} = a_i$, $f\big|_{M_i} = f_i$.

Theorem 3. [2] *Let* $R(\overrightarrow{M_1}, \overrightarrow{M_2}, \ldots, \overrightarrow{M_k})$ *be an* $\mathcal{R}$*-cross-section of semigroup* $\mathcal{I}_n$*,* $R_1, \ldots, R_k$ *be* $\mathcal{R}$*-cross-sections of semigroup* $\mathcal{I}_m$*. Then*

$$R = \varphi_\mu \Big((R_1 \wr_p R(\overrightarrow{M_1})) \times \ldots \times (R_k \wr_p R(\overrightarrow{M_k})) \Big)$$

is an $\mathcal{R}$*-cross-section of semigroup* $\mathcal{I}_m \wr_p \mathcal{I}_n$*.*

Moreover, every $\mathcal{R}$*-cross-section of semigroup* $\mathcal{I}_m \wr_p \mathcal{I}_n$ *is isomorphic to*

$$\Big(R_1 \wr_p R(\overrightarrow{M_1})) \times \ldots \times (R_k \wr_p R(\overrightarrow{M_k}) \Big).$$

A map $a \mapsto a^{-1}$ is an anti-isomorphism of semigroup $\mathcal{I}_m \wr_p \mathcal{I}_n$, that sends $\mathcal{R}$-classes to $\mathcal{L}$-classes. It is also clear that it maps $\mathcal{R}$-cross-sections to $\mathcal{L}$-cross-sections and vice-versa. Hence dualizing Theorem 3, one gets a description of $\mathcal{L}$-cross-sections.

Theorem 4. [2] *Let* R'*,* R'' *be* $\mathcal{R}$*-cross-sections of the semigroup* $\mathcal{I}_m \wr_p \mathcal{I}_n$*,* $\varphi\colon R' \to R''$ *be an isomorphism. Then there exists such an element* $\Theta = (\vartheta, \theta) \in \mathcal{S}_m \wr \mathcal{S}_n$ *that*

$$\varphi\big((f, a)\big) = \Theta^{-1}(f, a)\Theta.$$

In other words, if $(f, a) \in R'$ *and* $(g, b) = \varphi\big((f, a)\big)$*, then* $\operatorname{dom} b = \theta\big(\operatorname{dom}(a)\big)$ *and for any* $x \in \operatorname{dom}(a)$

$$\theta\big(a(x)\big) = b\big(\theta(x)\big),$$

$$g\big(\theta(x)\big) = \vartheta^{-1}(x) f(x) \vartheta\big(a(x)\big).$$

Corollary 1. *Semigroup* $\mathcal{I}_m \wr_p \mathcal{I}_n$ *contains*

$$(m!)^n \, n! \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} \left(\sum_{i=1}^{m} \frac{1}{i!} \binom{m-1}{i-1} \right)^k$$

different $\mathcal{R}$*-(*$\mathcal{L}$*-) cross-sections.*

Corollary 2. *The number of non-isomorphic* $\mathcal{R}$*-(*$\mathcal{L}$*-) cross-sections of* $\mathcal{I}_m \wr_p \mathcal{I}_n$ *is*

$$\sum_{\substack{j_1, j_2, \ldots, j_n \geqslant 0 \\ j_1 + 2j_2 + \cdots + nj_n = n}} \prod_{i=1}^{m} \binom{p_m + j_i - 1}{j_i},$$

where p_n *denotes the number of decompositions of* n *into the sum of positive integers, where the order of summands is not important.*

4 Maximal nilpotent subsemigroups and its combinatorics

Let S be a semigroup with a zero element 0. An element $a \in S$ is called *nilpotent* if for some $r \in \mathbb{N}$ $a^r = 0$. Semigroup S is called *nilpotent* if for some $r \in \mathbb{N}$

$$a_1 \cdot a_2 \cdot \ldots \cdot a_r = 0$$

for arbitrary $a_1, a_2, \ldots, a_r \in S$.

Usually, two classes of nilpotent subsemigroups are distinguished. First class includes nilpotent subsemigroup containing semigroup zero 0. In this case we have $H^r = \{0\}$ for some $r > 0$, $H \subset S$. Second class includes subsemigroups of S, which are nilpotent as semigroups, but their zero element may differ from 0. In this case we have $H^r = \{e\}$ for some idempotent and some $r \geqslant 1$. We will call these subsemigroups *proper nilpotent subsemigroups.*

A nilpotent subsemigroup $H \subset S$ is called *maximal nilpotent subsemigroup*, if it is not contained in any other nilpotent subsemigroup $H' \subset S$, $H \neq H'$.

