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Подвійна еквівалентність функцій Морса та градієнтно-подібних векторних полів на тривимірних многовидах

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Double equivalence of Morse functions and gradient-like vector fields on 3-manifolds

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Даються необхідні та достатні умови подвійної еквівалентності пари, що складається з функції Морса та градієнтно-подібного поля на тривимірному многовиді, тобто існування гомеоморфізму, що є одночасно топологічною еквівалентністю функцій і еквівалентністю полів. Будується повний топологічний інваріант пари.

Ключові слова: функція Морса, топологічна еквівалентність, градієнтно-подібне поле, діаграма Хегора.

Necessary and sufficient conditions for the double equivalence of pair consisting of a Morse function and a gradient-like field on the 3-manifold, i.e. existence of a homeomorphism, which is also the topological equivalence of functions and equivalence of fields are given in this article. A complete topological invariant of the pair is constructed using generalized ordered Heegaard diagram. The existence of a Morse function with given generalized ordered Heegaard diagram was investigated. Topological classification of MS-pairs, which cosist of a Morse function and gradient-like vector field for it, on closed oriented 3-manifolds was obtained. It was demonstated on some examples.

Key words: Morse function, topological equiva-lence, gradient-like field, Heegaard diagram.

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Let M be a smooth manifold, $f : M \to R$ is smoosh function. A vector field X is gradient-like if the the set of singular points of X coinside with the set of critical point of *f*. In this case we call pair (*f*, X) as gradient-like pair.

Two vector fields X, Y are called topologically equivalent if there is a homeomorphism $h : M \to M$, which maps the trajectory of X in the trajectory of Y, keeping their orientation.

In different ways topological classification of Morse - Smale vector fields on surfaces was obtained by E.A.Leontovych and A.H.Mayer, M.Paiksoto, V.V.Sharko, H.Vonh, H.Flaitas, E.Hiryk and others.

Topological classification of Morse-Smale vector fields on 3-manifolds without closed orbits was built by H.Fleytas [1], Ya.L.Umansky [2] and O.O.Pryshlyak [3].

The functions $f, g : M \to R$ are called topologically equivalent if there are homeomorphisms $h : M \to M$, $h' : R \to R$, for which $f \circ h = h' \circ g$ and homeomorphism h' preserves orientation of the line.

Global topological classification of Morse functions on surfaces and 1-connected manifolds of dimension greater than 5 was obtained by V.V.Sharko and A.T.Fomenko. Topological classification of Morse functions on closed three-dimensional manifolds constructed in [3].

Definition 1. A pair (*f*, X) is called *MS-pair* if *f* is a Morse function and X is gradient-like vector field for *f*.

MS-pairs (*f*, X) (g, Y) are called *double equivalent* if there exist a homeomorphism $h : M \to M$, $h' : R \to R$ for which $f \circ h = h' \circ g$ and in addition homeomorphism h maps the trajectory of X on the trajectory of Y.

The main aim of this paper is to obtain the classification of MS-pair on closed 3-manifolds with respect to double equivalence.

To solve this problem we apply the method of

Heegaard diagrams.

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1 MS-invariant

Let $H \bigcup H' = M$ be a Heegaard decomposition of 3-manifold M, $F = \partial H = \partial H'$ be the Heegaard surface.

Definition 2. A set $u = u_1, u_2, ..., u_n$ of closed curves on the surface F which do not intersect each other called a generalized system of meridians of H, if they restrict disks $D_i \subset H$, after cutting which wrom H we obtain union of three-dimensional disks.

Let $v = v_1, v_2, ..., v_n$ other generalized system of meridians pretzel H'.

Generalized Heegaard diagram (GHD) of the manifold M is a triple (F, u, v), consisting of closed surface and two generalized system of meridians

Diagrams (F, u, v) and (F', u', v') called homeomorphic if there exists a homeomorphism $h : F \to F'$, that $h(u) = u'$, $h(v) = v'$.

Let $U_1, U_2, ..., U_k$ are the areas on which the meridians $u_1, u_2, ..., u_k$ break surface F, and through $V_1, V_2, ..., V_l$ — those areas for meridians $v_1, v_2, ..., v_l$.

Definition 3. An *ordering* of the diagram is a map $\sigma : \{U_1, ..., U_k, u_1..., U_n, v_1, ..., v_m, V_1, ..., V_l\} \to \{1, 2, ..., N\}$. The diagram with odering is called *ordered* diagram.

For MS-function we construct generalized ordered Heegaard diagram (GOHD). Heegaard surface F is the boundary of the union of handles index 0 and 1.

Meridians are the intersections of the surface F with unstable manifolds of critical points of index 1 and the intersections of the surface F with stable manifolds of critical points of index 2. Let y_1, ..., y_N are the critical values of functions that are organized by increase ($y_1 < y_2 < ... < y_N$). We set σ $(y_i) = i$. Each critical point of index 0 corresponds to the area U_i, the point index 1corresponds to meridian u_i, point of index 2 corresponds to meridian v_i and the point of index 3 corresponds to region V_i. Thus, the performance mapping σ the set $\{U_1, U_2, ..., U_k, u_1, u_2, ..., u_n, v_1, v_2, ..., v_m, V_1, V_2, ..., V_l$ } into the set { 1, 2, ..., N }. Thus, each MC function GOHD sets. This GOHD we call GOHD that generated this MS function.

Two GOHD are called equivalent if there is a homeomorphism of GHD that keeps ordering.

Theorem 1. *Gradient-like pairs* (f, X), (g, Y) *are double equivalent if and only if their GOHD are equivalent.*

Proof. Nessersaty follows from the construction.

