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В.Б. Трошкі[1], аспірант

Ідеальне шумопоглинання для сигналів із φ-субгауссовими шумами

[1]Ужгородський національний університет, математичний факультет, 88016, Ужгород, вул. Університетська 14, e-mail: btroshki@ukr.net

V.B. Troshki[1], postgraduate

Ideal de-noising for signals in φ-sub-Gaussian noise

[1] Uzhhorod National University, Faculty of Mathematics, 88016, Uzhhorod, University str., 14, e-mail: btroshki@ukr.net

Донаго та Джонстоун розглядали ідеальне шумопоглинання для сигналів із гауссовими шумами, а Себастіан Феррандо та Рандал Пайк отримали результати для сигналів з строго субгауссовими перешкодами. Основною метою даної статті є – розширити класи векторів перешкод. Тут в якості шумів розглянуті випадкові величини з простору $Sub_\varphi(\Omega)$. Отримані оцінки для сигналів із шумами хвости розподілів яких є "важчими"або "легшими"за гауссові.

Ключові слова: шумопоглинання, простір φ-субгауссових випадкових величин, розподіл Вейбула.

Donoho and Johnstone introduced an algorithm and supporting inequality that allows the selection of an orthonormal basis for optimal denoising. They considered ideal de-noising for signals in Gaussian noise and Sebastian E. Ferrando with Randall Pyke obtained the results for signals in strictly sub-Gaussian noise. The present paper concentrates in extending and improving this result, the main contribution is to incorporate a wider class of noise vectors. We consider signals with φ-sub-Gaussian noise. In particular, we show that the random variables which have centered Weibull distribution belongs to $Sub_\varphi(\Omega)$. Estimates for the signals in which the tails of the distribution of noise is "lighter"or "heavier"than Gaussian are established.

Keywords: Signal de-noising, φ-sub-Gaussian space, Weibull distribution.

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Introduction

The subject of this paper is oracle based denoising. Let s be a signal embedded in noise, we are interested in estimators $\hat{s}$, for the signal s, obtained by thresholding coefficients in an ortonormal basis $\mathcal{B}$ of $\mathbb{R}^n$. We consider the problem of optimal basis selection when there is available a library $\mathcal{L}$ of such bases from which to choose from. We will look for estimators which satisfy the following oracle-type inequality with high probability

$$\| \hat{s} - s \|_2^2 \leqslant c \min_{\mathcal{B}\in\mathcal{L}} \mathcal{R}(s,\mathcal{B}). \quad (1)$$

$\mathcal{R}(s,\mathcal{B})$ is the oracle risk for the basis $\mathcal{B}$, this last quantity is the average quadratic error incurred by an oracle estimator. This last estimator makes use of knowledge of s and is of excellent quality but unavailable in practice. After proper re-scaling, it can be argued that an inequality of the above type is asymptotically optimal as the oracle risk decays in a best possible manner. Therefore, this type of inequality gives an apriori measure for the quality of the algorithm associated to estimators satisfying (1).

Ideal de-noising for signals with Gaussian noise is considered in [3]. The results for strictly sub-Gaussian noise are obtained in the paper [4].

Ideal de-noising for signals with φ-sub-Gaussian noise is considered in this paper. The notion of the $Sub_\varphi(\Omega)$ space is introduced in the paper [6]. The definition and properties of random variables are studied in [5] (see also the book [1]).

The main point of the present article is to extend the results obtained in [3] and [4] to a wider class of noise vectors. Our results have been thus obtained that the algorithmic content of the original results have been preserved, in particular the thresholding parameters used are the ones used in [4]. We generalize Gaussian and sub-Gaussian hypothesis and we require that the noise vector satisfies a φ-sub-Gaussian hypothesis. Estimates of the risk of signal detection that occurred in the case of Gaussian noise and sub-Gaussian noise holds for φ-sub-Gaussian random variables are studied. That is why the results, which are obtained in this paper, can be used for wider class of noise. In particular the random variables which have centered Weibull distribution belongs to $Sub_\varphi(\Omega)$.

