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## Температурна залежність надплинної компоненти та спектр елементарних збуджень

*У рамках запропонованої самоузгодженої мікроскопічної моделі надплинної нерелятивістської Бозе-рідини $^4$He з подавленим за рахунок багаточастинкових ефектів одночастинковим конденсатом та інтенсивним парним когерентним конденсатом наведено розрахунок спектру елементарних збуджень. Цей результат та розраховані в роботі температурні поправки до швидкостей першого та другого звуків використано для отримання температурної залежності густини конденсату.*

Ключові слова: *надплинність, мікроскопічна теорія, квазічастинковий спектр*

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## Temterature dependence of superfluid component and elementary excitation spectrum

*In frame of proposed selfconsistent microscopical model of the superfluid non-relativistic Bose liquid $^4$He with suppressed due to many-particle effects single-particle condensate and intensive pair coherent condensate the ab initio calculation of the quasiparticle spectrum was demonstrated. This result together with temperature corrections for the first and second speeds of sound calculated in this paper where used to obtain temperature dependence of the condensate density.*

Key Words: *superfluidity, microscopical theory, quasiparticle spectrum.*

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In series of papers renormalized perturbation theory was used, it was built on combined hydrodynamic variables with analytic normal and anomalous self-energy functions and a nonzero SF order parameter, proportional to the density of the SF component. On the base of this theory a closed system of nonlinear integral equations for the normal and anomalous self-energy parts were obtained. Unlike in the Bogoliubov theory of a quasi-ideal Bose gas, were small parameter is the ratio of the number of supracondensate excitations to the number of particles in an intensive BEC, the ratio of the BEC density to the total particle density of the Bose liquid was used as a small parameter of the model. Quasiparticle spectrum, obtained within this approach, is in a good agreement with experimental spectrum of elementary excitations in superfluid $^4$He. And, as it was shown, the roton minimum in the spectrum is associated with negative minimum of the Fourier component of the pair interaction potential.

This article holds a brief discussion on current status in the microscopic theory of superfluidity of the non-relativistic Bose liquid. In the basis of the theory, which describes superfluidity and superconductivity phenomena, lays an appearance, as additional complex macroscopic order parameter, of the wave function of bosons or fermionic Cooper pairs, which stays in the same quantum state, in other words - appearance of the coherent condensate is the foundation of the superfluid and superconductivity phenomena. In Bose-systems such condensate appears because of direct accumulation of bosons in the ground state, but in the Fermi-systems due to the formation of the Cooper pairs of fermions.

Great amount of experimental and theoretical investigations performed over last 80 years allowed to achieve a high level of understanding of the properties of non-relativistic superfluid state in a Bose liquid $^4$He. Most important achievements in the theory of superfluidity reached at a phenomenological level, in particular, for the description of the properties of superfluid $^4$He (so-called Ne-II) on the basis of two-fluid Landau hydrodynamics, according to which the superfluid helium can be divided into two components – the superfluid component with density $\rho_s$ and velocity $v_s$, that describes the non-dissipative motion of quantum fluid and normal component with density $\rho_n$ and velocity $v_n$, which describes dissipative flow of the gas of excitations (phonons and rotons). Numerical calculations of thermal conductivity and viscosity of superfluid helium, obtained according to this theory, are in good agreement with experiment. One of the greatest achievements of the phenomenological approach to the phenomenon of superfluidity is that Landau, basing on the temperature dependence of heat capacity of superfluid 4He on the basis of his superfluidity criteria for the quantum liquids predicted the form of the superfluid helium elementary excitation spectrum with a linear (phonon) dispersion law $E(p) \approx pc$ for the small momentum $p \to 0$ and so called "roton" minimum for $p \neq 0$. Later this form of the spectrum was brilliantly confirmed in the experiments for the scattering of the slow neutrons in the liquid helium.

But in the frame of phenomenological theory of superfluidity exact calculation (from first principles) of the elementary excitations spectrum in superfluid Bose liquid with strong interaction between particles may not be possible. This is the task for the microscopic theory of superfluidity of Bose liquids. Theoretical study of the properties of superfluid Bose liquid $^4$He at the microscopic level hampered by a number of fundamental problems associated with the strong interaction between bosons and complex quantum structure of effective coherent condensate, which is the main part of the superfluid component, unlike of almost ideal Bose gas, where this role is played by single-particle Bose Einstein condensate (BEC).