For a semigroup $\mathcal{I}_n$ there is a one-to-one correspondence between maximal nilpotent subsemigroups and partitions of the set $\{1, 2, \ldots, n\}$ into disjoint union of ordered blocks:

If e is an idempotent, which is the zero in a $T \subset \mathcal{I}_n$, and

$$\overline{\operatorname{dom}(e)} = \{a_1, a_2, \ldots a_k\},$$

then every maximal nilpotent subsemigroup corresponds to a permutation $b_1, b_2, \ldots, b_k$ of $a_1, a_2, \ldots, a_k$ and has the form:

$$T = \Big\{ \sigma \in \mathcal{I}_n \,|\, \operatorname{dom}(e) \subseteq \operatorname{dom}(\sigma); \\ \sigma(x) = x \text{ for all } x \in \operatorname{dom}(e); \\ \sigma(b_i) = b_j \text{ implies } i < j \text{ for all } b_i \notin \operatorname{dom}(e) \Big\}$$

Details can be found in [1, Chapter 8].

Consider maximal nilpotent subsemigroups (those containing the semigroup zero) in a slightly more general setting.

Let P be an inverse semigroup.

Lemma 1. *An element $(f, a) \in P \wr_p \mathcal{I}_n$ is nilpotent iff $a \in \mathcal{I}_n$ is nilpotent.*

Proposition 5. *Let S be a maximal nilpotent subsemigroup of the semigroup $\mathcal{I}_n$. Then subsemigroup $P \wr_p S$ is a maximal nilpotent subsemigroup of the semigroup $P \wr_p \mathcal{I}_n$. Moreover, every maximal nilpotent subsemigroup of semigroup $P \wr_p \mathcal{I}_n$ is of this form.*

Corollary 3. *Maximal nilpotent subsemigroup of semigroup $(\wr_p \mathcal{I}_n)^k$ are those having the form*

$$((\underset{p}{\wr} \mathcal{I}_n)^{k-1}) \wr_p S,$$

where S is a maximal nilpotent subsemigroup of the semigroup $\mathcal{I}_n$.

Now consider proper maximal nilpotent subsemigroups.

Let T be a proper nilpotent subsemigroup of $\mathrm{PAut}\, T$ with a zero $e \in E(\mathrm{PAut}\, T)$. Denote by T_x the maximal subtree of T such that its root is $x \in VT$ and none of the edge of T_x is in $\mathrm{dom}(e)$.

Theorem 5. *Let T be k-level n-regular rooted tree. Proper maximal subsemigroup of $\mathrm{PAut}\, T$ is (canonically) isomorphic to*

$$\prod_{x \in \mathrm{dom}(e)} Nilp_x,$$

where $Nilp_x$ is a maximal nilpotent subsemigroup of $\mathrm{PAut}\, T_x$.

Proposition 6. *The cardinality of a maximal subsemigroup of $P \wr_p \mathcal{I}_n$ is equal to*

$$|P|^n B_n \left(\frac{1}{|P|} \right),$$

where $B_n(x)$ denotes the nth Bell polynomial.

Consequently, the cardinality of a maximal subsemigroup of $(\wr_p \mathcal{I}_n)^k$ is

$$\left| (\underset{p}{\wr} \mathcal{I}_n)^{k-1} \right|^n B_n \left(\frac{1}{\left| (\wr_p \mathcal{I}_n)^{k-1} \right|} \right).$$

Proposition 7. *The number of proper maximal nilpotent subsemigroups of $\mathrm{PAut}\, T$ with a zero $e \in E(\mathrm{PAut}\, T)$ equals*

$$\prod_{x \in \mathrm{dom}(e)} (k_x)!,$$

where k_x is the number of vertices of the first level of the tree T_x.

Let $l(x)$ be the level of T, where the vertex x lies, let $h(x) = k - l(x) + 1$. The cardinality of every maximal proper nilpotent subsemigroup equals

$$\prod_{x \in \mathrm{dom}(e)} \left| (\underset{p}{\wr} \mathcal{I}_n)^{h(x)} \right|^{k_x} B_{k_x} \left(\frac{1}{\left| (\wr_p \mathcal{I}_n)^{h(x)} \right|} \right).$$

Список використаних джерел

1. Ganyushkin O., Mazorchuk V. Classical finite transformation semigroups. An introduction./Ганюшкін О. London: Springer, 2009. – xii, 314 p.

2. Kochubinska Ye. On isomorphisms of $\mathcal{R}$- and $\mathcal{L}$-cross-sections of wreath products of finite inverse symmetric semigroups./ Кочубінська Є. // Int. J. Algebra Comput., vol. 19 (2009), no. 8, pp. 999 – 1010.

3. Kochubinska Ye. Combinatorics of partial wreath power of finite inverse symmetric semigroup $\mathcal{IS}_d$./ Кочубінська Є. // Algebra and Discrete Mathematics, no. 1 (2007), pp. 49–61.

Надійшла до редколегії 14.03.2013