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Про один ітераційний метод знаходження розв'язків рівняння другого порядку з запізнюванням

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An iterative method of finding solutions of the second order equation with delay

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Розглядається лінійне стаціонарне диференціальне рівняння другого порядку. За допомогою методу кроків і створення ітераційної процедури записано розв'язок задачі Коші.

Ключові слова: рівняння із запізнюванням аргументу, задача Коші, ітераційна процедура.

Differential equations with delay is a powerful instrument for modeling many natural, technical and social processes. In particular, a lot of the dynamic processes in ecology, economics, medicine, population dynamics can not adequately describe the real state without aftereffect. Differential equations with aftereffect (in particular with delay) can be defined as differential equations in Banach space. And its studing encounters significant difficulties. In particular, linear vibration equation is frequently used in describing many mechanical (linear oscillator). Representation of its solutions is not difficult. But even if you enter one constant delay in the phase coordinate, the analytical representation of the solution become difficult.

In this paper, using an iterative procedure, made an attempt to write the solution of the Cauchy problem for a linear oscillator with a constant delay. Preformed form of the solution of the Cauchy problem for a second-order linear equation. Uses two special functions that define "fundamental system of solutions". For obtaining these functions uses iterative procedure. At each step of the iteration, the equation is an ordinary differential equation with known right-hand side depending on the prehistory. Recordes its solution.

Key Worda: differential equations with delay, problem of Cauchy, iteration procedure.

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Introduction

In this paper we consider the linear second-order equation with one constant delay. Second order differential equations are widely used in various applied sciences [1,2]. Aftereffects influence on the systems dynamics caused the investigation of functional differential systems [3-6]. However, obtain the solution of these equations in explicit form possible only to systems with pure delay [7]. The paper proposes an iterative method for finding solutions of second order equation with one constant delay.

1. General solution.

In this paper we consider a linear homogeneous equation with one constant delay.

$$x''(t)+a_1x'(t)+a_2x'(t-\tau)+b_1x(t)+b_2x(t-\tau)=f(t),$$
$$t\geq 0,\quad \tau>0, \tag{1.1}$$

with the initial condition

$$x(t) \equiv \phi(t), \quad x'(t) = \phi'(t), \; -\tau \le t \le 0. \quad (1.2)$$

Here a_1, a_2, b_1, b_2 constants, $f(t)$ - continuous function, $\phi(t)$ - random, twice continuously differentiable on the initial interval $-\tau \le t \le 0$. function determining the initial conditions. By a solution of the Cauchy problem (1.2) the equation (1.1) we mean defined at $t \ge 0$, twice continuously differentiable function $x(t)$, identically satisfying the eqution (1.1) and the initial conditions (1.2). As is known, the solution of the Cauchy problem (1.2) for the inhomogeneous equation (1.1) can be represented as a sum of

$$x(t) = x^0(t) + \bar{x}(t),$$

where $x^0(t)$ – the solution of homogeneous equation

$$x''(t) + a_1 x'(t) + a_2 x'(t-\tau) + b_1 x(t) + b_2 x(t-\tau) = 0,$$
$$t > 0, \quad \tau > 0, \quad (1.3)$$

with initial conditions (1.2), and $\bar{x}(t)$ – the solution of inhomogeneous equation (1.1) with zero initial conditions $\bar{x}(t) \equiv 0$, $\bar{x}'(t) \equiv 0$, $-\tau \le t \le 0$.

Denote $x_1(t)$ – the solution of the homogeneous differential equation with initial conditions

$$x_1(t) \equiv 1, \quad x_1'(t) \equiv 0, \; -\tau \le t \le 0, \quad (1.4)$$

and $x_2(t)$ - solution of the homogeneous differential equation (1.3) with initial conditions

$$x_2(t) \equiv t + \tau, \quad x_2'(t) \equiv 1, \; -\tau \le t \le 0. \quad (1.5)$$

We give an affirmation of representation of solution of the Cauchy problem (1.2) for the homogeneous equation (1.3), using functions $x_1(t)$, $x_2(t)$, satisfying the initial conditions (1.4), (1.5). The resulting dependence is similar to given in the work [4, p.69].