The first microscopic theory of superfluidity, based on a model of weakly non-ideal Bose gas, was proposed by N.N. Bogoliubov [1] more than 50 years ago. Bogoliubov theory has been improved further and in the most of these improved models of superfluidity by the selection of variational parameters one can achieve a good agreement between theoretical and experimental spectrum of elementary excitations in superfluid $^4$He for a certain range of the momentum. But such agreement is more coincidental and is not true, because, as it was shown in the later experimental studies, a part of single-particle BEC in the superfluid $^4$He is small and ranges from 2 to 10 percent, which is the opposite to a condition of weak non-ideality of the Bose-gas in the Bogoliubov theory. Therefore in order to give an adequate microscopic description of the Bose-liquid superfluid properties the most promising is the Greens function method, which for the first time was used in the papers of S.T. Beliaev and widely developed in the future.

Analysis of the modern experimental and theoretical works testifies to the fact that the study of unique phenomena of superfluidity of helium $^4$He is far from the end. There are number of contradictions between theory and experiment, which are related to both the hydrodynamics of superfluid Bose liquid $^4$He, and the form of quasi-particle spectra, which have not yet satisfactory explanation in frame of the microscopic theory (for more details see [2]).

Firstly, experiments on inelastic scattering of slow neutrons in liquid helium, which confirmed the form of proposed by Landau curve for the spectrum of elementary excitations, indicated a weak temperature dependence of the spectrum, as the consequence the size of the "roton" gap $\Delta_r = 8.65K$ at minimum of $E(p)$, that determines the critical velocity of normal fluid, changes only to the value of $\Delta_r = 5.24K$ near the temperature of the phase transition from superfluid to normal state, i.e. the so-called $\lambda$-point $T_\lambda = 2.17K$. At the same time, for the spectrum of quasiparticles in the electronic Fermi liquid in superconductors superfluidity criterion holds for the phase below critical temperature $T_c$, when there is a finite energy gap in the spectrum, and fail in the normal state at $T > T_c$, when gap in the spectrum becomes zero.

Moreover, obtaining of the quasiparticle spectrum from the "first principles" also fails. Using the latest calculation methods such as Monte-Carlo it is possible to obtain a good agreement between calculated and experimental data almost for all range of momentum, but remains unclear the physical reason for the roton minimum in the quasiparticle spectrum of the Bose-liquid.

Secondly, in the superfluid Bose-liquid, unlike the Bose gas, single-particle Bose-condensate (SPC) due to strong interaction between bosons must be significantly impoverished by the particles of zero energy and momentum (i.e. "depleted" condensate) even at $T = 0$ temperature. Analysis of the experimental data from neutron scattering [3] and quantum evaporation of the helium atoms [4] shows that superfluid $^4$He at low temperature in the Bose-condensate state contain less than 10% of the full density of liquid $^4$He, whereas density of the superfluid component at the temperatures $T < 1K$, according to classical measurements of the viscosity of superfluid helium [5], almost equal to the liquid helium density. This means that against widespread conviction superfluidity of the Bose-liquid $^4$He cannot be connected only with Bose-condensation phenomena. Microscopical structure of the superfluid component, as it was for the first time shown in [6], should have more complicated quantum nature in the form of effective coherent condensate.

Our approach, based on the microscopic model of superfluidity of the Bose-liquid with depleted BEC and intensive pair coherent condensate (PCC), proposed by Pashitskij E.A. and Nepomnyashij Yu.A [6]. Such PCC may occur at sufficiently high "effective attraction" between bosons in some ranges of momentum due to the effects of the quantum diffraction of Bosons in the process of their interaction, and is similar to attraction between fermions near the Fermi surface. The ratio of BEC density to the full density of the liquid was taken as a small parameter (i.e. $\rho_0/\rho << 1$), in contrast to the Bogoliubov [1] theory were small parameter is taken as ratio of the number of supra-condensate excitations to the number of particles in the intensive BEC. Due to a small BEC density it is possible to derive a closed self-consistent system of integral equations for the normal and anomalous self-energy parts by cutting an infinite series of perturbation theory and keeping only a first order terms by small parameter $\rho_0/\rho$:

$$\Sigma_{11}(\varepsilon, p) = n_0\Lambda(\varepsilon, p)V(\varepsilon, p) + n_1V(0) + \psi_{11}(\varepsilon, p),$$
$$\Sigma_{12}(\varepsilon, p) = n_0\Lambda(\varepsilon, p)V(\varepsilon, p) + \psi_{12}(\varepsilon, p),$$

