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Підгрупи напівгрупи відповідностей

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Subgroups of correspondences semigroup

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Описано максимальні підгрупи напівгрупи відповідностей. Одержано необхідні і достатні умови, коли напівгрупа відповідностей буде об'єднанням груп.

Ключові слова: максимальна підгрупа, об'єднання груп.

There are described the maximal subgroups of semigroups of correspondences. Necessary conditions for an element of semigroup of correspondences to be an idempotent are declared. There are received necessary and efficient conditions for the semigroup of correspondences to be an union of groups. Also it is discussed then the semigroup of correspondences will be the sheaf.

Key words: maximal subgroup, union of groups.

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Introduction

Let G be an universal algebra. As the subalgebra of $G \times G$ is viewed as a binary relation on G, so the set $S(G)$ of all subalgebras of $G \times G$ is a semigroup related to De Morgan's product of relations. The subgroup $S(G)$ is called *the semigroup of correspondences* of the algebra G. The problem of learning the semigroups of correspondences was set by Kurosh [2].

In the work [3] there is shown that when G is the group than the elements of the semigroup $S(G)$ can be identified by the fifths of the form $(H_1, G_1, H_2, G_2, \varphi)$, where

$$H_1 \lhd G_1 < G,\ H_2 \lhd G_2 < G,$$

and φ is the isomorphism of the factorgroup G_1/H_1 onto the factorgroup G_2/H_2. Herewith the related element of the semigroup $S(G)$ – as the subset out of $G \times G$ – has the form:

$$(H_1, G_1, H_2, G_2, \varphi) = \bigcup_{a \in G_1} (aH_1 \times \varphi(aH_1)).$$

Sets of the form $aH_1 \times bH_2$, where $bH_2 = \varphi(aH_1)$, we shall call the blocks of elements $A = (H_1, G_1, H_2, G_2, \varphi)$.

1 Maximal Subgroups of Correspondences Semigroup

In [4] there are given the necessary and efficient conditions for the element $(H_1, G_1, H_2, G_2, \varphi)$ of the semigroup of correspondences to be idempotent. But they are too huge. There are far the simplest conditions but only necessary are given by the following theorem.

Theorem 1. *Let G be a group. If the element $(H_1, G_1, H_2, G_2, \varphi)$ of the semigroup of correspondences $S(G)$ is idempotent, so there are fulfilled the following conditions:*

1) $H_1(G_1 \cap G_2) = G_1, \quad H_2(G_1 \cap G_2) = G_2;$

2) for any $g \in G_1 \cap G_2$ $\varphi(gH_1) = gH_2$.

Proof. Let the element

$$A = (H_1, G_1, H_2, G_2, \varphi)$$

be idempotent. Then for any block $aH_1 \times bH_2$ from A we have:

$$(aH_1 \times bH_2) \circ (aH_1 \times bH_2) = (aH_1 \times bH_2).$$

Hence

$$bH_2 \cap aH_1 = \varnothing\,.$$

Thus, for every adjacency class aH_1 of group G_1 to subgroup H_1 will be found such an element

$$c \in aH_1 \cap bH_2 \subseteq G_1 \cap G_2\,,$$

that $aH_1 = cH$. But then

$$H_1(G_1 \cap G_2) \subseteq H_1G_1 = G_1.$$

It can be similarly proved, that $H_2(G_1 \cap G_2) = G_2$. This proves Condition 1).

For any $g \in G_1 \cap G_2$ the element A has the blocks of the form $aH_1 \times gH_2$ i $gH_1 \times bH_2$. While $g \in gH_1 \times gH_2$, then $gH_1 \cap gH_2 \neq \varnothing$, so

$$(aH_1 \times gH_2) \circ (gH_1 \times bH_2) = (aH_1 \times bH_2).$$

On the other hand, from the idempotentness of the element A we have:

$$(aH_1 \times gH_2) \circ (aH_1 \times gH_2) = aH_1 \times gH_2.$$

So, A has the blocks $aH_1 \times bH_2$ and $aH_1 \times gH_2$.. The first components of blocks are similar, so $bH_2 = gH_2$. There is similarly proved that $aH_1 = gH_1$. But then

$$\varphi(gH_1) = \varphi(aH_1) = bH_2 = gH_2\,,$$

that proves Condition 2). □

For any idempotent $e \in S$ let us define by G_e the maximal subgroup of S, for which e is the unit.

Theorem 2. *Let G be the finite group, and the element $e = (H_1, G_1, H_2, G_2, \varphi)$ of the semigroup of correspondences $S(G)$ is the idempotent. Then*

$$G_e = \{(H_1, G_1, H_2, G_2, \psi\varphi) | \psi \in \mathrm{Aut}(G_1/H_1)\} =$$

$$= \{(H_1, G_1, H_2, G_2, \varphi\psi) | \psi \in \mathrm{Aut}(G_2/H_2)\}.$$

In particular, if $G_1 = G_2 = G$, $H_1 = H_2 = H$, $\varphi = \varepsilon$ is the identity automorphism, so

$$G_e = \{(G, H, G, H, \psi) | \psi \in \mathrm{Aut}(G/H)\}\,.$$

Proof. It is easy to check that the element $f = (H_1, G_1, H_1, G_1, \varepsilon)$, where ε is the identity automorphism of the factor group G_1/H_1, will be the idempotent.