1 Basic definitions and some preliminary results from the theory of $Sub_\varphi(\Omega)$

The definition and properties of random variables are studied in [5] (see also the book [1]). We now provide a number of results that we will use extensively in this paper.

Definition 1. ([2]) Let $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ be a continuous even convex function. φ is called an Orlicz N-function if $\varphi(0) = 0$, $\varphi(x) > 0$ as $x \neq 0$ and the following conditions hold

$$\lim_{x\longrightarrow 0} \frac{\varphi(x)}{x} = 0, \quad (2)$$

$$\lim_{x\longrightarrow \infty} \frac{\varphi(x)}{x} = \infty. \quad (3)$$

Lemma 1. *([6], [2]) For any N-function φ the following statements hold:*

a) $\varphi(| x | + | y |) \geqslant \varphi(x) + \varphi$ as $x \in \mathbb{R}, 0 \leqslant \alpha \leqslant 1$;

b) $\varphi(x) = \int\limits_0^{|x|} p(t)dt$, where the density $p = \{p(t), t \geqslant 0\}$ is right continuous non-decreasing $p(0) = 0$ and $p(t) \to \infty$ as $t \to \infty$.

Definition 2. ([2], [5]) Let $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ be an N-function. The function φ^* defined by

$$\varphi^* = \sup_{y\in\mathbb{R}} \left(xy - \varphi(y)\right),$$

is called the Young-Fenchel transform of φ

Remark 1. ([5]) The Young-Fenchel transform of an N-function is an N-function as well.

Example 1. ([5]) If

$$\varphi(x) = \frac{x^p}{p}, 1 < p \leqslant 2, \quad (4)$$

then

$$\varphi^*(x) = \frac{x^q}{q},$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Example 2. Let

$$\varphi(x) = \begin{cases} \frac{x^2}{p}, & | x |\leqslant 1, \\ \frac{x^p}{p}, & | x |> 1 \end{cases} \quad \text{when } p > 2, \quad (5)$$

Find $\varphi^*(x)$. From Lemma 1 it follows that for all $x > 0$ $\varphi(x) = \int\limits_0^x p(u)du$, where

$$p(u) = \begin{cases} \frac{2u}{p}, & 0 \leqslant u \leqslant 1, \\ u^{p-1}, & u > 1. \end{cases}$$

Then

$$p^{(-1)}(u) = \begin{cases} \frac{p}{2}u, & 0 \leqslant x < 2/p, \\ 1, & 2/p \leqslant x \leqslant 1. \\ u^{1/(p-1)}, & u > 1. \end{cases}$$

From Lemma 1 and remark 1 follows that $\varphi^*(x) = \int\limits_0^x p^{(-1)}(u)du$.

Therefore

$$\varphi^*(x) = \begin{cases} \frac{p}{4}x^2, & 0 < x \leqslant \frac{2}{p}, \\ x - 1/p, & \frac{2}{p} \leqslant x \leqslant 1, \\ \frac{p-1}{p}x^{p/(p-1)}, & x > 1. \end{cases}$$

Lemma 2. *Let $\varphi_1(x)$ and $\varphi_2(x)$ be two N-function, where $\varphi_1(x) < \varphi_2(x)$. Then $\varphi_2^*(x) \leqslant \varphi_1^*(x)$.*

Proof. From the Young-Fenchel inequality it follows

$$\varphi_1^*(x) = \sup_{y>0}(xy - \varphi_1(y)) \geqslant$$
$$\geqslant \sup_{y>0}(xy - \varphi_2(y)) = \varphi_2^*(x).$$

Thus $\varphi_2^*(x) \leqslant \varphi_1^*(x)$. □

Condition Q([5]): An N-function φ satisfies Q if

$$\liminf_{x\longrightarrow 0} \frac{\varphi(x)}{x^2} = c > 0.$$

Let $\{\Omega, \mathbf{B}, \mathcal{P}\}$ be a standard probability space.

