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Багатовимірні тріадно-усічені симплекси

Розглянуто тип опуклих многогранників, які названо тріадно-усіченими симплексами. З точки зору конструктивного об'єкту у векторних просторах розмірності чотири та вище зазначені многогранники є багатовимірними аналогами одного з класичних напівправильних многогранників, а саме усіченого тетраедру.

Представлено результати досліджень внутрішньої геометричної структури та комбінаторних характеристик повної сукупності граней тріадно-усічених симплексів у векторних просторах довільної розмірності.

Ключові слова: конструктивний геометричний об'єкт, багатовимірний аналог усіченого тетраедру, тріадно-усічений симплекс, комбінаторні характеристики.

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Multidimensional triangle-truncated simplexes

The type of convex polytopes, called triangle-truncated simplexes, is considered. As a constructive object in four and higher dimension's vector space such polytopes are multidimensional analogues of one classical semi-regular polytopes namely truncated tetrahedron.

In paper we present the results of investigations underlying geometrical structure and combinatorial characteristics of the complete assemblage of faces of triangle-truncated simplexes in vector spaces of arbitrary dimensionality.

Key Words: constructive geometrical object, multidimensional analogue of truncated tetrahedron, triangle-truncated simplex, combinatorial characteristics.

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1 Introduction

In the process of researches devoted to some types of constructive fractal simplex-sets [1] in four and higher dimension's vector spaces the direct involvement of the specified fractals with the truncated simplexes, representing multidimensional analogs of one of classical semiregular polytopes namely the truncated tetrahedron [2] as constructive object, was revealed.

On the whole, many types of multidimensional truncated polytopes are for today not only well-known, but also amply investigated [3, 4, 5, 6, 7]. At the same time, unlike three classical regular convex polytopes (multidimensional simplex, cube and cocube), for which combinatorial characteristics of the complete assemblage of $i-$faces, $i = \overline{0, n-1}$, are defined in an explicit form [3, 5, 7, 8, 9, 10, 11, 12], analogous expressions for the truncated polytopes, including the triangle-truncated simplexes, in vector spaces of arbitrary dimensionality to the author did not meet. Numerical characteristics of the complete assemblage of faces are known only for particular types of the truncated polytopes of small dimensions.

Since the multidimensional triangle-truncated simplexes turned out rather uncommon geometrical objects with the complex underlying structure, research of combinatorial characteristics of the last can claim for a role of an independent task.

2 The geometrical structure and combinatorial characteristics of multidimensional triangle-truncated simplexes

Let's enter the following definition of the multidimensional triangle-truncated simplex in vector space of arbitrary dimensionality.

Definition 2.1. The $n-$dimensional polytope, $n \geqslant 2$, being a convex hull of $2C_{n+1}^2$ points (vertices of the triangle-truncated simplex), partitioning $1-$faces of the arbitrary $n-$dimensional simplex in the $1:1:1$ proportion, is called an $n-$dimensional triangle-truncated simplex (${}_{tt}Simp^n$).

At the same time, from the point of view of constructive object, this polytope can be considered as result of truncation of corresponding multidimensional simplex.

Theorem 2.1. *The $n-$dimensional triangle-truncated simplex is a convex polytope, which $(n-1)-$faces are presented by equal quantity of polytopes of two types, namely*

— simplexes;

— triangle-truncated simplexes.

Thus quantity of $i-$faces, $i = \overline{0, n-1}$, of $n-$dimensional triangle-truncated simplex is equally:

$$\begin{cases} N_i({}_{tt}Simp^n) = (n-i+1) \cdot C_{n+1}^{n-i}, & i = \overline{1, n-1}, \\ N_0({}_{tt}Simp^n) = 2C_{n+1}^{n-1}. \end{cases} \tag{1}$$

This theorem gives complete description of underlying geometrical structure and the basic characteristics which unambiguously identify triangle-truncated simplexes in vector spaces of arbitrary dimensionality as independent geometrical objects.

Before to pass directly to proofs of Theorem (2.1) by the method of direct count and method of mathematical induction, will formulate a next lemma.

Lemma 1. a) *Each triangle-truncated simplex which is the $i-$face, $i = \overline{1, n-1}$, of $n-$dimensional triangle-truncated simplex, $n \geqslant 2$, belongs to the $(n-i)$ triangle-truncated simplexes being $(i+1)-$faces of specified geometrical object.*

б) *Each simplex which is the $i-$face, $i = \overline{0, n-2}$, of $n-$dimensional triangle-truncated simplex, $n \geqslant 2$, belongs to the $(n-i-1)$ triangle-truncated simplexes being $(i+2)-$faces of specified geometrical object.*

The proof of this lemma is carried out analogously to the proof of Lemma 1 [1] about internal belonging of $i-$faces, $i = \overline{0, n-1}$, of an $n-$dimensional middle-truncated simplex, $n \geqslant 2$.

Let's mark those statements of resulted lemma it is used in process of the proof of Theorem (2.1) by both a method of direct count and method of mathematical induction.

