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Критерій перевірки гіпотези про вигляд коваріаційної функції Гауссовоого стаціонарного процесу за умови існування спектральної щільності

Метод мажоруючих мір застосовується для побудови критерію перевірки гіпотези про вигляд коваріаційної функції Гауссовоого стаціонарного процесу.

Ключові слова: метод мажоруючих мір, коваріаційна функція, корелограма.

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Introduction

In this paper we propose some inequalities of the distribution for supremum of correlogram form of the stochastic processes. These estimates are applying for constructing the criterion of hypothesis testing on covariance functions of these processes.

Particular cases of problems considered in this paper were investigated by Kozachenko and Moklyachuk [2], Sergiienko [4].

1 Definitions and properties of some elements from the Orlicz space

Assume that $X = (X(t), t \in [0, T], T > 0)$ is a separable real-valued stationary Gaussian process defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, with mean zero and a continuous correlation function

$$B(h) = \mathbf{E}X(t)X(t+h), \quad h \in \mathbf{R}.$$

The so-called correlogram

$$\widehat{B}_T(h) = \frac{1}{T} \int_0^T X(t)X(t+h)dt, \quad h \geq 0,$$

will be used as an estimate of $B(h)$.

Let consider process $Z_T(h) = \widehat{B}_T(h) - B(h)$, $h \geq 0$. It is easy to see that $Z_T(h)$ is square Gaussian stochastic process [4, p.17], indeed, $\widehat{B}_T(h) = l.i.m. \Delta t_i \rightarrow \sum_{i=1}^{\infty} X(t_i + h)X(t_i) \frac{\Delta t_i}{T}$.

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How to test the hypothesis concerning the form of covariance function of Gaussian stationary process when exist spectral density

Method of majorizing measures is applying for constructing the criterion for testing the hypothesis concerning the form of covariance function of Gaussian stochastic process.

Key Words: method of majorizing measures, covariance function, correlogram.

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Let the process X have a square integrable spectral density $f = (f(\lambda), \lambda \in \mathbf{R})$, that is,

$$\int_{-\infty}^{\infty} f^2(\lambda)d\lambda < \infty.$$

By the definition of spectral density, the function f is Lebesgue integrable and by Bochner-Khintchine theorem, we have

$$B(h) = \int_{-\infty}^{\infty} \exp\{i\lambda h\} f(\lambda)d\lambda$$

for all $h \in \mathbf{R}$.

Lemma 1. For all $T > 0$ and all $h_1, h_2 \geq 0$, we have the following inequality:

$$\begin{aligned} \mathbf{E}|Z_T(h_1) - Z_T(h_2)|^2 &\leq \\ &\leq \frac{8\pi}{T} \left[\int_{-\infty}^{\infty} f^2(\lambda) \sin^2 \frac{\lambda(h_1 - h_2)}{2} d\lambda \right] + \\ &+ \frac{8\pi}{T} \|f\|_2 \left[\int_{-\infty}^{\infty} f^2(\lambda) \sin^2 \frac{\lambda(h_1 - h_2)}{2} d\lambda \right]^{1/2}, \end{aligned}$$

where

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} f^2(\lambda) d\lambda \right)^{\frac{1}{2}}.$$

Proof. This inequality is corollary from the lemma 4.1 [1, p.179] if we put $Y_T(h) = \sqrt{T}Z_T(h)$. \square

Lemma 2. The next inequality holds true while

$$0 < \alpha \leq 1$$

$$\left| \sin \frac{u}{v} \right| \leq \left(\frac{\ln(1 + |u|)}{\ln(1 + |v|)} \right)^\alpha.$$

Proof of lemma 2. you can see for example in [5, p.167].

Theorem 1. Let the next condition holds true for $0 < \alpha \leq 1$

$$\int_{-\infty}^{\infty} \ln^{2\alpha} \left(1 + \frac{|\lambda|}{2} \right) f^2(\lambda) d\lambda < \infty.$$

Then the next equation holds true

$$\sup_{|h_1 - h_2| < h} (\mathbf{E}|Z_T(h_1) - Z_T(h_2)|^2)^{\frac{1}{2}} \leq \frac{C_{\alpha,T}}{\ln^{\frac{\alpha}{2}}(1 + \frac{1}{h})},$$

where

$$C_{\alpha,T} = \frac{8\pi}{T} (B_\alpha^2 + B_\alpha \|f\|_2),$$

$$B_\alpha^2 = \int_{-\infty}^{\infty} \ln^{2\alpha} \left(1 + \frac{|\lambda|}{2} \right) f^2(\lambda) d\lambda.$$