here *V(p)* – the Fourier component of the inoculating two-particle interaction potential of bosons, $V(\varepsilon,p)=V(p)[1-V(p)\Pi(\varepsilon,p)]^{-1}$– the renormalized ("shielded") due to many-particle collective effects Fourier component of non-local interaction, $\Pi(\varepsilon, p)$ – bosonic polarization operator, $n_1 = n - n_0$ – the number of "supra-condensate" particles. Functions $\psi_{11}(\varepsilon, p)$ and $\psi_{12}(\varepsilon, p)$, taking into account poles of single-particle Green functions, are defined by the following expressions:

$$\psi_{11}(\varepsilon, p) = -1/2\int d^3k(2\pi)^{-3}\Gamma(\varepsilon, p, k, E(k))\times$$
$$\times V(p-k, \varepsilon - E(k))\left[A(k, E(k))E^{-1}(k) - 1\right],$$
$$\psi_{12}(\varepsilon, p) = -1/2\int d^3k(2\pi)^{-3}\Gamma(\varepsilon, p, k, E(k))E^{-1}(k)\times$$
$$\times V(p-k, \varepsilon - E(k))\times$$
$$\times\left[n_0\Lambda(k, E(k))V(k, E(k)) + \psi_{12}(k, E(k))\right],$$

here $\Gamma(\varepsilon, p, k, \omega)$ and $\Lambda(\varepsilon, p) = \Gamma(\varepsilon, p, 0, 0)$ – the vortex functions, which describes many-particle correlations; function $A(p, E(p))$ defined as

$$A(p, E(p)) = n_0\Lambda(p, E(p))V(p, E(p)) + p^2/2m +$$
$$+\psi_{11}(p, E(p)) - \psi_{12}(0,0) + \psi_{11}(0,0),$$

and $E(p)$ – quasiparticle spectrum, which in our approximation have the following form

$$E(p) = \begin{pmatrix} A^2(p, E(p)) - \\ [n_0\Lambda(p, E(p)V(p, E(p)) + \psi_{12}(p, E(p))]^2 \end{pmatrix}^{-1/2}$$
$$+1/2\left(\psi_{11}(p, E(p)) - \psi_{12}(p, E(p))\right).$$

From the last expression, because of analiticity of the $\psi(\varepsilon, p)$ functions follows the fact that quasiparticle spectrum is acoustic at $p \to 0$ and its structure at $p \neq 0$ is strongly dependent from the properties of the renormalised two-particle interaction between bosons. In case when BEC is absent ($n_0 = 0$) integral equation for the $\psi_{12}(\varepsilon, p)$ function became homogeneous and degenerate over the phase of the function. Thus it is similar to the momentum space Bethe-Goldstone integral equation for a pair of particles with zero binding energy, which has a nontrivial solution $\psi \neq 0$ only in case of attraction $V(p) < 0$ in a sufficiently wide region of values of the transmitted momentum *p*. Follow this analogy, function $\psi_{12}(\varepsilon, p)$ can be considered as order parameter for bosonic PCC, which describe condensation of bosonic pairs in the momentum space (identically to the Cooper condensate of the fermion pairs).

In this connection, pair interaction between bosons was chosen in the form of regularized repulsion potentials in models of "hard" and "semitransparent spheres", which Fourier components, due to diffraction of particles of one another, are oscillating and alternating functions of transmitted momentum $p$, and can be determined by the spherical Bessel functions of zero and first orders.

Many-particle collective effects in Bose liquid lead to a significant renormalization of the pair interaction, which determines the normal and anomalous self-energy parts (Fig. 1). An important feature of the renormalized interaction is that, as was shown in [7], in those areas of the phase volume $(\varepsilon, p)$, were the real part of the bosonic polarization operator $\mathrm{Re}\,\Pi(p,\omega)$ is negative, repulsion become weaker (when $V(p) > 0$) and attraction effectively increased (when $V(p) < 0$) (see Fig. 1). The key point in the behavior of the renormalized (screened) potential $V(p, E(p))$, as it was shown by the numerical calculations in [8], is that for all values of momentum $p > 0$ the real part of the bosonic polarization operator is negative ($\mathrm{Re}\,\Pi(p,\omega) < 0$, upper corner of Fig. 1) if quasiparticle spectrum is stable with respect to decay on a pairs of quasiparticles.