Theorem 1.1. Solution of Cauchy problem for the homogeneous equation (1.3) with initial conditions (1.2) can be written as

$$x(t) = \phi(-\tau) x_1(t) + \phi'(-\tau) x_2(t) + \int_{-\tau}^{0} x_2(t-\tau-s)\phi''(s)ds, \quad (1.6)$$

where $x_1(t)$ the solution of equation (1.3), satisfying the conditions (1.4), $x_2(t)$ the solution of equation (1.3), satisfying conditions (1.5).

Proof. The solution of equation (1.3) we will find in the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \int_{-\tau}^{0} x_2(t-\tau-s) c''(s) ds, \quad (1.7)$$

where c_1, c_2 – constants, and $c(t)$ – twice continuously differentiable function. Substituting (1.7) into equation (1.3), we obtain

$$\frac{d^2}{dt^2}\left[c_1 x_1(t) + c_2 x_2(t) + \int_{-\tau}^{0} x_2(t-\tau-s) c(s) ds \right] +$$
$$+a_1 \frac{d}{dt}\left[c_1 x_1(t) + c_2 x_2(t) + \int_{-\tau}^{0} x_2(t-\tau-s) c(s) ds \right] +$$
$$+a_2 \frac{d}{dt}\left[c_1 x_1(t-\tau) + c_2 x_2(t-\tau) + \int_{-\tau}^{0} x_2(t-2\tau-s) c(s) ds \right] +$$
$$+b_1 \left[c_1 x_1(t) + c_2 x_2(t) + \int_{-\tau}^{0} x_2(t-\tau-s) c(s) ds \right] +$$
$$+b_2 \frac{d}{dt}\left[c_1 x_1(t-\tau) + c_2 x_2(t-\tau) + \int_{-\tau}^{0} x_2(t-2\tau-s) c(s) ds \right] = 0$$

Rewrite the resulting equation in the form

$$c_1\left[x_1''(t) + a_1 x_1'(t) + a_2 x_1'(t-\tau) + b_1 x_1(t) + b_2 x_1(t-\tau) \right] +$$
$$+c_2\left[x_2''(t) + a_1 x_2'(t) + a_2 x_2'(t-\tau) + b_1 x_2(t) + b_2 x_2(t-\tau) \right] +$$
$$+\int_{-\tau}^{0} \left[x_2''(t-\tau-s) + a_1 x_2'(t-\tau-s) + a_2 x_2'(t-2\tau-s) + \right.$$
$$\left. + b_1 x_2(t-\tau-s) + b_2 x_2(t-2\tau-s) \right] c''(s) ds \equiv 0.$$

Since, $x_1(t)$ and solutions of the homogeneous equation, then for random constants c_1 and c_2 equation in square brackets identically equal to zero And obtained identity shows that (1.6) is a solution of equation (1.3) for arbitrary constants c_1, c_2 and arbitrary function $c(t)$.

We show that if we put $c_1 = \phi(-\tau)$, $c_2 = \phi'(-\tau)$, $c(t) = \phi(t)$, that will satisfy the initial conditions

$$x(t) = \phi(t), \quad x'(t) = \phi'(t), \; -\tau \le t \le 0,$$

So the Cauchy problem (1.2) for the homogeneous equation (1.3) has the form (1.6).

