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Пошарова та топологічна еквівалентність m-функцій

Вивчається пошарова та топологічна еквівалентність m-функцій на двовимірному диску, що еквівалентна топологічній класифікації векторного поля косого градієнта sgrad f.

Побудувані графи Ріба для m-функцій з n критичними точками, що задають пошарову еквівалентність, та написано алгоритм для побудови двох топологічно неізоморфних графів, що задають топологічну еквівалентність.

Ключові слова: граф Ріба, атом, молекула

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Layers and topological equivalent of m-functions.

The layering and topological equivalence of m-functions on the two-dimensional disc are studied, which is equivalent to the topological classification of gradient vector field oblique sgrad f.

Rib's graphs of m-functions with n critical points that asks layered equivalence are constracted and an algorithm to construct two topologically non-isomorph graphs that asks topological equivalence is written.

Keywords: Rib's graph, atom, molecule

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Introdution

Let M is a closed two-dimensional manifold, f – smooth function on M. Consider the Hamiltonian dynamical system, given by the equation $\frac{dx}{dt} = sgradf(x),\ x \in M$. Then its trajectory lying on the components of the line-level function f. These components are called layers. Homeomorphism surface reflecting layers on layers, called layers equivalence. Thus, the layered classification function sets the topological classification of Hamiltonian dynamical systems. In general function can be quite complex structure. However, the set of all functions can be divided into open everywhere dense subset consisting of simple Morse functions. Konrod A. and G. Rib for research functions have filled graph obtained from the surface after charging each layer to the point. This graph is a complete topological invariant of simple Morse functions. For arbitrary Morse functions except this column should additional information.

In the work O.V. Bolsinov and A. T. Fomenko proposed layered neighborhood of critical level to call atom and Rib's graph, in which vertices correspond to atoms and edges correspond components of atoms, to call molecule. Then it was possible to build a layered and arbitrary topological classification of Morse functions. For manifolds with boundary analogue of Morse functions is m-functions. It is a function, in which all critical points are nondegenerate and such that the restriction of the function on the border is a Morse function. Topological properties of m-functions studied in the works of S. Maksymenko for topological classification of m-functions used graphs with an involution.

The aim is to complete layering and topological classification of m-functions on the two-dimensional disc, which is equivalent to the topological classification of gradient vector field of oblique sgrad f. The objective of the work is to construct Rib's graphs of m-functions with n critical points that asks layered equivalence and write an algorithm to construct two topologically non-isomorph graphs that asks topological equivalence.

1 Basic notions

We consider all m-function, in which there are no internal critical points, but there are only critical points on a border.

a is a regular value of the function, c is a critical value of the function.

Arbitrarily small movements of the m-function can achieve that at each critical level c, the set of points x, for which $f(x) = c$, lying exactly one critical point. In other words, the critical points that were on the same level, can be made to lying on nearby level.

Definition: m- function which has exactly one critical point in each level is called simple.

Let f is m-function on the two-dimensional disk. Consider an arbitrary segment level $f^{-1}(a)$ and its connectivity components that are called layers. As a result, the disk is partitioned in association layers and get a bundle of features. Got a space layer Γ. For m- function space Γ is a graph.

Definition: Graph Γ is called Rib's graph of m-function f on the two-dimensional disk G. Vertex of Rib's graph is called finite if it is the end of exactly one edge of the graph. All other vertex is called internal.

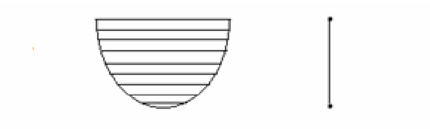
Let f is m-function on two-dimensional disk G such that $f: G \to \mathbb{R}$. Let g is the other m-function on the other two-dimensional disk H such that $g: H \to \mathbb{R}$. Consider the question of equivalence of two m-functions. Consider the pair (G, f) and (H, g).

Definition: The functions f, g on the corresponding two-dimensional layer discs will be called equivalent if there exists a diffeomorphism $\lambda: G \to H$, which switches connected components of level lines of f in connected components of level lines function g.

Definition: The functions f, g on the corresponding two-dimensional disk will be called topologically equivalent if there are homeomorphisms $h: G \to H, h': \mathbb{R} \to \mathbb{R}$, for which there is equality $f \circ h = h' \circ g$.

If a is a regular value, then the corresponding line level is a segment. These segments are rearranged when passing through a particular level of function.

Atom A: neighborhood of local extremum of function.