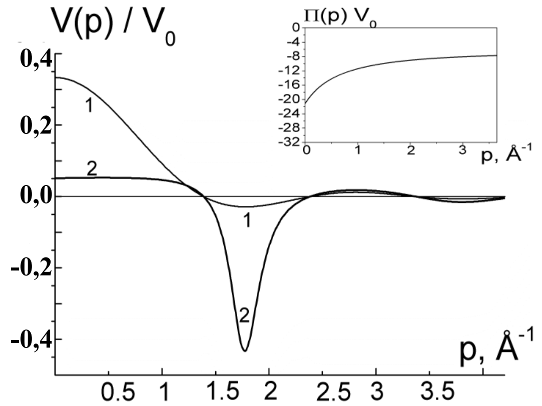


Fig. 1. Fourier components of the non-renormalized (curve 1) and renormalized (curve 2) interactions in the model of "semitransparent spheres". In the upper corner depicted the momentum dependence of the real part of the bosonic polarization operator.

In the numerical calculations [8], based on the Fourier-component of the "hard spheres" potential $V(p) = V_0 \sin(pa)/pa$, were used a simplified model of the renormalized potential in the form

$$V(p,\omega) = V_0 \sin(pa)/\left(pa + \alpha \sin(pa)\right),$$

here $\alpha = -V_0\Pi = const$, and $\Pi$ – the value of the polarization operator on the mass surface $\omega = E(p)$ averaged over the momentum $p$, which was taken as negative constant value in case if quasiparticle spectrum not decay. Resulting spectrum qualitatively agrees with experimental spectrum $E_{exp}$ in superfluid $^4$He (Fig. 2), but numerical correspondence of the minimal and maximal values of quasiparticle energies are not satisfactory. In addition, calculated value of the speed of the first (hydrodynamical) sound $c$=2.08×10$^4$ cm/sec appeared to be lower than experimental value $c$=2.36×10$^4$ cm/sec, and total concentration of particles is higher – $n$=2.57×10$^{22}$ cm$^{-3}$; the value of the BEC particle concentration is lower $n_0 = 3\%n$ then obtained from experiment.

Further calculations, carried in papers [9,10], show that oscillating pseudo-potential in the model of "semitransparent spheres"

$$V(p) = V_0\left(\sin(pa) - pa\cos(pa)\right)/(pa)^{-3}$$

is more appropriate than "hard sphere" potential both for the stability of the spectrum and its correspondence to the empirical spectrum in $^4$He. In the "semitransparent spheres" model the explicit momentum dependence of the polarization operator $\Pi(p,\omega)$ is used.

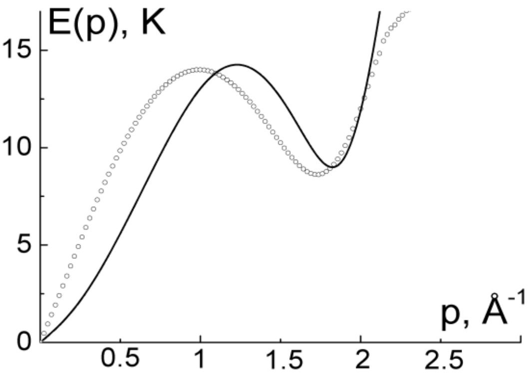


Fig. 2. Solid curve - spectrum obtained in the model of "hard spheres", dot curve - experimental data [11] - [14].

Iterative numerical calculations of the self-energy and bosonic polarization operator, the two-particle order parameter and the quasiparticle spectrum at $T = 0$ allowed to find conditions when theoretical spectrum $E(p)$ is in a good agreement with experimental spectrum of elementary excitations in $^4$He. The roton minimum of the quasiparticle spectrum $E(p)$ in the Bose liquid, as it was shown in [10], clearly associated with the first negative minimum of the Fourier-component renormalized potential. The only fitting parameter in these calculations was the amplitude of the starting pseudo-potential with the parameter $a = 2.44\,A^{-1}$, which is equal to the twice of a quantum radius of the $^4$He atom. For the calculations we take an experimental value of the BEC density $n_0 = 9\%n = 1.95\times10^{22}\,cm^{-3}$ [3, 4].