Definition 3. ([5]) Let φ be an N-function satisfying condition Q. The random variable ξ belongs to the space $Sub_\varphi(\Omega)$ if $\mathrm{E}\xi = 0, \mathrm{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$

$$\mathrm{E}\exp\{\lambda\xi\} \leqslant \exp\{\varphi(\lambda a)\}.$$

Consider now the following functional, defined on the space $Sub_\varphi(\Omega)$:

$$\tau_\varphi(\xi) = \inf(a \geqslant 0 : \mathrm{E}\exp\{\lambda\xi\} \leqslant \leqslant \exp\{\varphi(\lambda a)\}, \lambda \in \mathbb{R}).$$

Theorem 1. *([5]) The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm $\tau_\varphi(\cdot)$.*

Lemma 3. *([5]) Let $\xi \in Sub_\varphi(\Omega), \tau_\varphi(\xi) > 0, \varepsilon > 0$. The following inequality holds*

$$P\{|\ \xi\ |> \varepsilon\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}.$$

Theorem 2. *([5]) The random variable $\xi \in Sub_\varphi(\Omega)$ if and only if $E\xi = 0$ and there exist two constants $C > 0$ and $D > 0$ such that*

$$P\{|\ \xi\ |> x\} \leqslant C\exp\left\{-\varphi^*\left(\frac{x}{D}\right)\right\} \qquad (6)$$

for any $x > 0$. If (6) holds then $\tau_\varphi(\xi) \leqslant (1 + C)DS_\varphi e^{\frac{49}{48}}$, where $S_\varphi = \max\limits_{i=\overline{1,3}} \gamma_i^{-1}$, γ_1 the root of the equation $\gamma = \lambda_0\sqrt{c_0(1-\gamma)}$; γ_2 the root of the equation $\gamma^3 - 2(1-\gamma) = 0$; γ_3 the root of the equation $\gamma = \varphi^{(-1)}(2)\sqrt{c_0(1-\gamma)}$, where $\lambda_0 > 0$ be any number and $c_0 = \inf\limits_{0<|\lambda|\leqslant\lambda_0} \frac{\varphi(\lambda)}{\lambda^2}$.

2 Main results

Lemma 4. *Let $I(t) = \int\limits_0^\infty \exp\left\{tu - \frac{u^{q/2}}{q}\right\}du$, $q > 2$. Then*

$$I(t) \leqslant 2^{2/q}C_q \exp\left\{t^{\frac{q}{q-2}}C_{1/2,q}\right\},$$

where $C_q = \int\limits_0^\infty \exp\left\{-\frac{s^{q/2}}{q}\right\}ds$ and $C_{1/2,q} = 4^{2/(q-2)}(q-2)/q$.

Proof. From the Young-Fenchel inequality follows that for any $s > 0$, $v > 0$ holds

$$sv \leqslant \varphi(s) + \varphi^*(v). \qquad (7)$$

Consider

$$tu - \frac{u^{q/2}}{q} = tu - \frac{u^{q/2}}{2q} - \frac{u^{q/2}}{2q};$$

$$tu - \frac{u^{q/2}}{2q} = \frac{1}{4}\left(tu - \frac{u^{q/2}}{q/2}\right).$$

Let $\varphi(s) = \frac{s^{q/2}}{q/2}$, then $\varphi^*(s) = \frac{s^r}{r}$, where $\frac{1}{r} + \frac{1}{q/2} = 1$. It follows that $r = \frac{q}{q-2}$. Therefore $\varphi^*(s) = \frac{q-2}{q}s^{q/(q-2)}$. Then from (7) obtain

$$tu - \frac{1}{2}\frac{u^{q/2}}{q/2} \leqslant \frac{1}{4}(4t)^{\frac{q}{q-2}}\frac{q-2}{q} = t^{\frac{q}{q-2}}C_{\frac{1}{2},q}.$$

Hence,

$$I(t) \leqslant \exp\left\{t^{\frac{q}{q-2}}C_{\frac{1}{2},q}\right\}\int\limits_0^\infty \exp\left\{-\frac{u^{q/2}}{2q}\right\}du = = 2^{2/q}C_q\exp\left\{t^{\frac{q}{q-2}}C_{\frac{1}{2},q}\right\}.$$

□

Remark 2. If $q = 2$ then $Sub_\varphi(\Omega) = Sub(\Omega)$, thus this occasion does not regarded.