Proof. Using lemma 1 and lemma 2 we can obtain

$$\begin{aligned} \mathbf{E}|Z_T(h_1) - Z_T(h_2)|^2 &\leq \frac{8\pi B_\alpha^2}{T} \frac{1}{\ln^{2\alpha} \left(1 + \frac{1}{|h_1 - h_2|} \right)} + \\ &+ \frac{8\pi B_\alpha \|f\|_2}{T} \frac{1}{\ln^\alpha \left(1 + \frac{1}{|h_1 - h_2|} \right)}. \end{aligned}$$

Our statement holds true after condition $|h_1 - h_2| < e - 1$. \square

Let (\mathbf{T}, d) be a compact metric space and let \mathcal{A} be a Borel σ -algebra on (\mathbf{T}, d) . Let $\mu(\cdot)$ be a finite measure on $(\mathbf{T}, \mathcal{A})$. Let $X = \{X(t), t \in \mathbf{T}\}$ be a separable stochastic process such that the sample path of X are measurable with respect to \mathcal{A} and $X \in L_U(\Omega)$, where $U(x)$ is a \mathbb{C} -function. Let $d(u, v) = \|X(u) - X(v)\|_U$ and \mathbf{S} be a set from \mathcal{A} such that $(\mu \times \mu)\{(u, v) \in \mathbf{S} \times \mathbf{S}, d(u, v) \neq 0\} > 0$.

Theorem 2. Let $X = \{X(t), \mathbf{T}\}$ be a separable stochastic process from the Orlicz space $L_U(\Omega)$ that is generated by \mathbb{C} -function $U(x) = \exp\{x\} - 1$. Let $\rho(v)$ be a not decreasing function. Then for all $x \geq \hat{\rho} = \sup_{u,v \in \mathbf{S}} \rho(d(u, v))$ the following inequality holds true

$$\mathbf{P}\{\eta > x\} \leq 2 \exp \left\{ -\frac{x}{\sqrt{2}\hat{\rho}} \ln \left(1 + \frac{1}{A(\mathbf{S})} \right) \right\} \times$$

$$\times \sqrt{1 + \frac{\sqrt{2}x}{\hat{\rho}} \ln \left(1 + \frac{1}{A(\mathbf{S})} \right)}, \quad (1)$$

where

$$\eta = \left\| \frac{(X(u) - X(v))\gamma(d(u, v))}{d(u, v)} \right\|_{U, \mu \times \mu}^{\mathbf{S} \times \mathbf{S}},$$

and

$$A(\mathbf{S}) = \int_{\mathbf{S}} \int_{\mathbf{S}} \frac{\rho(l(u, v))}{\hat{\rho}} d(\mu(u) \times \mu(v)).$$

Proof. It is easy to obtain from [4, p.16] for $q > 1$ that

$$\mathbf{P}\{\eta > x\} \leq \left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^q \times$$

$$\int_{\mathbf{S}} \int_{\mathbf{S}} \frac{\rho(d(u, v))}{\hat{\rho}} \mathbf{E} \exp \left\{ \frac{q|(X(u) - X(v))\hat{\rho}|}{d(u, v)x} \right\} \frac{d\hat{\mu}}{A(\mathbf{S})}$$

where $\hat{\mu} = \mu(u) \times \mu(v)$.

From [1, p.171] we will obtain that

$$\mathbf{E} \exp \left\{ \frac{q|(X(u) - X(v))\hat{\rho}|}{d(u, v)x} \right\} \leq 2L_0 \left(\frac{\sqrt{2}q\hat{\rho}}{x} \right),$$

where

$$L_0(s) = \frac{1}{\sqrt{1 - |s|}} \exp \left\{ -\frac{|s|}{2} \right\}.$$

That is why

$$\mathbf{P}\{\eta > x\} \leq \left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^q \times$$

$$\times \int_{\mathbf{S}} \int_{\mathbf{S}} \frac{\rho(d(u, v))}{\hat{\rho}} 2L_0 \left(\frac{\sqrt{2}q\hat{\rho}}{x} \right) \frac{d\hat{\mu}}{A(\mathbf{S})} =$$

$$= \left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^q 2L_0 \left(\frac{\sqrt{2}q\hat{\rho}}{x} \right) =$$

$$= \left(\frac{A(\mathbf{S})}{1 + A(\mathbf{S})} \right)^q \frac{2}{\sqrt{1 - \frac{\sqrt{2}q\hat{\rho}}{x}}} \exp \left\{ -\frac{\frac{\sqrt{2}q\hat{\rho}}{x}}{2} \right\}. \quad (2)$$

Let denote $K = \frac{A(\mathbf{S})}{1 + A(\mathbf{S})}$, $a = \frac{x}{\sqrt{2}\hat{\rho}}$, $q = sa$, $R = K^a$. Then, this inequality changes to

$$\mathbf{P}\{\eta > x\} \leq 2R^s \frac{1}{\sqrt{1 - s}} \exp \left\{ -\frac{s}{2} \right\} = f(s).$$

