УДК 519.21

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Вплив випадкових періодичних збурень на поведінку неавтономної коливної системи третього порядку у резонансному випадку

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Influence of random periodical disturbances on the behavior of non-autonomous third order oscillating system in the resonance case

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У роботі вивчається асимптотична поведінка неавтономної коливної системи, яка описується диференціальним рівнянням третього порядку з малими періодичними нелінійними зовнішніми збуреннями типу багатовимірного "білого" та Пауссонівського шумів. Розглядається резонансний випадок.

Ключові слова: метод усереднення, неавтономна коливна система, стохастичне диференціальне рівняння, резонансний випадок.

The asymptotic behavior of non-autonomous oscillating system describing by differential equation of third order with small periodical non-linear external perturbations of multidimensional "white noise" and "Poisson noise" types is studied. Every term of external perturbations has own order of small parameter ε. If small parameter is equal to zero, then general solution of obtained non-stochastic third order differential equation has an oscillating part. We consider given differential equation with external stochastic periodic perturbations as the system of stochastic differential equations and study the limit behavior of its solution at the time moment t/ε^k, as $\varepsilon \to 0$. The system of averaging stochastic differential equations is derived and its dependence on the order of small parameter in every term of external perturbations is studied. The resonance case is considered.

Key Words: averaging method, non-autonomous oscillating system, stochastic differential equation, resonance case.
Communicated by Prof. Kozachenko Yu.V.

1 Introduction

The averaging method proposed by N.M.Krylov, N.N.Bogolyubov and Yu.A.Mytropolskij ([1], [2]) is one of the main tool in studying of the deterministic oscillating systems under the action of small non-linear perturbations. The case of small random "white noise" type disturbances in oscillating systems of second order is considered in papers of Yu.A.Mytropolskij, V.G.Kolomiets ([3]). The autonomous and non-autonomous oscillating systems of second order under the action of "white noise" and Poisson type noise perturbations are studied in the papers of O.V.Borysenko ([4], [5]). Particular case of the third order oscillating systems are investigated in articles of O.D.Borysenko, O.V.Borysenko ([6]), O.D.Borysenko, O.V.Borysenko and I.G.Malyshev ([7], [8]). Limit behavior of autonomous and non-autonomous (non-resonance case) third order oscillating system under the action of external small nonlinear random disturbances such as multidimensional "white noise" and "Poisson noise" was studied in ([9], [10]).

This paper deals with investigation of the behavior, as $\varepsilon \to 0$, of the general type third order non-autonomous oscillating system driven by stochastic differential equation

$$\begin{aligned} x'''(t) + ax''(t) + b^2x'(t) + ab^2x(t) = \\ = \varepsilon^{k_0} f_0(\mu_0 t, x(t), x'(t), x''(t)) + \\ + f_\varepsilon(t, x(t), x'(t), x''(t)) \end{aligned} \tag{1}$$

with non-random initial conditions $x(0) = x_0, x'(0) = x'_0, x''(0) = x''_0$, where $\varepsilon > 0$ is a small parameter, $f_\varepsilon(t,x,x',x'')$ is a random function such that

$$\int_0^t f_\varepsilon(s,x(s),x'(s),x''(s))\,ds = \sum_{i=1}^m \varepsilon^{k_i}\times \times\int_0^t f_i(\mu_i s,x(s),x'(s),x''(s))\,dw_i(s) + \varepsilon^{k_{m+1}}\times$$
$$\times\int_0^t\int_{\mathbb{R}} f_{m+1}(\mu_{m+1}s,x(s),x'(s),x''(s),z)\,\nu(ds,dz),$$

$k_i > 0, i = \overline{0,m+1}$; f_i are non-random functions, periodic on $\mu_i t$, $i = \overline{0,m+1}$ with period 2π; $w_i(t), i = \overline{1,m}$ are independent one-dimensional Wiener processes; $\nu(dt,dy)$ is the Poisson measure independent on $w_i(t), i = \overline{1,m}$, $E\nu(dt,dy) = \Pi(dy)dt$, $\tilde\nu(dt,dy) = \nu(dt,dy) - \Pi(dy)dt$, $\Pi(A)$ is a finite measure on Borel sets $A \in \mathbb{R}$, $a > 0, b > 0$.

We will consider the equation (1) as the system of stochastic differential equations

$$\begin{aligned}
&dx(t) = x'(t)dt,\\
&dx'(t) = x''(t)dt,\\
&dx''(t) = [-ax''(t) - b^2x'(t) - ab^2x(t)+\\
&+\varepsilon^{k_0} f_0(\mu_0 t,x(t),x'(t),x''(t)) + \varepsilon^{k_{m+1}}\times\\
&\times\int_{\mathbb{R}} f_{m+1}(\mu_{m+1}t,x(t),x'(t),x''(t),z)\Pi(dz)]dt+\\
&+\sum_{i=1}^m \varepsilon^{k_i} f_i(\mu_i t,x(t),x'(t),x''(t))dw_i(t) + \varepsilon^{k_{m+1}}\times\\
&\times\int_{\mathbb{R}} f_{m+1}(\mu_{m+1}t,x(t),x'(t),x''(t),z)\tilde\nu(dt,dz),\\
&x(0) = x_0,\ x'(0) = x'_0,\ x''(0) = x''_0.
\end{aligned} \quad (2)$$

In what follows we will use the constant $K > 0$ for the notation of different constants, which are not depend on ε.