Lemma 5. *Let $\xi \in Sub_\varphi(\Omega)$ and $\varphi(x)$ be determined in (4), that is $\varphi^*(x) = \frac{x^q}{q}$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any t the following inequality holds*

$$\mathrm{E}\exp\left\{\frac{t\xi^2}{\tau^2}\right\} \leqslant 1 + t2^{\frac{q+2}{q}}C_q\exp\left\{t^{\frac{q}{q-2}}C_{\frac{1}{2},q}\right\}. \qquad (8)$$

Proof. Let $\varphi^*(x)$ be the Young-Fenchel transform of function $\varphi(x)$. Denote $\eta = \xi^2$, then $\eta \geqslant 0$. Let $F_\eta(x)$ be distribution function of random variable η. From Lemma 6 it follows that

$$1 - F_\eta(x) = P\{\eta > x\} = P\left\{\xi^2 > x\right\} = = P\left\{|\ \xi\ |> \sqrt{x}\right\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{\sqrt{x}}{\tau}\right)\right\}. \qquad (9)$$

Consider

$$\mathrm{E}\exp\left\{\frac{t\xi^2}{\tau^2}\right\} = \mathrm{E}\exp\left\{\frac{t\eta}{\tau^2}\right\} = = \int\limits_0^\infty \exp\{\tfrac{tu}{\tau^2}\}dF_\eta(u) = = -\int\limits_0^\infty \exp\left\{\frac{tu}{\tau^2}\right\}d(1 - F_\eta(u)).$$

Calculate this integral by parts, then

$$\mathrm{E}\exp\left\{\frac{t\xi^2}{\tau^2}\right\} = -\exp\left\{\frac{tu}{\tau^2}\right\}(1 - F_\eta(u))\ |_0^\infty + +\frac{t}{\tau^2}\int\limits_0^\infty (1 - F_\eta(u))\exp\left\{\frac{tu}{\tau^2}\right\}du.$$

The first summand in upper bound converting in origin if $u \to \infty$, since

$$(1 - F_\eta(u))\exp\left\{\frac{tu}{\tau^2}\right\} \leqslant 2\exp\left\{-\frac{u^{q/2}}{q\tau^q}\right\}\exp\left\{\frac{tu}{\tau^2}\right\} = \exp\left\{\frac{tu}{\tau^2} - \frac{u^{q/2}}{q\tau^q}\right\} \to 0,$$

and in lover bound converting in one, therefore that $1-F_\eta(0)=1$. Then from inequality (9) and Lemma 3 it follows that

$$\mathrm{E}\exp\left\{\frac{t\xi^2}{\tau^2}\right\}\leqslant 1+\frac{2t}{\tau^2}\int_0^\infty \exp\left\{\frac{tu}{\tau^2})\right\}\times \times\exp\left\{-\frac{u^{\frac{q}{2}}}{q\tau^q})\right\}du=1+ \frac{2t}{\tau^2}\int_0^\infty \exp\left\{\frac{tu}{\tau^2}-\frac{u^{\frac{q}{2}}}{q\tau^q}\right\}du= 1+2t\int_0^\infty \exp\left\{tu-\frac{u^{\frac{q}{2}}}{q}\right\}du\leqslant 1+t2^{(q+)/q}C_q\exp\left\{t^{q/(q-2)}C_{1/2,q}\right\}.$$

□

Lemma 6. *Let $\xi\in Sub_\varphi(\Omega)$ and $\varphi(x)$ be determined in (5), namely for $p>2$ (see Example 1.2)*

$$\varphi^*(x)=\begin{cases}\frac{p}{4}x^2, & 0<x\leqslant 2/p,\\ x-1/p, & 2/p\leqslant x\leqslant 1,\\ \frac{p-1}{p}x^{\frac{p}{p-1}}, & x>1.\end{cases}$$

Then if $t<1$ the following inequality holds

$$\mathrm{E}\exp\left\{\varphi^*\left(\frac{t\xi}{\tau}\right)\right\}\leqslant\frac{2}{1-t^q}. \qquad (10)$$