The function $f(s)$ has minimum at $s = 1 - \frac{1}{1-2\ln R}$. and
Then,

$$\mathbf{P}\{\eta > x\} \leqslant 2R^{\left\{1-\frac{1}{1-2\ln R}\right\}} \sqrt{1-2\ln R} \times$$

$$\times \exp\left\{-\frac{1}{2}\right\} \exp\left\{\frac{1}{2(1-2\ln R)}\right\} =$$

$$= 2R\sqrt{1-2\ln R} \exp\left\{-\frac{1}{2}\right\} \times$$

$$\times \exp\left\{-\frac{\ln R}{1-2\ln R}\right\} \exp\left\{\frac{1}{2(1-2\ln R)}\right\} =$$

$$= 2R\sqrt{1-2\ln R} = 2K^{\frac{x}{\sqrt{2}\hat{\rho}}} \times$$

$$\times \sqrt{1-2\ln K^{\frac{x}{\sqrt{2}\hat{\rho}}}} =$$

$$= 2 \exp\left\{\frac{x \ln K}{\sqrt{2}K^{\frac{x}{\sqrt{2}\hat{\rho}}}}\right\} \sqrt{1-2\frac{x}{\sqrt{2}\hat{\rho}} \ln K} =$$

$$= 2 \exp\left\{-\frac{x}{\sqrt{2}\hat{\rho}} \ln\left(1 + \frac{1}{A(\mathbf{S})}\right)\right\} \times$$

$$\times \sqrt{1 + \frac{\sqrt{2}x}{\hat{\rho}} \ln\left(1 + \frac{1}{A(\mathbf{S})}\right)}.$$

□

Corollary 1. Let $Y = \{Y(t), t \in \mathbf{T}\}$ be the square Gaussian process and let $\rho(\cdot)$ be a not decreasing function. Then for all $x > 0$ and $0 < p < 1$ the following inequality holds true

$$\begin{aligned} \mathbf{P}\left\{\sup_{t \in S} \left|Y(t) - \int_S Y(u) \frac{d\mu(u)}{\mu(\mathbf{S})}\right| > x\right\} &\leqslant \\ &\leqslant 2 \exp\left\{-\frac{x}{\sqrt{2}|a_p|\hat{\rho}} \ln\left(1 + \frac{1}{A(\mathbf{S})}\right)\right\} \times \\ &\quad \times \sqrt{1 + \frac{\sqrt{2}x}{\hat{\rho}|a_p|} \ln\left(1 + \frac{1}{A(\mathbf{S})}\right)}, \end{aligned} \quad (3)$$

where

$$a_p = 2 \sup_{t \in S} \frac{1}{p(1-p)} \int_0^{\delta_1(t)p} \frac{1}{\nu_t^2(t)} \frac{1}{V^{(-1)}\left(\frac{1}{\nu_t^2(t)}\right)} dt.$$

Here

$$V(y) = \begin{cases} y(\ln y - 1) + 1, & y > 1; \\ 0, & 0 < y \leqslant 1 \end{cases}$$

$$A(\mathbf{S}) = \int_{\mathbf{S}} \int_{\mathbf{S}} \frac{\rho(d(u, v))}{\hat{\rho}} d(\mu(u) \times \mu(v)),$$

$$\delta_1(t) = z\left(2\sigma_t(\sup_{t \in \mathbf{S}} \rho(t, s))\right),$$

$$\nu_t(u) = \mu\left(B\left(t, \sigma_t^{(-1)}\left(z^{(-1)}(u)/2\right)\right) \cap \mathbf{S}\right).$$

Theorem 3. Let $\mathbf{S} = [0, S], S > 0$. The next inequality holds true for correlogram $\widehat{B}_T(\tau)$ of correlation function $B(\tau)$

$$\begin{aligned} \mathbf{P}\left\{\sup_{0 \leqslant \tau \leqslant S} |B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S} \int_0^S B(u) du + \right. \\ \left. + \frac{1}{S} \int_0^S \widehat{B}_T(u) du| > x\right\} \leqslant \\ \leqslant 2 \exp\left\{-\frac{x}{\sqrt{2}|a_p|\hat{\rho}} \ln\left(1 + \frac{1}{A(S)}\right)\right\} \times \\ \times \sqrt{1 + \frac{\sqrt{2}x}{\hat{\rho}|a_p|} \ln\left(1 + \frac{1}{A(S)}\right)}, \end{aligned}$$

where $x \geqslant 1$,

$$A(S) = \int_0^S \int_0^S \frac{\rho(d(u, v))}{\hat{\rho}} d(\mu(u) \times \mu(v))$$

$$\begin{aligned} a_p = \frac{2}{p(1-p)} \times \\ \times \int_0^z \left(\frac{2C_{\alpha,T}}{\ln^{\frac{\alpha}{2}}(1+\frac{1}{S})}\right)^p \ln\left(1 + \frac{1}{\left(\frac{C_{\alpha,T}}{z^{(-1)}(\frac{u}{2})}\right)^{\frac{2}{\alpha}} - 1}\right) du \end{aligned}$$

Proof. Here we have $\sigma_t(h) = \frac{C_{\alpha,T}}{\ln^{\frac{\alpha}{2}}(1+\frac{1}{h})}$ and $\sigma_t^{(-1)}(h) = \frac{1}{\left(\frac{C_{\alpha,T}}{h}\right)^{\frac{2}{\alpha}} - 1}$.