In [5] it was shown that the elements

$$A' = (H_1', G_1', H_2', G_2', \varphi')$$

and

$$A'' = (H_1'', G_1'', H_2'', G_2'', \varphi'')$$

belong to one class $\mathcal{D}$ of the semigroup $S(G)$ then and only then when

$$G_1'/H_1' \simeq G_2'/H_2' \simeq G_1''/H_1'' \simeq G_2''/H_2''.$$

If $G_1'/H_1' \simeq F$, we shell say that the correspondent $\mathcal{D}$ class is defined by the factor F.

That's why the idempotents e and f will be $\mathcal{D}$ equivalent. According to the theorem by Green (see [1], theorem 4.7.5) the maximal subgroups G_e and G_f are isomorphic. That's why there is some sense at first to analyze the final part of the theorem.

Therefore, let $e = (G, H, G, H, \varepsilon)$, where ε is identity automorphism of the factorgroup G/H. In accordance with upshot 1 out of theorem 1 from [5] $\mathcal{H}$ is the class $\mathcal{H}(e)$, which coincides with G_e, contains $|\mathrm{Aut}(G/H)|$ elements. That's why to prove the final part of the theorem it is enough to show that every element of the look $A = (G, H, G, H, \varphi)$, where $\varphi \in Aut(G/H)$, belongs to G_e. We have:

$$(gH, \varphi(gH)) \circ (\varphi(gH), gH) = (gH, gH). \qquad (1)$$

On the other hand, if $gH \neq g'H$, то $gH \cap g'H = \varnothing$, where $\varphi(gH) \cap \varphi(g'H) = \varnothing$ and

$$(gH, \varphi(gH)) \circ (\varphi(g'H), g'H) = \varnothing. \qquad (2)$$

Out of the equalities (1) and (2) it follows that for the element

$$A^{-1} = (G, H, G, H, \varphi^{-1})$$

there will be $A \circ A^{-1} = e$. That's why $A \in G_e$.

Now let $e = (G_1, H_1, G_2, H_2, \varphi)$ be the arbitrary idempotent. Out of the proved above it follows that

$$|G_e| = |\{(G_1, H_1, G_2, H_2, \varphi\psi) | \psi \in \mathrm{Aut}(G_2/H_2)\}|. \qquad (3)$$

On the other hand, as for the element

$$A = (G_1, H_1, G_2, H_2, \varphi\psi)$$

and the idempotent $e = (G_1, H_1, G_2, H_2, \varphi)$ four (G_1, H_1, G_2, H_2) is one and the same, so in accordance with upshot 1 out of theorem 1 with [5] $A \in \mathcal{H}(e)$. As $\mathcal{H}(e) = G_e$, so altogether with the equality (3) this finishes the proof of the theorem. □

2 Association of Maximal Subgroups

Lemma 1. *Let R be the cyclic group of the order free of squares, $N \trianglelefteq R$ is the subgroup. Then there exists such a subgroup $M \leqslant R$, that $R = N \times M$.*

Proof. As R is cyclic, so R contains the subgroup M of the order $|R|/|N|$. Except that, the numbers $|N|$ i $|R|/|N|$ are mutually simple. That's why $N \cap M = E$. On the other hand, $|N| \cdot |M| = |R|$. That's why $R = N \times M$. □

Let us note that if R, N and M are the same as in the lemma, then elements of the subgroup M can be taken as the representatives of the classes of the adjacency of the group R by the subgroup N.

Theorem 3. *Let G be the finite group. The semigroup of correspondences $S(G)$ will be the union of the groups only and only then when G is the cyclic group of the order free of squares.*

Proof. Adequacy. Let G be the cyclic group of the order free of squares. Let us show that every element $A = (H_1, G_1, H_2, G_2, \varphi)$ of $S(G)$ is an element of the group.

All the factors of the cyclic group are cyclic, that's why

$$G_1/H_1 \simeq G_2/H_2 \simeq C_k$$

for some k. It follows from lemma 1 that the representatives of the class of the adjacency of the group G_i by the semigroup H_i can be taken the elements of some subgroup $B_i \leqslant G_i$ of the order k. In particular, $B_1, B_2 \leqslant G$. But the group G contains only one subgroup of the order k. That's why $B_1 = B_2$. Let $B_1 = B_2 =< a >$. Let the block of the group A with the first projection $H_1 a$ have the form $H_1 a \times H_2 a^r$. As the class $H_2 a^r$ is to be the forming element of the factorgroup G_2/H_2, so r is mutually simple with k. Besides, we have:

$$A = \bigcup_{j=0}^{k-1} \left(H_1 a^j \times H_2 a^{jr}\right).$$

That's why the isomorphism

$$\varphi : G_1/H_1 \to G_2/H_2$$

acts as follows:

$$\varphi(H_1 a^j) = H_2 a^{jr} .$$

Let us analyze now the element $A^2 = A \circ A$:

$$A^2 = \bigcup_{j=0}^{k-1} \left(H_1 a^j \times H_2 a^{jr}\right) \circ \bigcup_{j=0}^{k-1} \left(H_1 a^j \times H_2 a^{jr}\right).$$