Proof. Let $F_\xi(x)$ be distribution function of random variable ξ and $\varphi^*(x)$ be the Young-Fenchel transform of function $\varphi(x)$. Since $\varphi^*(x)$ is even (it follows from remark 1) then

$$\mathrm{E}\exp\left\{\varphi^*\left(\frac{t|\xi|}{\tau}\right)\right\}=\mathrm{E}\exp\left\{\varphi^*\left(\frac{t\xi}{\tau}\right)\right\}= \int_0^\infty \exp\left\{\varphi^*\left(\frac{tu}{\tau}\right)\right\}d\left(F(u)\right).$$

Therefore

$$\begin{aligned}&\mathrm{E}\exp\left\{\varphi^*\left(\tfrac{t\xi}{\tau}\right)\right\}=\\ &=-\int_0^\infty \exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\left(1-F(u)\right)=\\ &=-\int_0^{2\tau/tp} \exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\left(1-F(u)\right)-\\ &-\int_{2\tau/tp}^{\tau/t} \exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\left(1-F(u)\right)-\\ &-\int_{\tau/t}^{\infty} \exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\left(1-F(u)\right)=\\ &=-\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}\Big|_0^{2\tau/tp}+\\ &+\int_0^{2\tau/tp}\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\varphi^*\left(\tfrac{tu}{\tau}\right)-\\ &-\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}\Big|_{2\tau/tp}^{\tau/t}+\\ &+\int_{2\tau/tp}^{\tau/t}\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\varphi^*\left(\tfrac{tu}{\tau}\right)-\\ &-\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}\Big|_{\tau/t}^{\infty}+\\ &+\int_{\tau/t}^{\infty}\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\varphi^*\left(\tfrac{tu}{\tau}\right)=\\ &=-\left(1-F\left(2\tau/tp\right)\right)\exp\left\{\varphi^*\left(2/p\right)\right\}+\\ &1-\left(1-F\left(\tau/t\right)\right)\exp\left\{\varphi^*\left(1\right)\right\}+\\ &+\left(1-F\left(2\tau/tp\right)\right)\exp\left\{\varphi^*\left(2/p\right)\right\}+\\ &\left(1-F\left(\tau/t\right)\right)\exp\left\{\varphi^*\left(1\right)\right\}+\end{aligned}$$

$$\begin{aligned}&+\int_0^{2\tau/tp}\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\varphi^*\left(\tfrac{tu}{\tau}\right)+\\ &+\int_{2\tau/tp}^{\frac{\tau}{t}}\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\varphi^*\left(\tfrac{tu}{\tau}\right)+\\ &+\int_{\tau/t}^{\infty}\left(1-F(u)\right)\exp\left\{\varphi^*\left(\tfrac{tu}{\tau}\right)\right\}d\varphi^*\left(\tfrac{tu}{\tau}\right).\end{aligned}$$

From Lemma 3 it follows that

$$\begin{aligned}&\mathrm{E}\exp\left\{\varphi^*\left(\tfrac{t\xi}{\tau}\right)\right\}\leqslant 2+2\int_0^{\frac{2\tau}{tp}}\exp\left\{-\tfrac{pu^2}{4\tau^2}\right\}\times\\ &\times\exp\left\{\tfrac{p(tu)^2}{4\tau^2}\right\}d\left(\tfrac{p(tu)^2}{4\tau^2}\right)+\\ &+2\int_{2\tau/tp}^{\tau/t}\exp\left\{-(\tfrac{u}{\tau}-\tfrac{1}{p})\right\}\exp\left\{\tfrac{tu}{\tau}-\tfrac{1}{p}\right\}d\left(\tfrac{tu}{\tau}\right)\\ &+4\int_{\tau/t}^{\infty}\exp\left\{-\tfrac{1}{q}\left(\tfrac{u}{\tau}\right)^q\right\}\exp\left\{\tfrac{1}{q}\left(\tfrac{tu}{\tau}\right)^q\right\}d\left(\tfrac{1}{q}\left(\tfrac{tu}{\tau}\right)^q\right)\end{aligned}$$