Substituting the relation (1.6) the initial conditions (1.2), we obtain

$$\phi(t)=\phi(-\tau)x_1(t)+\phi'(-\phi)x_2(t)+\int_{-\tau}^{0}x_2(t-\tau-s)\phi''(s)ds,$$
$$-\tau\le t\le 0.$$

We change the variables in the integral: $t-\tau-s=\xi$. Then we obtain

$$\phi(t)=\phi(-\tau)x_1(t)+\phi'(-\tau)x_2(t)+\int_{t-\tau}^{t}x_2(\xi)\phi''(t-\tau-\xi)d\xi,$$
$$-\tau\le t\le 0.$$

We split the integral into two

$$\phi(t)=\phi(-\tau)x_1(t)+\phi'(-\tau)x_2(t)+\int_{t-\tau}^{-\tau}x_2(\xi)\phi''(t-\tau-\xi)d\xi+$$
$$+\int_{-\tau}^{t}x_2(\xi)\phi''(t-\tau-\xi)d\xi.$$

Since for $t-\tau\le\xi\le-\tau$ will be $x_1(t)\equiv 0$, first integral disappears and remains

$$\phi(t)=\phi(-\tau)x_1(t)+\phi'(-\tau)x_2(t)+\int_{-\tau}^{t}x_2(\xi)\phi''(t-\tau-\xi)d\xi,$$
$$-\tau\le t\le 0.$$

Take the integral by parts

$$\phi(t)=\phi(-\tau)x_1(t)+\phi'(-\tau)x_2(t)-$$
$$-x_2(\xi)\phi'(t-\tau-\xi)\Big|_{\xi=-\tau}^{\xi=t}+\int_{-\tau}^{t}x_2'(\xi)\phi'(t-\tau-\xi)d\xi=$$
$$=\phi(-\tau)x_1(t)+x_2(-\tau)\phi'(t)+$$
$$\int_{-\tau}^{t}x_2'(\xi)\phi'(t-\tau-\xi)d\xi.$$

Since on the interval $-\tau\le t\le 0$ will be $x_1(t)\equiv 1$, $x_2(-\tau)=0$, $x_2'(t)\equiv 1$, it remains

$$\phi(t)=\phi(-\tau)+\int_{-\tau}^{t}\phi'(t-\tau-\xi)d\xi=$$
$$=\phi(-\tau)-\phi(t-\tau-\xi)\Big|_{\xi=-\tau}^{\varsigma=t}=\phi(t).$$

Therefore first initial condition is satisfied. We shall show that the second initial condition is satisfied. Differentiating (1.6), we obtain

$$x'(t)=\phi(-\tau)x_1'(t)+\phi'(-\tau)x_2'(t)+\int_{-\tau}^{0}x_2'(t-\tau-s)\phi''(s)ds.$$

After the change of variables in the integral, we have

$$x'(t)=\phi(-\tau)x_1'(t)+\phi'(-\tau)x_2'(t)+\int_{t-\tau}^{t}x_2'(\xi)\phi''(t-\tau-\xi)d\xi.$$

Considering, that on the initial interval $x_1'(t)\equiv 0$, $x_2'(t)\equiv 1$, we obtain

$$x'(t)=\phi'(-\tau)+\int_{-\tau}^{t}\phi''(t-\tau-\xi)d\xi=$$
$$=\phi'(-\tau)-\phi'(t-\tau-\xi)\Big|_{\xi=-\tau}^{\xi=t}=\phi'(t).$$

Therefore it is shown that the second initial condition is satisfied.

The theorem is proved.

2. Construction iterative procedure.

Thus, to obtain the solution of the Cauchy problem (1.2) the equation (1.1) should have two functions $x_1(t)$, $x_2(t)$, are solutions of the homogeneous equation with special initial conditions.

Theorem 2.1. The solution $x_1(t)$, $x_2(t)$ homogeneous equation with delay (1.3) with initial conditions (1.4), (1.5) can be written as

$$x_1(t)=\begin{cases} x_{1,0}(t), & -\tau\le t<0,\\ x_{1,1}(t), & 0\le t<\tau,\\ \dots & .\\ x_{1,n}(t) & (n-1)\tau\le t<n\tau\end{cases},$$

$$x_2(t)=\begin{cases} x_{2,0}(t), & -\tau\le t<0,\\ x_{2,1}(t), & 0\le t<\tau,\\ \dots & .\\ x_{2,n}(t) & (n-1)\tau\le t<n\tau\end{cases} \qquad (2.1)$$