Picture 1. Atom A and its Rib's graph

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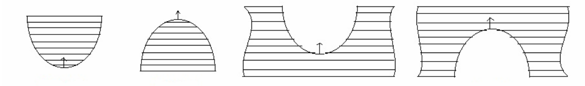
Picture 2. Atom B and its Rib's graph

That is, for each of the two atoms have two cases:

1) the gradient field directed inward;
2) the gradient field directed outward.

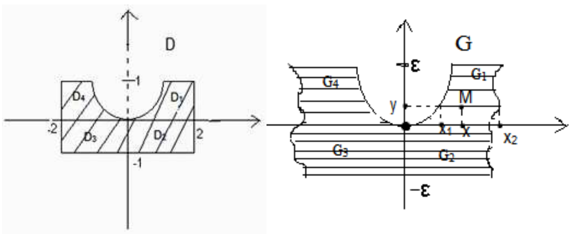


Picture 3. Atoms A_1, A_2, B_1, B_2

2 Preliminary results

Theorem (*about atom B***).** *For any point which is not a local extremum of m-function exists a homomorphism φ,which reflecting the neighborhood of this point (region G) on the standard area $D = \{x, y \mid x^2 + (y-1)^2 \geq 1,\ x \in [-2,2],\ y \in [-1,1]\}$.*

Proof. Consider the region $D = \{x, y \mid x^2 + (y-1)^2 \geq 1,\ x \in [-2,2],\ y \in [-1,1]\}$, is shown at the picture 4.



Picture 4. Regione D and region G.

Consider the point $M(x, y) \in G_1$. Let this point is located at the level y, where $x_1 = f^{-1}(f(x)) \cap S^1$, $x_2 = f^{-1}(f(x)) \cap \partial M$.

Construct a homomorphism that will translate the point $M(x, y)$ into the point $M'(x', y') \in D$. For coordinate y' get $y' = \frac{y - f(x_0,\ y_0)}{\varepsilon}$.

For each x' and y' we know, that $\{x, y \mid (x')^2 + (y'-1)^2 \geq 1\}$ are the points from the border, therefore $x' \geq \pm\sqrt{1-(y'-1)^2}$.

Let divide the region G into 4 regions G_1, G_2, G_3, G_4, corresponding to quadrants of the plane OXY.

First, let consider I the quadrant, scilicet region G_1. Therefore here $x' \geq \sqrt{1-(y'-1)^2}$, reflected on D_1. Hence we obtain $x_1 = \sqrt{1-(y'-1)^2}$. Then $x' = \frac{\rho(x_1, x)}{\rho(x_1, x_2)} \cdot (2 - x_1) + x_1$.

Next, consider the second quadrant, that is the region that reflected on D_2 (standard region). Consider $M(x, y) \in G_2$, located at the level y, where $x_1 = f^{-1}(f(x)) \cap S^1$, $x_2 = f^{-1}(f(x)) \cap \partial M$ is the end of a level, which belong to G_2. Then

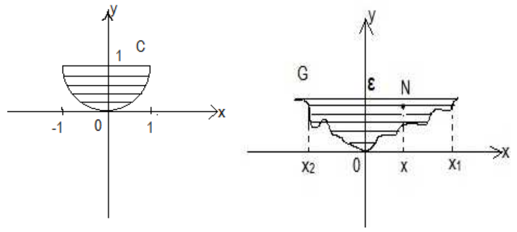
$y' = \frac{y-f(x_0,\ y_0)}{\varepsilon}$, $x' = \frac{\rho(x_1 x)}{\rho(x_1 x_2)} \times 2$. That got the formulas of transition.

For the third quadrant have the following formulas of transition: $y' = \frac{y-f(x_0,\ y_0)}{\varepsilon}$, $x' = \frac{\rho(x_2 x)}{\rho(x_1 x_2)} \times (-2)$.

For fourth quadrant get $y' = \frac{y-f(x_0,\ y_0)}{\varepsilon}$, $x' = \frac{\rho(x_1 x)}{\rho(x_1 x_2)} \times (-2 - x_1) + x_1$. Theorem is proofed.

Theorem (*about atom A).* *For any point of local extremum of m – function exists a homomorphism ψ, reflecting the neighborhood of this point by the standard area* $C = \{x, y | x^2 + (y-1)^2 \le 1, y \in [0,1]\}$

Proof. Let Build a homomorphism that translates region G into the region C.



Picture 5. Region C and Region D.

Define this homomorphism separately for the coordinates y and x. For example let considerate point $N(x,y) \in G$, which translate into the point $N(x',y') \in C$.