This theorem is a consequence of the theorem 2 for $Z_T(\tau) = \widehat{B}_T(\tau) - B(\tau)$, where $\mathbf{S} = [0, S]$. Here $Z_T(\tau)$ is square Gaussian stochastic process and $\|Z_T(\tau)\| \leqslant \sqrt{\frac{C_r}{\ln^{\frac{\alpha}{2}}(1+\frac{1}{S})}} \left(\mathbf{E} Z_T(\tau)^2\right)^{1/2}$, where $C_r = r^{-2} |\ln(1-r)| - r^{-1}$. □

Let H be a hypothesis which for $0 \leqslant \tau \leqslant T$ lie down that correlation function of Gaussian stochastic process is equals to $B(\tau)$. We take $\widehat{B}_T(\tau)$ as an estimate of $B(\tau)$. Let consider next test.

Criterion 1. Let find x_γ for fixed γ , $0 \leq \gamma \leq 1$ that

$$f(x_\gamma) = \gamma,$$

where

$$f(x) = 2 \exp \left\{ -\frac{x}{\sqrt{2}|\check{a}_p| \hat{\rho}} \ln \left(1 + \frac{1}{A(S)} \right) \right\} \times \\ \times \sqrt{1 + \frac{\sqrt{2}x}{\hat{\rho}|\check{a}_p|} \ln \left(1 + \frac{1}{A(S)} \right)}.$$

Hypothesis that the covariance function of the process $Y(t)$ is $B(\tau)$ for $0 \leq \tau \leq T$ is accepted if

$$\sup_{0 \leq \tau \leq S} \left| B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S} \int_0^S B(u) du + \right. \\ \left. + \frac{1}{S} \int_0^S \widehat{B}_T(u) du \right| < x_\gamma$$

and is rejected otherwise.

Example 1. Let put $z(x) = x$, $\rho = 1$ and $\alpha = 1$. Then $A(S) = S^2$ and

$$\check{a}_p = \frac{2}{p(1-p)} \times \\ \times \int_0^{\frac{2C_{1,T}}{\ln^{\frac{1}{2}}(1+\frac{1}{S})} p} \ln \left(1 + \frac{1}{\left(\frac{2C_{1,T}}{u} \right)^2 - 1} \right) du = \\ = \frac{2}{p(1-p)} \left[\frac{2C_{1,T}p}{\ln^{\frac{1}{2}}(1+\frac{1}{S})} \ln \left| \frac{\ln(1+\frac{1}{S})}{\ln(1+\frac{1}{S}) - p^2} \right| + \right. \\ \left. + \frac{4C_{1,T}p}{\ln^{\frac{1}{2}}(1+\frac{1}{S})} + 2C_{1,T} \ln \left| \frac{\ln^{\frac{1}{2}}(1+\frac{1}{S}) + p}{\ln^{\frac{1}{2}}(1+\frac{1}{S}) - p} \right| \right].$$

Criterion 2. Let's find such x_γ for fixed α , $0 \leq \gamma \leq 1$ that

$$f(x_\gamma) = \gamma,$$

where

$$f(x) = 2 \exp \left\{ -\frac{x}{\sqrt{2}|\check{a}_p|} \ln \left(1 + \frac{1}{S^2} \right) \right\} \times \\ \times \sqrt{1 + \frac{\sqrt{2}x}{|\check{a}_p|} \ln \left(1 + \frac{1}{S^2} \right)}.$$

Hypothesis that the covariance function of the process $Y(t)$ is $B(\tau)$ for $0 \leq \tau \leq T$ is accepted if

$$\sup_{0 \leq \tau \leq S} \left| B(\tau) - \widehat{B}_T(\tau) - \frac{1}{S} \int_0^S B(u) du + \right. \\ \left. + \frac{1}{S} \int_0^S \widehat{B}_T(u) du \right| < x_\gamma$$

and is rejected otherwise.

Conclusions

In this paper we have obtained estimates for distribution of square Gaussian stochastic processes. We have constructed the criterion how to test the hypothesis concerning the form of covariance function of Gaussian stochastic process.

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