$$= \frac{2t^2}{t^2-1} \int\limits_0^{2\tau/tp} \exp\left\{\frac{pu^2}{4\tau^2}(t^2-1)\right\} d\left(\frac{pu^2}{4\tau^2}(t^2-1)\right)$$
$$+2 + \frac{2t}{t-1} \int\limits_{\frac{2\tau}{tp}}^{\frac{\tau}{t}} \exp\left\{\frac{u}{\tau}(t-1)\right\} d\left(\frac{u}{\tau}(t-1)\right) +$$
$$\frac{2t^q}{t^q-1} \int\limits_{\frac{\tau}{t}}^{\infty} \exp\left\{\frac{1}{q}\left(\frac{u}{\tau}\right)^q (t^q-1)\right\} \times$$
$$\times d\left(\frac{1}{q}\left(\frac{u}{\tau}\right)^q (t^q-1)\right) =$$
$$= 2 + \frac{2t^2}{t^2-1} \exp\{\frac{pu^2}{4\tau^2}(t^2-1)\} \Big|_0^{\frac{2\tau}{tp}} +$$
$$\frac{2t}{t-1} \exp\{\frac{u}{\tau}(t-1)\} \Big|_{\frac{2\tau}{tp}}^{\frac{\tau}{t}} + \frac{2t^q}{t^q-1}\times$$
$$\times \exp\left\{\frac{1}{q}\left(\frac{u}{\tau}\right)^q (t^q-1)\right\} \Big|_{\frac{\tau}{t}}^{\infty} =$$
$$= 2 + \frac{2t^2}{t^2-1}\left(\exp\left\{\frac{t^2-1}{t^2}\right\} - 1\right) +$$
$$\frac{2t}{t-1}\left(\exp\left\{\frac{t-1}{t}\right\} \exp\left\{\frac{2(t-1)}{pt}\right\}\right) -$$
$$-\frac{2t^q}{t^q-1} \exp\left\{\frac{t^q-1}{qt^q}\right\}.$$

Since $p > 2$ and $0 \leqslant t < 1$ thus

$$\exp\left\{\frac{t-1}{t}\right\} \leqslant \exp\left\{\frac{2(t-1)}{pt}\right\};$$
$$\exp\left\{1 - \frac{1}{t^2}\right\} \leqslant \exp\left\{\frac{1}{q}\left(1 - \frac{1}{t^q}\right)\right\};$$
$$\exp\left\{1 - \frac{1}{t^2}\right\} < 1;$$
$$\frac{t^q}{t^q-1} \leqslant \frac{t^2}{t^2-1}.$$

From these inequalities it follows that

$$\mathrm{E}\exp\left\{\varphi^*\left(\frac{t\xi}{\tau}\right)\right\} \leqslant 2 + \frac{2t^q}{t^q-1}\left(\exp\left\{1-\frac{1}{t^2}\right\} - 1\right)$$
$$-\frac{2t^q}{t^q-1}\exp\left\{\frac{1}{q}\left(1-\frac{1}{t^q}\right)\right\} \leqslant 2 - \frac{2t^q}{t^q-1} +$$
$$+\frac{2t^q}{t^q-1}\left(\exp\left\{1-\frac{1}{t^2}\right\} - \exp\left\{\frac{1}{q}\left(1-\frac{1}{t^q}\right)\right\}\right)$$
$$\leqslant 2\left(1 - \frac{t^q}{t^q-1}\right) = \frac{-2}{t^q-1} = \frac{2}{1-t^q}.$$

□

Let $\mathcal{L}$ be a library of orthonormal bases of $\mathbb{R}^n$, $\mathcal{M}_n$ the set of distinct vectors in $\mathcal{L}$. M_n will denote the cardinality of $\mathcal{M}_n$. If $\mathcal{B}$ is orthonormal bases of $\mathbb{R}^n$ then by $s_k[\mathcal{B}]$ denote the k^{th} coordinate of s in the basis $\mathcal{B}$.