As a result, after numerical calculations it was possible to get a quite satisfactory agreement of the theoretical $E(p)$ and experimental $E_{\exp}(p)$ spectrum under condition $p < 2.5\,A^{-1}$ (Fig. 3, curve 1). In calculations of $E(p)$ the only fitting parameter was taken in order to satisfy a condition that quasiparticle phase velocity $E(p)/p$ at $p \to 0$ coincide with hydrodynamical speed of sound $c_1 \cong 236\,m/\sec$ in the liquid $^4$He, which corresponds to the value $U_0 = V_0/4\pi a^3 = 1552\,K$ for the amplitude of the "semitransparent spheres" potential at $a = 2.44\,A^{-1}$. In the range $p > 2.5\,A^{-1}$ theoretical spectrum $E(p)$ lies slightly above $E_{\exp}(p)$. This is, most likely, due to the fact that vertex function $\Gamma(k,p)$, which decay with increasing of the momentum $p$, was taken as constant $\Gamma = 1.5$. This value was obtained from the exact asymptotic of the polarization operator

$\Pi(0,0) = -n/mc^2$. In this connection vertex function on the interval $2.1A^{-1} < p < 3.8A^{-1}$ was approximated by the linear function, which slowly changed from $\Gamma = 1.5$ to $\Gamma = 1.1$. Spectrum, obtained within this approximation (Fig. 3, curve 2), is in good agreement with experimental curve for all range of the momentum.

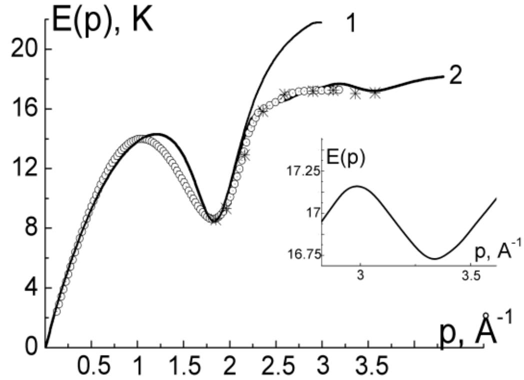


Fig. 3. Curve 1 – theoretical quasiparticle spectrum $E(p)$, obtained with model of the "semitransparent spheres" for the constant value of the vertex function $\Gamma = 1.5$. Curve 2 – theoretical quasiparticle spectrum $E(p)$, obtained with a model of the "semitransparent spheres" for the weakly decaying (from $\Gamma = 1.5$ to $\Gamma = 1.1$) vertex function, on the interval $2.1A^{-1} < p < 3.8A^{-1}$. In both cases the value of the fitting parameter is $U_0 = V_0/4\pi a^3 = 1552K$. Circles show experimental spectrum, obtained by inelastic neutron scattering in the liquid $^4$He [11-14], stars – show the results of the experiment [15] beyond the roton minimum $2A^{-1} < p < 3.6A^{-1}$. Inserted box shows weak oscillations of the spectrum with maximum $E_{\max} = 17.2K$ at $p = 2.99A^{-1}$ and minimum $E_{\min} = 16.7K$ at $p = 3.39A^{-1}$.

Self-consistency of the given model is confirmed by the value of the total particle density $n_{teor} = 2.12\times10^{22}\,cm^{-3}$, which is close to the experimental value of the particle density $n = 2.17\times10^{22}\,cm^{-3}$ of the liquid $^4$He (when $n_0 = 9\% n$). On the other hand, independent calculation of the over-condensate density of particles $n_1$ for the given parameters gives a value of about $93\% n$, which also agrees with experiment under condition that BEC density is determined with accuracy of $\pm1\%$.

Thus, proposed in [6] model of superfluid Bose liquid with a suppressed BEC and intensive PCC, which is based on renormalized perturbation field theory with combined variables [16], allow to cut an infinite series by the low BEC density and obtain "truncated", closed system of nonlinear integral equations for the self-energy parts $\Sigma_{1j}(\varepsilon, p), (j = 1,2)$.

On the other hand, the oscillating character of the renormalized Fourier component of the potential leads to a non-monotonic behavior of the momentum dependencies of the mass operators $\Sigma_{1j}(\varepsilon, p), (j = 1,2)$ and, as a result, to the appearance of the roton minimum in the spectrum of quasiparticles $E(p)$, which definitely connected with first deepest negative minimum of the renormalized Fourier component of the potential. Thus, for sufficiently large values of the amplitude of the initial potential $V_0$ quasiparticles excitation spectrum become unstable in some domain of momenta $p \neq 0$, where $E^2(p) < 0$.