where: $x_{1,0}(t)\equiv 1$, $x_{1,0}'(t)\equiv 0$, $x_{2,0}(t)\equiv t+\tau$, $x_{2,0}'(t)\equiv 1$, $-\tau\le t\le 1$;

$$x_{i,n}(t)=x_{i,n-1}((n-1)\tau)x_1^0(t)+\frac{d}{dt}x_{i,n-1}(t)\Big|_{t=(n-1)\tau}x_2^0(t)+$$
$$+\int_0^t K(t,s)F_{i,n}(s)ds, \qquad (2.2)$$

$$F_{i,n}(t)=-a_2\frac{d}{dt}x_{i,n-1}(t-\tau)-b_2x_{i,n-1}(t-\tau),$$
$$i=\overline{1,2},\ n=1,2,3,\dots \qquad (2.3)$$

$$K(t,s)=\frac{x_1^0(s)x_2^0(t)-x_1^0(t)x_2^0(s)}{W\left[x_1^0(s),x_2^0(s)\right]},$$

$$W\left[x_1^0(s),x_2^0(s)\right]=\begin{vmatrix} x_1^0(s) & x_2^0(s) \\ \frac{d}{ds}x_1^0(s) & \frac{d}{ds}x_2^0(s)\end{vmatrix}. \quad (2.4)$$

$x_1^0(t)$, $x_2^0(t)$ solutions of the homogeneous equation without delay

$$x''(t)+a_1x'(t)+b_1x(t)=0,$$

satisfying the conditions $x_1^0(0)=1$,

$$\frac{d}{dt}x_1^0(0)=0,\ x_2(0)=0,\ \frac{d}{dt}x_2(0)=1.$$

Proof. To find solutions $x_1(t)$, $x_2(t)$ of homogeneous equation with delay (1.3) satisfying the single initial conditions (1.4), (1.5), we use the method of steps [4].

It is known that the solution of the linear nonhomogeneous equation with constant coefficients of the second order without delay

$$x''(t)+a_1x'(t)+b_1x(t)=F(t), \quad (2.5)$$

satisfying the zero initial conditions, has the form

$$x(t)=\int_t^t K(t,s)F(s)ds,$$

where $K(t,s)$ – the solution of homogeneous equation satisfying the initial conditions

$$K(t,s)\big|_{t=s}=0,\ K'_x(t,s)\big|_{t=s}=1.$$

Since equation (2.5) with constant coefficients, then, depending on the eigenvalues of the characteristic equation, the corresponding homogeneous equation

$$x''(t)+a_1x'(t)+b_1x(t)=0$$

has two linearly independent solutions that are represented as exponent, the exponent with trigonometric functions or exponent with multiplier t. Let the solution of the homogeneous equation satisfying the initial conditions $x(t_0)=1$, $x'(t_0)=0$ is $x_1^0(t)$, and the solution that satisfies the initial conditions $x(t_0)=0$, $x'(t_0)=1$ is $x_2^0(t)$. Then the solution of the nonhomogeneous equation (2.5) satisfying the nonzero initial conditions $x(t_0)=x_0$, $x'(t_0)=x'_0$ has the form

$$x(t)=x_0x_1^0(t)+x'_0x_2^0(t)+\int_t^t K(t,s)F(s)ds,$$

$$K(t,s)=\frac{x_1^0(s)x_2^0(t)-x_1^0(t)x_2^0(s)}{W\left[x_1^0(s),x_2^0(s)\right]},$$

$$W\left[x_1^0(s),x_2^0(s)\right]=\begin{vmatrix} x_1^0(s) & x_2^0(s) \\ \left(x_1^0(s)\right)' & \left(x_2^0(s)\right)'\end{vmatrix}. \quad (2.6)$$

Go back to finding solutions $x_1(t)$, $x_2(t)$ equation with delay (1.3) with initial values. We obtain pre-form solutions of the homogeneous second order differential equation with a constant delay (1.3) corresponding to the initial conditions $x(t)\equiv\phi(t)$, $x'(t)=\phi'(t)$, $-\tau\le t\le 0$, where $\phi(t)$ random continuously differentiable function.