Let considerate distances $\rho(x_1 x_2)$ and $\rho(x_1 x)$. The picture shows that the first distance is greater than the second. Then $\frac{\rho(x_1 x_2)}{\rho(x_1 x)} \le 1$. Let $x_* = \sqrt{1-(y-1)^2}$ – points from ∂C, then we obtain the formula of the transition x into x': $x' = \frac{\rho(x_1 x_2)}{\rho(x_1 x)} \cdot (1 - x_*) + x_*$. For coordinates y : $y' = \frac{y-f(x_0 y_0)}{\varepsilon}$, where $f(x_0, y_0)$- value of the function at the critical point. Theorem is proofed.

Theorem. *Any simple atom coincides with the atom A_1, A_2, B_1, B_2.*

*Proof***.** There are two cases: either the point is a local extremum or not. So then we get 4 cases. Consider the case when the point is a local extremum, then the vector directed either inward or outward. Then the previous theorem showed that this object is homeomorphic standard field and thus coincide respectively with the corresponding atoms A_1, A_2, B_1, B_2. Theorem is proofed.

Let f is a simple $m-$ function on two-dimensional disk G. Consider its Rib's graph Γ. Its vertices correspond to the critical layers m-function. Let's replace neighborhood of these vertexes with corresponding atoms A_1, A_2, A_3, A_4. The resulting graph will be called a molecule W.

The concept of simple molecules actually no different from Rib's graph for simple $m-$ function, but for complex $m-$ function, molecule will carry a more information than Rib's graph. Consider a finite connected graph with vertices of degrees 1,2 and 3. In each vertex of multiplicity 1 place the atom $A_1 A_2$ (depending on the minimum or maximum). At each vertex of multiplicity 3 we put B_1, B_2.

2.1 Layers equivalence of $m-$ functions

Theorem *(implementation for layering equivalence). For each molecule W exists a function that defines this molecule up to a layer-equivalence.*

Proof. Consider the molecule W, which is embedded in $\mathbb{R}^2$. Consider the neighborhood of the molecule and the function of the height $f(x,y) = y$.

So, to get a function that sets this molecule, we use the following. We divide the molecule under the following simple molecules (picture 6).

Picture 6. Simple molecules.

These four cases are considering getting a small neighborhood of the vertex of the molecule that is incident to three edges. In particular, the last two - a small neighborhood of a vertex that is incident to one edge.

Then have the following four parts (picture 3) corresponding $m-$ function (m- function constraints per atom).

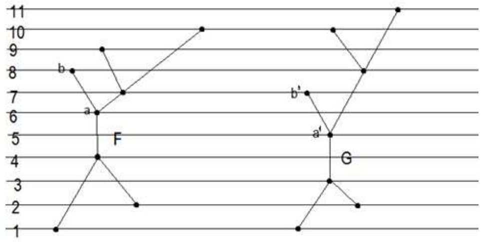
Indeed, a molecule tells us that of which pieces we need to stick together border of m-function, which components of its border should be taken. Theorem is proofed.

Definition: We say that graphs are layers isomorphic if for each vertex from the fact that $f(x_i) < f(x_j)$ implies that $f(x_i') < f(x_j')$,the edges have the same monotony. A similar definition for layers isomorphic molecules.

Definition: We say that graphs are topologically isomorphic if for every vertex $f(x_i, y_i) = f(x_i', y_i) = y_i$. A similar definition for topologically isomorphic molecules.

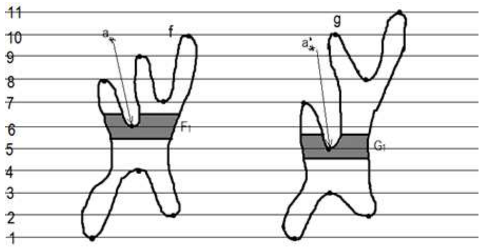
Theorem *(Layered equivalence of m –functions)***.** *Let $W(G,f)$ and $W(H,g)$ – simple molecules of two simple m –functions on the two-dimensional disks G і H. If the molecules are layers* *isomorphic, the functions f and g are layers equivalent.*

Proof. Consider two molecules F and G (Picture 7) are isomorphic, exists homomorphism that transits a vertex a into vertex a', b into vertex b'.



Picture 7. Molecules F and G.

Consider the functions f and g. For each vertex we have a molecule corresponding (picture 8) to the extremum point of the function $a \to a_*$, $a' \to a'_*$.

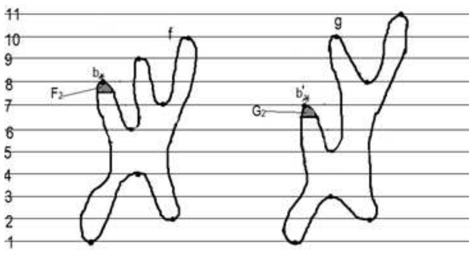


Picture 8. Corresponding *m*-function for F and G molecules.