2 Main result

We will study the possible resonance case: $\mu_j = \frac{p_j}{q_j} b$ for some $j = \overline{0,m+1}$, where p_j and q_j are relatively prime integers. Let us consider the following representation of processes $x(t), x'(t), x''(t)$:

$$x(t) = C(t)e^{-at} + A_1(t)\cos(bt) + A_2(t)\sin(bt),$$
$$x'(t) = -aC(t)e^{-at} - bA_1(t)\sin(bt) + bA_2(t)\cos(bt),$$
$$x''(t) = a^2C(t)e^{-at} - b^2A_1(t)\cos(bt) - b^2A_2(t)\sin(bt),$$
$$N(t) = C(t)\exp\{-at\}.$$

Then

$$\begin{aligned}
N(t) &= \frac{b^2x(t) + x''(t)}{a^2+b^2},\\
A_1(t) &= \cos\alpha\cos(bt+\alpha)x(t) - \frac{\sin bt}{b}x'(t) - \\
&-\frac{\sin\alpha\sin(bt+\alpha)}{b^2}x''(t),\\
A_2(t) &= \cos\alpha\sin(bt+\alpha)x(t) + \frac{\cos bt}{b}x'(t) + \\
&+\frac{\sin\alpha\cos(bt+\alpha)}{b^2}x''(t),
\end{aligned}$$

where $\alpha = \operatorname{arctg}(b/a)$. We can apply Ito formula [11] to stochastic process $\xi(t) = (N(t), A_1(t), A_2(t))$ and obtain for the process $\xi(t)$ the system of stochastic differential equations

$$\begin{aligned}
dN(t) &= -aN(t)dt + \frac{1}{a^2+b^2}dH(t)\\
dA_1(t) &= -\frac{\sin\alpha\sin(bt+\alpha)}{b^2}dH(t)\\
dA_2(t) &= \frac{\sin\alpha\cos(bt+\alpha)}{b^2}dH(t)
\end{aligned} \quad (3),$$

$$\begin{aligned}
N(0) &= \frac{b^2x_0 + x''_0}{a^2+b^2},\\
A_1(0) &= \frac{a^2x_0 - x''_0}{a^2+b^2},\\
A_2(0) &= \frac{ax''_0 + (a^2+b^2)x'_0 + ab^2x_0}{b(a^2+b^2)},
\end{aligned}$$

where

$$\begin{aligned}
&dH(t) = [\varepsilon^{k_0}\tilde f_0(\mu_0 t, N(t), A_1(t), A_2(t), t) + \varepsilon^{k_{m+1}}\times\\
&\times\int_{\mathbb{R}} \tilde f_{m+1}(\mu_{m+1}t, N(t), A_1(t), A_2(t), t, z)\Pi(dz)]dt+\\
&+\sum_{i=1}^m \varepsilon^{k_i}\tilde f_i(\mu_i t, N(t), A_1(t), A_2(t), t)dw_i(t) + \varepsilon^{k_m+1}\times\\
&\times\int_{\mathbb{R}} \tilde f_{m+1}(\mu_{m+1}t, N(t), A_1(t), A_2(t), t, z)\tilde\nu(dt,dz),
\end{aligned}$$

$\tilde f_i(\mu_i t, N, A_1, A_2, t) = f_i(\mu_i t, N + A_1\cos bt + A_2\sin bt, -aN - bA_1\sin bt + bA_2\cos bt, a^2N - b^2A_1\cos bt - b^2A_2\sin bt)$, $i = \overline{0,m}$, $\tilde f_{m+1}(\mu_{m+1}t, N, A_1, A_2, t, z) = f_{m+1}(\mu_{m+1}t, N + A_1\cos bt + A_2\sin bt, -aN - bA_1\sin bt + bA_2\cos bt, a^2N - b^2A_1\cos bt - b^2A_2\sin bt, z)$.