1. Consider the first interval $T_1: 0\le t<\tau$. On this interval the equation (1.3) is a linear nonhomogeneous differential equation without delay and has the form

$$x''(t)+a_1x'(t)+b_1x(t)=F_1(t),$$
$$F_1(t)=-a_2x'(t-\tau)-b_2x(t-\tau),\ T_1: 0\le t<\tau. \quad (2.7)$$

Suppose that the initial conditions take the form

$$x(0)=\phi(0),\ x'(0)=\phi'(0). \quad (2.8)$$

Then it follows from (2.6), the solution of the equation (2.7) satisfying conditions (2.8), is

$$x_{1,\phi}(t)=x_{0,\phi}(0)x_1^0(t)+\frac{d}{dt}x_{0,\phi}(t)\Big|_{t=0}x_2^0(t)+\int_0^t K(t,s)F_1(s)ds$$

$$x_{0,\phi}(0)=\phi(0),\ \frac{d}{dt}x_{0,\phi}(t)\Big|_{t=0}=\phi'(0),$$

$$F_1(t)=-a_2\phi'(t-\tau)-b_2\phi(t-\tau). \quad (2.9)$$

2. Consider the second interval $T_2: \tau\le t<2\tau$. On this interval the equation (1.3) is also a linear nonhomogeneous differential equation without delay and has the form that is different from previous one only by the right part

$$x''(t)+a_1x'(t)+b_1x(t)=F_2(t),$$

$$F_1(t)=-a_2\frac{d}{dt}x_{1,\phi}(t-\tau)-b_2x_{1,\phi}(t-\tau),$$

$$T_2: \tau\le t<2\tau. \quad (2.10)$$

The initial conditions have the form

$$x(\tau) = x_{1,\phi}(\tau),\ x'(\tau) = \frac{d}{dt}x_{1,\phi}(t)\Big|_{t=\tau}. \quad (2.11)$$

As follows from (2.6), a solution of the equation (2.10), satisfying the conditions (2.11) is

$$x_{2,\phi}(t) = x_{1,\phi}(\tau)x_1^0(t) + \frac{d}{dt}x_{1,\phi}(t)\Big|_{t=\tau} x_2^0(t) + \int_\tau^t K(t,s)F_2(s)ds$$

$$F_2(t) = -a_2\frac{d}{dt}x_{1,\phi}(t-\tau) - b_2 x_{1,\phi}(t-\tau). \quad (2.12)$$

Continuing this process further, we find that on the interval $T_n : (n-1)\tau \le t < n\tau$. The equation (1.3) has the form

$$x''(t) + a_1 x'(t) + b_1 x(t) = F_n(t),$$
$$F_n(t) = -a_2\frac{d}{dt}x_{n-1,\phi}(t-\tau) - b_2 x_{n-1,\phi}(t-\tau),$$
$$T_n : (n-1)\tau \le t < n\tau. \quad (2.13)$$

The initial conditions for the solutions of this equation have the form

$$x(\tau) = x_{n-1,\phi}((n-1)\tau),$$
$$x'(\tau) = \frac{d}{dt}x_{n-1,\phi}(t)\Big|_{t=(n-1)\tau}, \quad (2.14)$$

and solution of the equation on the interval $T_n : (n-1)\tau \le t < n\tau$, satisfying the conditions of continuity is

$$x_{n,\phi}(t) = x_{n-1,\phi}((n-1)\tau)x_1^0(t) +$$
$$+\frac{d}{dt}x_{n-1}(t)\Big|_{t=(n-1)\tau} x_2^0(t) + \int_{(n-1)\tau}^t K(t,s)F_n(s)ds,$$
$$F_n(t) = -a_2\frac{d}{dt}x_{n-1}(t-\tau) - b_2 x_{n-1}(t-\tau). \quad (2.15)$$

To obtain linearly independent solutions $x_1(t)$ и $x_2(t)$, satisfying the initial conditions (1.4), (1.5) we put initial values for $x_1(t)$:

$$x_{1,0}(t) \equiv 1,\ \frac{d}{dt}x_{1,0}(t) \equiv 0,\ -\tau \le t \le 0, \quad (2.16)$$

And for $x_2(t)$:

$$x_{2,0}(t) \equiv t+\tau,\ \frac{d}{dt}x_{2,0}(t) \equiv 1,\ -\tau \le t \le 0. \quad (2.17)$$

From (2.15), we obtain Theorem 1.2.