For each such point $(a_*\ and\ a'_*)$ take a neighborhood, then get atoms homeomorphic to the standard atom B_1. By theorem about atom B such homomorphism exists. Let $\varphi_1: F_1 \to B_1, \varphi_2: G_1 \to B_1$.

Then $\varphi = \varphi_1 \circ \varphi_2^{-1}: F_1 \to G_1$.

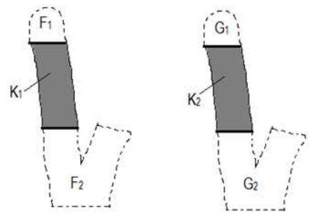
Similarly, construct a homomorphism to a neighborhood of each critical point functions f and g. The corresponding figures are shown at the picture 9.



Picture 9. The neiborhood of critical point.

$\psi_1: F_2 \to A_2$, $\psi_2: G_2 \to A_2$. Hence we obtain the reflection $\psi = \psi_1 \circ \psi_2^{-1}: F_2 \to G_2$, that translates a neighborhood of the critical point of the function f in the corresponding neighborhood of the critical point of the function g.

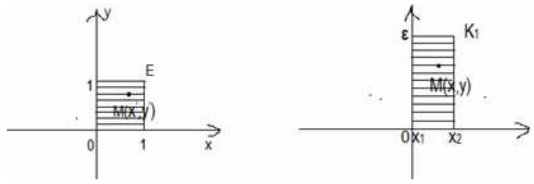
Consider further parts of molecules that are shown in the pictures 10.



Picture 10. Some parts of *m*-function.

It is clear that we have built an reflection except homomorphism $\gamma: K_1 \to K_2$. Let construct it.

Place K_1 in the coordinate plane. And transits it on the standard square $E = \{(x,y) | x \in [0,1], y \in [0,1]\}$(pic.11)



Picture 11. Region E and K_1

We have reflection γ_1 such as $x' = \frac{\rho(x_1,x)}{\rho(x_1,x_2)}$, $y' = \frac{y - f(x_0,y_0)}{\varepsilon}$, where $f(x_0,y_0)$ is a value of function in this case at the point $(0,0)$, because in that way we placed K_1. So have $\gamma_1: K_1 \to E$. Similarly, we obtain $\gamma_2: K_2 \to E$. Hence $\gamma = \gamma_1 * \gamma_2^{-1}: K_1 \to K_2$. Prove that $\varphi\ and\ \gamma$ coincide on the border. Studying these formulas homomorphisms, we see that they are given so that the stored lengths of proportionality.

So get a reflection that is layered equivalency. That is, applying φ, ψ, γ in turns according to the type (kind) of the critical point and stick the edge get $\lambda = \varphi * \psi * \gamma$ – diffeomorphism that is layered equivalency.

Theorem is proofed.

2.2 Topological equivalence of m – function

Theorem *(implementation for topological equivalence). For each molecule W exists a function that defines this molecule up to topological equivalence.*

Proof: Consider the molecule W, which is embedded in $\mathbb{R}^2$. So to get the function to this molecule sets, we use the following picture 6.

These four cases we get when considerate a small neighborhood of the vertex of the molecule that is incident to three edges. In particular, the last two - a small neighborhood of a vertex that is incident to one edge. Then have the following four parts corresponding m-function (m- function constraints per atom, picture 3).

Indeed, a molecule tells us that of which pieces we need to stick border of m-function, which the components of its boundary should be taken. For getting m –function we need:

1) Each vertex placed at their level;

2) In each of the points we see, which part of the molecule, corresponding to the neighborhood of this point;

3) Draw at this point the restriction of m – functions per atom;

4) Knowing how to point, which were connected on the molecule, for a function that would mean monotony appropriate.

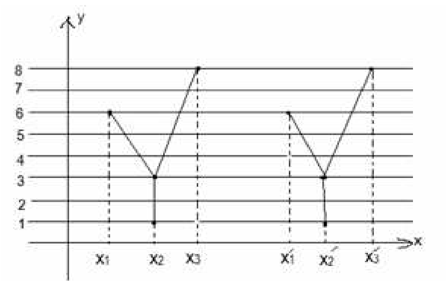
5) In paragraph 3, we still have only some fragments of atoms, so now by paragraph 4 (knowing the monotony), then paste $I \times [0,1]$, where I – segment of length equal to the number of levels between critical points whose atoms we put together. For example, the first point is located on the first level 1, and the second on the 5th, then I=$[1+\varepsilon, 5-\varepsilon]$.

Function is built.

Theorem is proofed.