Sufficiency. Due to the fact that we can change the function by homeomorphisms of the line, without limiting the generality, we assume that the critical value functions coincide with the function σ. Homeomorphism of the chart defines a bijection between the trajectories of field gradients. Equality of values of functions at each homeomorphism between the sets of trajectories trajectories that collectively define the desired 3-manifold homeomorphism.

2 The implementation of invariant by MS-pair

Let M is a n-manifold with boundary ∂M.

Definition 4. n-disk H is called *handle of index* λ (or λ - handle) if there is a homeomorphism $\varphi : D^\lambda \times D^{n-\lambda} \to H$ such that $\varphi(\partial D^\lambda \times D^{n-\lambda}) = H \bigcap M \subset \partial M$. φ is called *characteristic* map and restriction $\psi = \varphi|_{\partial D^\lambda \times D^{n-\lambda}}$ *attaching map.* We say that the manifold $M' = M \bigcup H = M \bigcup_\psi D^\lambda \times D^{n-\lambda}$ obtained from M attaching λ-handles and write $M' = M \bigcup_\psi H^{(\lambda)}$. The set $\varphi(D^\lambda \times 0)$ is called *core*, $\varphi(0 \times D^{n-\lambda})$ is *cocore*. $\varphi(\partial D^\lambda \times 0)$ is *a-sphere*, $\varphi(0 \times \partial D^{n-\lambda})$ is *b- sphere.*

Handle decomposition of manifold M is decomposition $M = H_0 \bigcup H_1 \bigcup \ldots \bigcup H_m$, where H_0 is n-disk and H_i is handle on $M_{i-1} = \bigcup_{j<i} H_j$. Two handle decompositions of manifolds M and M' are called isomorphic if there is a homeomorphism $M \to M'$, which maps handles to handles, cores and cocores in cores and cocores, respectively.

For MS-pair uniquely up to homeomorphism can be build a handle decomposition: 0-handle is a regular neighborhood of critical points of index 0. Then by induction k-handle is regular neighborhood of stable manifold of the critical point of index n in the complement to handels of smaller indexes. Heegaaard surface is the boundary union of handles of index 0 and 1. Meridians u is b-spheres of 1-handles and meridians v is a-spheres of 2-handles.

We describe some properties of the function σ. If $u_i \subset \partial U_j$ then attaching area of H_i^1 intersects the cocore of H_j^0. This means that the handle is H_i^1 is attached after H_j^0. Hence, $\sigma(u_i) > \sigma(U_j)$. Similarly,

1) if $u_i \subset \partial U_j$, then σ (u_i)> σ (U_j);
2) if $v_i \subset \partial V_j$, then σ (v_i) <σ (V_j);
3) if $u_i \bigcap v_j \neq \emptyset$, then σ (u_i) <σ (v_j);
4) if $U_i \bigcap v_j \neq \emptyset$, then σ (U_i) <σ (v_j);
5) if $u_i \bigcap V_j \neq \emptyset$, then σ (u_i) <σ (V_j);
6) if $U_i \bigcap V_j \neq \emptyset$, then σ (U_i) <σ (V_j).

Definition 5. A function σ, which satisfies properties 1) - 6) will be called *admissible*.

The order of each meridian of the first type less order of any meridian the second type.

Theorem 2. *For each admissible GOHD there is a MS-pair that generates this GOHD.*

Proof. GHD sets the handle decomposition of the manifolds as in [3]. Function σ defines the procedure for gluing handles. After gluing each handle will stick collar. As a result, we get handle decomposition with the collars. Compressing each handle in the (critical) point, we obtain a manifolds and a function on it that is defined by the projections collar to segment in sum with constants such that these functions were agreed at the edges of collars. Any MC-function, which is topologically equivalent to the constructed function is search, because the inverse construction leads to the desired GOHD.

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3 Examples

A) Suppose there is a meridian of the first type and a meridian of second type on Heegaard surface of genus one.

1) Suppose there is a point of intersection between the meridians. Then $\sigma(U) = 1$, $\sigma(u) = 2$, $\sigma(v) = 3$, $\sigma(V) = 4$.

Thus, there is only one (up to double equivalence) MS -pair with Heegaard surface of genus one and one point of intersection between.

2) Similar to 1, there is one MS-pair with Heegaard surface of genus one and with three points of intersection between the meridians.

B) Suppose there are two meridians u_1, u_2 first type and one meridian v of the second type on the Heegaard surface of genus one. We consider the case of two intersection points between the meridians (ie v intersect each meridian u_1, u_2), then the function σ is one of:
1) $\sigma(\mathrm{U}_2) = 2, \sigma(u_1) = 3, \sigma(\mathrm{u}_2) = 3, \sigma(v) = 4$,
2) $\sigma(\mathrm{U}_2) = 2, \sigma(u_1) = 3, \sigma(\mathrm{u}_2) = 4, \sigma(v) = 5$,
3) $\sigma(\mathrm{U}_2) = 2, \sigma(u_1) = 4, \sigma(\mathrm{u}_2) = 3, \sigma(v) = 5$,
4) $\sigma(\mathrm{U}_2) = 1, \sigma(u_1) = 2, \sigma(\mathrm{u}_2) = 3, \sigma(v) = 4$,
5) $\sigma(\mathrm{U}_2) = 1, \sigma(u_1) = 2, \sigma(\mathrm{u}_2) = 2, \sigma(v) = 3$.
$\sigma(\mathrm{U}_1) = 1, \sigma(V) = \sigma(v) + 1$.

Thus, the number of double non-equivalent MS-pair with genus 1 and two points of intersection between the meridians is **5**.

Summary

Topological classification of MS-pairs on closed oriented 3-manifolds was obtained. It was demonstated on some examples. The author hopes that this result will be generelized to functions and vector fields with isolated critical points.

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