If equation (1.3) has special form

$$x''(t) + \omega_1^2 x(t) + \omega_2^2 x(t-\tau) = 0,\ t \ge 0,\ \tau > 0, \quad (2.18)$$

so $a_1 = a_2 = 0$, $b_1 = \omega_1^2$, $b_2 = \omega_2^2$, тthen linearly independent solutions $x_1(t)$, $x_2(t)$, satisfying the initial conditions (1.4) and (1.5) have more concrete form [8].

Theorem 2.2. The solution $x_1(t)$, $x_2(t)$ of the equation (2.18) with initial conditions (1.4), (1.5) can be written in the form

$$x_1(t) = \begin{cases} x_{10}(t), & -\tau \le t < 0, \\ x_{11}(t), & 0 \le t < \tau, \\ \dots . & \\ x_{1n}(t) & (n-1)\tau \le t < n\tau \end{cases},$$

$$x_2(t) = \begin{cases} x_{20}(t), & -\tau \le t < 0, \\ x_{21}(t), & 0 \le t < \tau, \\ \dots . & \\ x_{2n}(t) & (n-1)\tau \le t < n\tau, \end{cases}, \quad (2.19)$$

where

$$x_{1,n}(t) = x_{1,n-1}((n-1)\tau)\cos\omega_1(t-(n-1)\tau) +$$
$$+\frac{x'_{1,k-1}((n-1)\tau)}{\omega_1}\sin\omega_1(t-(n-1)\tau) -$$
$$-\omega_2^2\int_{(n-1)\tau}^t \sin\omega_1(t-s)x_{1,n-1}(s)ds,$$
$$(n-1)\tau \le t < n\tau,\ x_{10}(t) = 1,\ x'_{10}(t) = 0,$$
$$x_{2,n}(t) = x_{2,n-1}((k-1)\tau)\cos\omega_1(t-(n-1)\tau) +$$
$$+\frac{x'_{2,n-1}((n-1)\tau)}{\omega_1}\sin\omega_1(t-(n-1)\tau) -$$
$$-\omega_2^2\int_{(n-1)\tau}^t \sin\omega_1(t-s)x_{2,n-1}(s)ds,$$
$$(n-1)\tau \le t < n\tau,\ x_{2,0}(t) = t+\tau,\ x'_{2,0}(t) = 1.$$

Proof. To find solutions $x_1(t)$, $x_2(t)$ the equation (2.18) we use the results of the previous theorem. Solution of the homogeneous equation

$$x''(t) + \omega_1^2 x(t) = 0, \quad (2.21)$$

satisfying the initial conditions $x(0) = 1$, $x'(0) = 0$ is $x_1^0(t) = \cos\omega_1 t$, and solution satisfying the conditions $x(0)=0$, $x'(0)=1$, will be $x_2^0(t)=\frac{1}{\omega_1}\sin\omega_1 t$. Those

$$W\left[x_1^0(s),x_2^0(s)\right]\equiv 1,\ K(t,s)=\frac{1}{\omega_1}\sin\omega_1(t-s).$$