Theorem *(topological equivalence of* m *– function). Let* $W(G,f)$ *and* $W(H,g)$ *– simple molecules of two simple* m*-functions on the two-dimensional disks* G *and* H*. If the molecules are topologically isomorphic, the functions f and g are topologically equivalent.*

Proof: Consider two topologically isomorphic molecules and the corresponding m –functions.



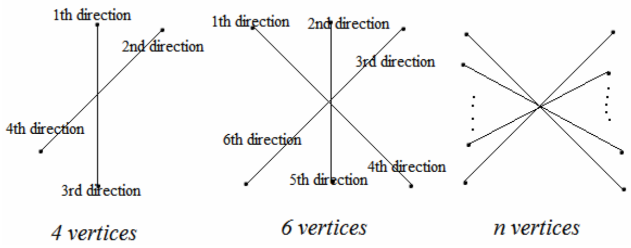
Exsist homomorphisms $\varphi: F_2 \to G_2$, $\psi: F_1 \to G_1$, $\gamma: F_3 \to G_3$, which when pasting provide topologically equivalent m-functions by the theorem implementation for topological equivalence.

Theorem is proofed.

3 Algorithm of finding number of topological nonequivalent m –functions.

The purpose of writing a program is to count the number of topologically nonequivalent m-functions, which Rib's graph containing n vertexes.

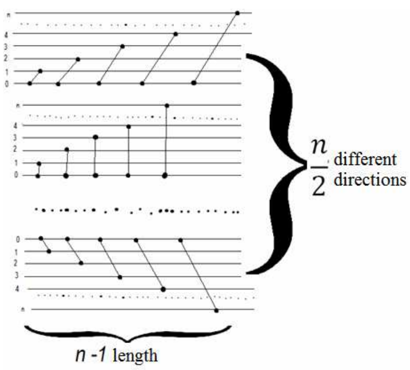
Part 1. Total there are n directions for edges of the Rib's graph of m -functions. There is an example on the picture 13.



Picture 13. Possible directions .

That is, for each vertex we have $\frac{n}{2}$ different directions and $\frac{n}{2}$ their opposite directions. For each edge fix its maximum length that can equal numbers from 1 to n.

Getting $\frac{n}{2} \times (n-1)$ different edges.



Picture 14. Possible edges

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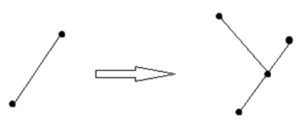
j=1,…,$\frac{n}{2}$ (choose from ready edges), add new edge. An exsample is shown on the picture 15.



Picture 15. An example.

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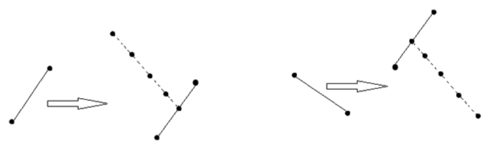
Picture 16. An example.

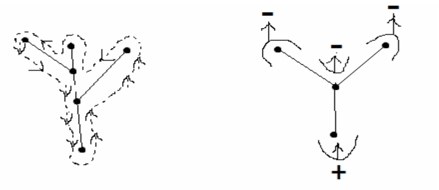
And for that edge that we have added select where it will be situated. So choose one of the verteces (levels) that still free(Picture 17).



Picture 17. Possible positions for the edge.

Then go from j to j +1 and write these columns in the array.

Part 3. All these graphs have $n-1$ edges. Next, define the m –function, which corresponds to the resulting graph. We pass on all graphs defining the substitution of «+» and « - ».



Picture 18. First step for building *m*-function.

point (vertex of graph).

Then check if the number of vertices, which we avoided is new. If so, add its number in permutation.

The vertex is new if program passed it by one of the following two ways (an example is on the picture 19), when the top has changed to the opposite direction (downward).

Picture 19. An example.

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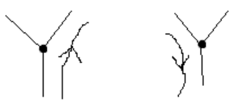

Picture 20. An example.

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appropriate cell fits if realized this permutation on the corresponding set of «+» i « - ».

Then based two-dimensional table with those implemented. Print a table on the screen.

4 Result: number of m –functions

The number of layers non-equivalent m -functions with the appropriate number of critical points.

4 critical points – 2 layers non-equivalent m -functions;

6 critical points – 17 layers non-equivalent m -functions;

8 critical points – 134 layers non-equivalent m -functions.

The number of topologically nonequivalent m-functions with the appropriate number of critical points.

4 critical points – 4 topologically nonequivalent m-functions

6 critical points – 72 topologically nonequivalent m-functions

8 critical points – 2960 topologically nonequivalent m-functions.

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