It is easy to check that

$$(H_1 a^j \times H_2 a^{jr}) \circ (H_1 a^t \times H_2 a^{yr}) =$$

$$= \begin{cases} \varnothing, \ \text{if} jr \neq t; \\ H_1 a^j \times H_2 a^{jr^2}, \ \text{if} jr = t. \end{cases}$$

Really, let

$$H_2 a^{jr} \cap H_1 a^t \neq \varnothing ,$$

then

$$h_2 a^{jr} = h_1 a^t \ \text{ for some } \ h_i \in H_i \, , \ i = 1, 2 \, .$$

Then $a^{jr-t} = h_2^{-1} h_1$, so that a^{jr-t} belongs to the subgroup $H_2 \cdot H_1$. But the order of the subgroup $< a >$ is mutually simple with the order of each of the group H_1 i H_2, and that's why with the order of the subgroup $|H_2 \cdot H_1|$. That's why

$$< a > \cap (H_1 \cdot H_2) = E \, , a^{jr-t} = e$$

and $a^{jr} = a^t$.

Besides, out of the mutual simplicity i k it outcomes that the congruence $xr \equiv t(\bmod k)$ has the only solution for the arbitrary t. That's why

$$A^2 = \bigcup_{j=0}^{k-1} \left(H_1 a^j \times H_2 a^{jr^2}\right).$$

The isomorphism $\varphi : H_1 a^j \mapsto H_2 a^{jr}$ can be identified with automorphism $\varphi : a^j \mapsto a^{jr}$ of the group $< a >$. As $a^{jr^2} = \varphi^2(a^j)$, so $A^2 = (H_1, G_1, H_2, G_2, \varphi^2)$.

By the induction on m it is analogically proved that

$$A^m = (H_1, G_1, H_2, G_2, \varphi^m) \, .$$

But φ, as the automorphism of the finite group has the finite order q. That's why $\varphi^{q+1} = \varphi$ and $A^{q+1} = A$. So, it is generated by the element A the cyclic subsemigroup is the cyclic group of the order q, and the element A is a group. That is why $S(G)$ is the union of groups.

Necessity. It is enough to show that for every group G, that satisfies one of the conditions:

a) G is not cyclic;

b) G is cyclic and its order $|G|$ is divided into square p^2 of a simple number p, semigroup of correspondences $S(G)$ is not the union of the groups.

For this it is enough to show that in each of these two cases the semigroup $S(G)$ contains the nongroup element.

a) Let the group G be not the cyclic one. Then there exists such a number k, that the group G contains 2 different cyclic subgroups of the order k. Let it be G_1 and G_2 and let $H = G_1 \cap G_2$. There is the isomorphism: $G_1 \to G_2$, which on the subgroup H acts identically. Let us view the element $A = (E, G_1, E, G_2, \varphi)$. Then $A^2 = (E, H, E, H, \varepsilon)$, where ε is the identified automorphism of the group H. While transferring from A to A^2 both of the projections have lessened so there does not exists such q that $A^q = A$. So, the element A is not of the group one.

b) Let $< a >$ be a cyclic subgroup of the order p^2 із G. Let us take $G_1 =< a^p >$, $G =< a >$, $H_1 = E$, $H_2 =< a >$. Then $A = \{(a^{pk}, a^k) \mid 0 \leqslant k \leqslant p^2\}$ is the element out of $S(G)$. Besides, $A^2 = \{(e, p^k) \mid 0 \leqslant k < p^2\}$.

But for the arbitrary $m > 1$ the first projection of the element A^m will be equal to $\{e\}$. So, A is not the group element. □

Corollary 1. *If G is the group, so the semigroup of correspondences $S(G)$ will be the sheaf then and only then when $|G| \leqslant 2$.*

Proof. The each element is a group, so according to theorem 2 the group G should be cyclic. If $|G| > 2$, so the group G should have a nontrivial automorphism φ.

Let us choose such an element $g \in G$, that $\varphi(g) \neq g$. Let us view in $S(G)$ the element $A = \{(g, \varphi(g))|g \in G\}$.

For this element $H_1 = H_2 = E$, $G_1 = G_2 = G$. Since $\varphi(gH_1) = \varphi(g) \neq g = gH_2$, so for A there is not fulfilled condition 2 of theorem 1. That's why A is not the idempotentне and i $S(G)$ is not the link.

If $|G| \leqslant 2$, it is easy to check that $S(G)$ is the link. □

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