Proof of Theorem (2.1) *by a method of direct count.*

In case of $n = 2, 3$ this theorem is obviously true as the two-dimensional triangle-truncated simplex is a hexagon, three-dimensional — one of classical semiregular polytopes namely a triangle-truncated tetrahedron [2].

An underlying geometrical structure of $n-$dimensional triangle-truncated simplex, $n \geqslant 4$, on the whole is analogously to the underlying geometrical structure of $n-$dimensional middle-truncated simplex [1]. The single substantial difference consists that unlike the second polytope, which $2-$faces are triangles, $2-$faces of the first are hexagons. The last causes different quantities of $1-$ and $0-$faces.

I, II stages. The quantity of $i-$faces, $i = \overline{n-1, 2}$, of a polytope ${}_{tt}Simp^n$ obviously coincides with the quantity of corresponding faces of a polytope ${}_{mt}Simp^n$ and is defined analogously (see I, II stages of the first proof of Theorem 1 [1] according to approaches of common methodology of definition combinatorial characteristics of $n-$dimensional, $n \geqslant 4$, middle/triangle-truncated simplex), i.e. $N_i({}_{tt}Simp^n) = (n-i+1) \cdot C_{n+1}^{n-i}$, $i = \overline{n-1, 2}$. Let's mark that in the process of definition of $i-$faces, $i = \overline{n-2, 2}$, dependence of item a) of Lemma (1) is used.

Unlike an $n-$dimensional middle-truncated simplex, representing the limiting case of truncation initial, that entails degeneration of its $1-$faces being middle-truncated simplexes in points (i.e. in fact in a $0-$faces), $1-$faces of $n-$dimensional triangle-truncated simplex are presented both simplexes and triangle-truncated simplexes as its $2-$sides are hexagons. Thus, the quantity of $1-$faces of a polytope ${}_{tt}Simp^n$ also is defined according to common methodological approaches of definition combinatorial characteristics:

$$N_1({}_{tt}Simp^n) = C_{n+1}^{n-2} \cdot \left[3 + \frac{3}{n-1}\right] = n \cdot C_{n+1}^{n-1}.$$

III stage. The quantity of vertices ($0-$faces) of the triangle-truncated simplex ${}_{tt}Simp^n$ coincides with the doubled number of edges of an initial set of construction, in which it is inscribed entered, by definition: $N_0({}_{tt}Simp^n) = 2 \cdot C_{n+1}^{n-1}$.

As a result, we will receive the statement (1).

IV stage: verification of authenticity of results. It is known that in case of convex polytopes of an arbitrary dimension $n \geqslant 2$ the formula of Euler-Poincare is true [3, 7, 9, 10, 12].

Proceeding from the got results in part of combinatorial characteristics of the multidimensional triangle-truncated simplex, it is simple to make sure that in this case the formula of Euler-Poincare really is true. □

Corollary 1. *$i-$faces, $i = \overline{1, n-1}$, of the $n-$dimensional triangle-truncated simplex are presented by $(n-i)\cdot C_{n+1}^{n-i}$ simplexes and the C_{n+1}^{n-i} triangle-truncated simplexes.*

Also as well as in case of the middle-truncated simplex [1] this corollary naturally follows from the first proof of the Theorem (2.1).

Proof of Theorem (2.1) *by a method of mathematical induction.* In case of $n = 2, 3$ this theorem is obviously true as the two-dimensional triangle-truncated simplex is a hexagon, three-dimensional — one of classical semiregular polytopes namely a triangle-truncated tetrahedron [2].

Let's suppose that in a vector space $V^n(\Re)$, $n \geqslant 4$, an $n-$dimensional triangle-truncated simplex has the following combinatorial characteristics:

$$\begin{cases} N_i({}_{tt}Simp^n) = (n-i+1)\cdot C_{n+1}^{n-i}, & i = \overline{1, n-1}, \\ N_0({}_{tt}Simp^n) = 2C_{n+1}^{n-1}. \end{cases}$$

Thus for $i-$faces, $i = \overline{0, n-1}$, conditions of the Lemma (1) are satisfied.

Let's consider an $(n+1)-$dimensional triangle-truncated simplex in a vector space $V^{n+1}(\Re)$, $n \geqslant 4$. We will prove that its combinatorial characteristics look like:

$$\begin{cases} N_i({}_{tt}Simp^{n+1}) = (n-i+2)\cdot C_{n+2}^{n-i+1}, & i = \overline{1, n}, \\ N_0({}_{tt}Simp^{n+1}) = 2C_{n+2}^{n}. \end{cases} \tag{2}$$

$(n+1)-$dimensional triangle-truncated simplex is entered in the $(n+1)-$dimensional simplex. Therefore, its $n-$faces are presented by equal quantity of polytopes of two types: $(n+2)$ simplexes and $(n+2)$ triangle-truncated simplexes. From here, $N_n({}_{tt}Simp^{n+1}) = 2\cdot(n+2) = 2\cdot C_{n+2}^{1}$.