Theorem 2.1. *Let $\Pi(\mathbb{R}) < \infty$, $t \in [0,t_0]$, $k = \min(k_0, 2k_1, \ldots, 2k_m, k_{m+1})$. Let us suppose, that functions $f_j, j = \overline{0,m+1}$ bounded and satisfy Lipschitz condition on x, x', x''. If given below matrix $\bar\sigma^2(A_1,A_2)$ is non-negative definite, then*

1. Let $\mu_j = \frac{p_j}{q_j} b$ for all $j = \overline{0,m+1}$, where p_j and q_j are some relatively prime integers. If $k_0 = 2k_i = k_{m+1}$, $i = \overline{1,m}$ then the stochastic *process $\xi_\varepsilon(t) = \xi(t/\varepsilon^k)$ weakly converges, as $\varepsilon \to 0$, to the stochastic process $\bar{\xi}(t) = (0, \bar{A}_1(t), \bar{A}_2(t))$, where $\bar{A}(t) = (\bar{A}_1(t), \bar{A}_2(t))$ is the solution to the system of stochastic differential equations*

$$\begin{array}{l} d\bar{A}(t) = \bar{\alpha}(\bar{A}(t))dt + \bar{\sigma}(\bar{A}(t))d\bar{w}(t), \\ \bar{A}(0) = (A_1(0), A_2(0)), \end{array} \tag{4}$$

where $\bar{\alpha}(\bar{A}) = (\bar{\alpha}^{(1)}(A_1, A_2), \bar{\alpha}^{(2)}(A_1, A_2))$,

$$\bar{\alpha}^{(1)}(A_1, A_2) = -\frac{1}{4\pi^2 b(a^2+b^2)} \Bigg[\sum_{p_0 n + q_0 l = 0} \int_0^{2\pi}\int_0^{2\pi} \hat{f}_0(\psi, A_1, A_2, \phi) \times (a\sin\psi + b\cos\psi) e^{-i(n\psi + l\phi)}\, d\phi\, d\psi + \sum_{p_{m+1} n + q_{m+1} l = 0} \int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} \hat{f}_{m+1}(\psi, A_1, A_2, \phi, z) \times (a\sin\psi + b\cos\psi) e^{-i(n\psi + l\phi)}\, \Pi(dz)\, d\phi\, d\psi \Bigg],$$

$$\bar{\alpha}^{(2)}(A_1, A_2) = \frac{1}{4\pi^2 b(a^2+b^2)} \Bigg[\sum_{p_0 n + q_0 l = 0} \int_0^{2\pi}\int_0^{2\pi} \hat{f}_0(\psi, A_1, A_2, \phi) \times (a\cos\psi - b\sin\psi) e^{-i(n\psi + l\phi)}\, d\phi\, d\psi + \sum_{p_{m+1} n + q_{m+1} l = 0} \int_0^{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} \hat{f}_{m+1}(\psi, A_1, A_2, \phi, z) \times (a\cos\psi - b\sin\psi) e^{-i(n\psi + l\phi)}\, \Pi(dz)\, d\phi\, d\psi \Bigg],$$

$$\bar{\sigma}(A_1, A_2) = \left\{ \bar{B}(A_1, A_2) \right\}^{\frac{1}{2}} = \Bigg\{ \frac{1}{4\pi^2 b^2 (a^2+b^2)^2} \times \sum_{j=1}^{m} \sum_{p_j n + q_j l = 0} \int_0^{2\pi}\int_0^{2\pi} \hat{f}_j^2(\psi, A_1, A_2, \phi) \times e^{-i(n\psi + l\phi)} B(\psi)\, d\phi\, d\psi \Bigg\}^{\frac{1}{2}},$$

$$B(\psi) = (B_{ij}(\psi), i, j = 1, 2), \quad B_{11}(\psi) = (a\sin\psi + b\cos\psi)^2,$$

$$B_{12}(\psi) = B_{21}(\psi) = -(a\sin\psi + b\cos\psi)(a\cos\psi - b\sin\psi),$$

$$B_{22}(\psi) = (a\cos\psi - b\sin\psi)^2,$$

$$\hat{f}_j(\psi, A_1, A_2, \phi) = \tilde{f}_j(\psi, 0, A_1, A_2, \phi), \ j = \overline{0, m}$$

$$\hat{f}_{m+1}(\psi, A_1, A_2, \phi, z) = \tilde{f}_{m+1}(\psi, 0, A_1, A_2, \phi, z),$$

$\bar{w}(t) = (\bar{w}_i(t), i = 1, 2)$, $\bar{w}_i(t), i = 1, 2$ *are independent one-dimensional Wiener processes.*

2. If $k < k_0$ then in the averaging equation (4) we must put $\hat{f}_0 \equiv 0$; if $k < 2k_j$ for some $1 \leqslant j \leqslant m$, then in the averaging equation (4) we must put $\hat{f}_j \equiv 0$ for all such j; if $k < k_{m+1}$ then in the averaging equation (4) we must put $\hat{f}_{m+1} \equiv 0$.

3. If $\mu_j \neq \frac{p_j}{q_j} b$ for some $j = \overline{0, m+1}$ and any relatively prime integers p_j and q_j, then in averaging coefficients in (4) we must put $l = n = 0$ in corresponding sums containing $\hat{f}_j$.

Proof is the similar to the proof of the Theorem 3.1 in [10].

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Received: 10.11.2013