For the conclusion it is necessary to emphasize that mentioned property of the polarization operator $\Pi(p, E(p)) < 0$ is typical only for Bose-systems, where single-particle and many-particle spectrum are coincide and have a common zero energy reference point, unlike Fermi-systems, where single-particle excitation spectrum due to the Pauli principle is counted from the Fermi energy. Therefore, corresponding effective increasing of the negative values of the polarization operator cannot take place for the Fermi-liquid $^3$He, so formation of the Cooper pairs is possible only for non-zero values of the orbital momentum and a real Van der Waals attraction between fermions. Probably, this is the reason for the three order difference between critical temperatures of superfluid transition in $^4$He and $^3$He.

Our calculation of the temperature dependence of superfluid component density is based on the model of the coherent structure of the condensate of the Bose liquid $^4$He. Temperature dependence of the superfluid component density in this model is given by the following expression [10]

$$\frac{\rho_s(T)}{\rho} = \frac{\psi_0(T)}{\tilde{V}(0)n}\left[1 - \frac{\psi_s(T)}{\tilde{V}(0)n}\right]^{-1}, \rho \equiv mn,$$

here

$$\psi_0(T) = -\frac{1}{2}\int\frac{d\mathbf{q}}{(2\pi)^3}\tilde{V}(\mathbf{q})\left[\frac{A_{12} - D_{12}}{c_1\mathbf{q}}\coth\left(\frac{c_1\mathbf{q}}{2T}\right) + \frac{B_{12}}{c_2\mathbf{q}}\coth\left(\frac{c_2\mathbf{q}}{2T}\right)\right],$$

$$\psi_s(T) = -\frac{1}{2}\int\frac{d\mathbf{q}}{(2\pi)^3}\tilde{V}(\mathbf{q})\left[\frac{D_{12}}{c_1\mathbf{q}}\coth\left(\frac{c_1\mathbf{q}}{2T}\right) - \frac{B_{12}}{c_2\mathbf{q}}\coth\left(\frac{c_2\mathbf{q}}{2T}\right)\right].$$

In the last expression $\tilde{V}(\mathbf{q})$ – the Fourier component of the renormalized due to many-particle effects a pair interaction potential. For the calculation of the temperature dependences we use a model potentials ("hard sphere" and "semitransparent sphere" potential), Aziz potential and expressions for the velocity of the first and second sounds in liquid

$^4$He and they temperature corrections, obtained in [18] from the solutions of a kinetic type equations:

$$E_1^2(\mathbf{q}) = c_1^2\mathbf{q}^2 = c_0^2\left(1+\frac{29}{16}\frac{\rho_n}{\rho}\right)\mathbf{q}^2,$$

$$E_2^2(\mathbf{q}) = c_2^2\mathbf{q}^2 = \frac{c_0^2}{3}\left(1-\frac{33}{8}\frac{\rho_n}{\rho}\right)\mathbf{q}^2,$$

Temperature dependence of the superfluid component, calculated in this paper is given on Fig. 4.

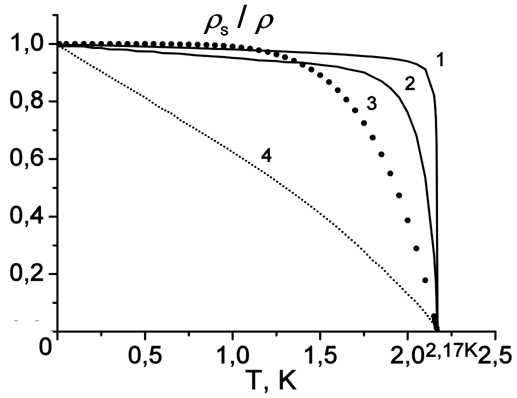


Fig. 4. Temperature dependencies of the Bose-liquid $^4$He superfluid density obtained in [10] (curve 1); present calculations (curve 2); experimental data obtained by Andronikashvilli E.L. [20] (curve 3); result of Ginzburg V.L. [19] (curve 4).

**Conclusions**

In this paper using the elementary excitation spectum obtained in frame of proposed in [17] the selfconsistent microscopical model of the superfluid non-relativistic Bose liquid $^4$He and temperature corrections to the first and second speeds of sound calculated within approach proposed in [18] we obtain the temperature dependence of the condensate density of strongly interacting Bose-system. Our calculations, shown on Fig.4 are in qualitative agreement with experiment.

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