Consider the first interval $T_1: 0\le t<\tau$. On this interval equation (1.5) is a linear nonhomogeneous differential equation without delay and has the form

$$x''(t)+\omega_1^2 x(t)=-\omega_2^2 x_{10}(t-\tau) \qquad (2.22)$$

the right part is a known function $x_{10}(t)=1$. As follows from (2.9), the solution of (1.12) on the interval $T_1: 0\le t<\tau$ will be

$$x_{11}(t)=x_{10}(0)x_1^0(t)+x_{10}'(0)x_2^0(t)-$$
$$-\omega_2^2\int_0^t K(t,s)x_{10}(s-\tau)ds=\cos\omega_1 t-$$
$$-\omega_2^2\int_0^t\frac{1}{\omega_1}\sin\omega_1(t-s)ds=\cos\omega_1 t+\frac{\omega_2^2}{\omega_1^2}(1-\cos\omega_1 t)=$$
$$=\frac{\omega_2^2}{\omega_1^2}+\left(1-\frac{\omega_2^2}{\omega_1^2}\right)\cos\omega_1 t \qquad (2.23)$$

Consider the second interval $T_2:\tau\le t<2\tau$. On this interval equation (1.5) has the form

$$x''(t)+\omega_1^2 x(t)=-\omega_2^2 x_{11}(t-\tau). \qquad (2.24)$$

Solution of equation (2.24) on the interval $T_2:\tau\le t<2\tau$ will be

$$x_{12}(t)=x_{11}(\tau)\cos\omega_1(t)+\frac{x_{11}'(\tau)}{\omega_1}\sin\omega_1(t)-$$
$$-\omega_2^2\int_\tau^t\frac{1}{\omega_1}\sin\omega_1(t-s)x_{11}(s-\tau)ds=$$
$$=\left[\frac{\omega_2^2}{\omega_1^2}+\left(1-\frac{\omega_2^2}{\omega_1^2}\right)\cos\omega_1\tau\right]\cos\omega_1(t-\tau)-$$
$$-\left(1-\frac{\omega_2^2}{\omega_1^2}\right)\sin\omega_1\tau\sin\omega_1(t-\tau)-$$
$$-\omega_2^2\int_\tau^t\frac{1}{\omega_1}\sin\omega_1(t-s)\left[\frac{\omega_2^2}{\omega_1^2}+\left(1-\frac{\omega_2^2}{\omega_1^2}\right)\cos\omega_1(s-\tau)\right]ds$$
.

Continuing this process further, we find that on the interval $T_n:(n-1)\tau\le t<n\tau$ solution of the homogeneous equation (2.21) is

$$x_{1,n}(t)=x_{1,n-1}((n-1)\tau)\cos\omega_1(t-(n-1)\tau)+$$
$$+\frac{x_{1,n-1}'((n-1)\tau)}{\omega_1}\sin\omega_1(t-(n-1)\tau)-$$
$$-\omega_2^2\int_{(k-1)\tau}^t\sin\omega_1(t-s)x_{1,n-1}(s-\tau)ds.$$

Those, $x_1(t)$ has the form

$$x_1(t)=\begin{cases}x_{1,0}(t), & -\tau\le t<0,\\ x_{1,1}(t), & 0\le t<\tau,\\ \dots & .\\ x_{1,n}(t) & (n-1)\tau\le t<n\tau.\end{cases}$$

Solution $x_2(t)$ of the equation (1.5) with initial conditions $x_2(t)\equiv t+\tau$, $x_2'(t)\equiv 1$, $-\tau\le t\le 0$ can be written in a similar form

$$x_{2,n}(t)=x_{2,n-1}((n-1)\tau)\cos\omega_1(t-(n-1)\tau)+$$
$$+\frac{x_{2,n-1}'((n-1)\tau)}{\omega_1}\sin\omega_1(t-(n-1)\tau)-$$
$$-\omega_2^2\int_{(n-1)\tau}^t\sin\omega_1(t-s)x_{2,n-1}(s)ds,$$
$$x_{2,0}(t)=t+\tau,\ x_{2,0}'(t)=1.$$

$$x_2(t)=\begin{cases}x_{2,0}(t), & -\tau\le t<0,\\ x_{2,1}(t), & 0\le t<\tau,\\ \dots & .\\ x_{2,n}(t) & (n-1)\tau\le t<n\tau,\end{cases}$$

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