Analogously to a case of $(n+1)-$dimensional middle-truncated simplex [1]: *to define the quantity of $i-$faces, $i = \overline{0, n-1}$, of $(n+1)-$dimensional triangle-truncated simplex, it is necessary to define the corresponding quantity of $i-$faces of $(n+2)$ $n-$dimensional middle-truncated simplexes which agglutinate on $(n-1)-$dimensional middle-truncated simplexes.*

The quantity of $i-$faces, $i = \overline{1, n-1}$, each of the $(n+2)$ $n-$dimensional middle-truncated simplexes is equally: $N_i({}_{tt}Simp^n) = (n-i+1)\cdot C_{n+1}^{n-i}$.

Thus each assemblage of $i-$faces of the specified polytopes, $i = \overline{1, n-1}$, is presented by $(n-i)\cdot C_{n+1}^{n-i}$ simplexes and C_{n+1}^{n-i} triangle-truncated simplexes (see. Corollary (1)).

After agglutinating of the $(n+2)$ $n-$dimensional triangle-truncated simplexes each such $i-$dimensional simplex belongs to the $(n-i)$ polytopes ${}_{tt}Simp^{i+2}$, each $i-$dimensional triangle-truncated simplex belongs to $(n-i+1)$ polytopes ${}_{tt}Simp^{i+1}$.

This statement can be proved analogously to a case of the middle-truncated simplex [1] with use of the corresponding dependences of Lemma (1).

Thus, as in the process of agglutinating of the $(n+2)$ $n-$dimensional triangle-truncated simplexes each $i-$face of such polytope being a simplex is initially considered $(n-i)$ times and each $i-$face being a triangle-truncated simplex is considered $(n-i+1)$ times, we will receive the following expression for definition of quantity of $i-$faces, $i = \overline{1, n-1}$, of the $(n+1)-$dimensional triangle-truncated simplex:

$$N_i\Big({}_{tt}Simp^{n+1}\Big) = \\ = (n+2)\cdot\left[\frac{(n-i)\cdot C_{n+1}^{n-i}}{n-i} + \frac{C_{n+1}^{n-i}}{n-i+1}\right] = \\ = (n-i+2)\cdot C_{n+2}^{n-i+1},\ i = \overline{1, n-1}.$$

Thus specified $i-$faces, $i = \overline{1, n-1}$, are obviously presented by $(n-i+1)\cdot C_{n+2}^{n-i+1}$ simplexes and the C_{n+2}^{n-i+1} triangle-truncated simplexes.

At determining the quantity of $0-$faces of $(n+1)-$dimensional triangle-truncated simplex it is necessary to consider that its $2-$faces being triangle-truncated simplexes represent the hexagons, which $1-$faces are presented by six segments: three one-dimensional simplexes and three one-dimensional triangle-truncated simplexes.

As it is known that a common face of polytopes like a simplex and triangle-truncated simplex is the simplex (in this case zero-dimensional, i.e. a point) it is necessary to use dependence of the Lemma (1) for a zero-dimensional simplex.

Following the Lemma (1), for each of $(n+2)$ $n-$dimensional triangle-truncated simplexes it is executed: $Simp^0$ belongs to $(n-1)$ polytopes ${}_{tt}Simp^2$.

After agglutinating of $(n+2)$ $n-$dimensional triangle-truncated simplexes on the $(n-1)-$ dimensional triangle-truncated simplexes we have the following dependence: zero-dimensional simplex $Simp^0$ belongs to n two-dimensional triangle-truncated simplexes ${}_{tt}Simp^2$.

Thus, $N_0({}_{tt}Simp^{n+1}) = 2 \cdot C_{n+2}^{n}$.

On the basis of foregoing, combinatorial characteristics of the $(n+1)-$dimensional triangle-truncated simplex in a vector space $V^{n+1}(\Re)$, $n \geqslant 4$, really look like (2). $\square$

Validity of the Corollary (1) is confirmed by this proof also.

Corollary 2. *A degree of the arbitrary vertex of $n-$dimensional triangle-truncated simplex, $n \geqslant 2$, is equal to n.*

Corollary 3. *The polytope, dual to the $n-$dimensional triangle-truncated simplex, $n \geqslant 2$, has the following combinatorial characteristics:*

$$\begin{cases} N_{n-1} = 2 \cdot C_{n+1}^{n-1}, \\ N_i = (i+2) \cdot C_{n+1}^{i+1}, \quad i = \overline{n-2, 0}. \end{cases}$$

Thus the single case of self-duality of the triangle-truncated simplex is the two-dimensional triangle-truncated simplex.

3 Conclusion

The main result of presented matters is the theorem about combinatorial characteristics of multidimensional triangle-truncated simplexes, which unambiguously identifies the last as independent geometrical objects.

As the considered polytopes have a direct connection with specific types of constructive fractal simplex-sets, they can be of interest not only to experts in the field of multidimensional geometry, but also and the fractal analysis.

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