Put

$$\delta_n(\lambda) = 4\lambda(1 + 2\log M_n)) \tag{11}$$

and

$$\Lambda_n = \Lambda_n(\lambda) = \tau^2\delta_n(\lambda), \lambda > 2$$

The data is given in the form

$$y = s + z \tag{12}$$

where s is the deterministic signal and z is a noise vector whose coordinates are assumed to be i.i.d. φ-sub-gaussian random variables.

Introduce the empirical entropy for defined best orthogonal basis as as in [3]

$$\varepsilon_\lambda(y, \mathcal{B}) = \sum_i \min(y_i^2[\mathcal{B}], \Lambda_n)$$

Let $\hat{\mathcal{B}}$ be the best orthogonal basis according to this entropy

$$\hat{\mathcal{B}} = \arg\min_{\mathcal{B}\in\mathcal{L}} \varepsilon_\lambda(y, \mathcal{B})$$

For determined thresholding z consider function what defined for all real c as follows

$$\eta_z(c) = c\mathcal{I}_{\{|c|>z\}},$$

where $\mathcal{I}_{\{|c|>z\}}$ is the characteristic function of the set $(-\infty; -z) \cup (z; \infty)$. Apply hard thresholding to obtain the empirical best estimate for signal s in the basis $\hat{\mathcal{B}}$

$$\hat{s}_i^*[\hat{\mathcal{B}}] = \eta_{\sqrt{\Lambda_n}}(y_i[\hat{\mathcal{B}}])$$

Consider the complexity functional as well as it was done in [4]

$$K(\tilde{s,} s) = \| \tilde{s} - s \|_2^2 + \Lambda_n \min_{\mathcal{B}\in\mathcal{L}} \sum_{\{i, \tilde{s}_i[\mathcal{B}]\neq 0\}} 1 =$$
$$= \| \tilde{s} - s \|_2^2 + \Lambda_n N_{\mathcal{L}}(\tilde{s})$$

where

$$N_{\mathcal{L}}(\tilde{s}) = \min_{\mathcal{B}} \#\{e_i \in \mathcal{B} : \tilde{s}_i[\mathcal{B}] \neq 0\}.$$

Let s^0 denote a signal of minimum theoretical complexity i.e.

$$K(s^0, s) = \min_{\tilde{s}} K(\tilde{s}, s)$$

.

Let $j_0 \equiv \max(N_{\mathcal{L}}(s^0), 1)$.

Using the result obtained above it can be shown that the estimate from [4]

$$\| \hat{s^* - s} \|_2^2 \leqslant \frac{2\lambda\delta_n(\lambda)}{\lambda - 2}\mathcal{R}^*(s, \mathcal{L}),$$

where

$$\mathcal{R}^*(s, \mathcal{L}) = \min_{\mathcal{B}} \sum_i \min\left(s_i^2[\mathcal{B}], \tau^2\right) \tag{13}$$

with a certain probability holds for φ-sub-gaussian random variables.

Theorem 3. *Let $\varphi(x)$ be defined as in (4) namely $\varphi^*(x) = \frac{x^q}{q}$, where q is such that equality $\frac{1}{p}+\frac{1}{q}=1$ holds. If the data is presented as $y = s + z$, where $z \in Sub_\varphi(\Omega)$ (see (12)) then when $t = 1$*

$$P\left\{\| \hat{s}^* - s \|_2^2 \geqslant \frac{2\lambda\delta_n(\lambda)}{\lambda-2}\mathcal{R}^*(s,\mathcal{L})\right\} \leqslant \leqslant \frac{\exp\{C-j_0\}}{M_n^{j0}} \quad (14)$$

where $\delta_n(\lambda) = 4\lambda(1 + 2\log M_n))$, $\mathcal{R}^*(s,\mathcal{L}) = \min_{\mathcal{B}} \sum_i \min\left(s_i^2\left[\mathcal{B}\right],\tau^2\right)$, $C = 1 + 2^{\frac{q+2}{q}} C_q \exp\left\{C_{\frac{1}{2},q}\right\}$, *constants* C_q *and* $C_{\frac{1}{2},q}$ *are defined in Lemma 4 .*

Proof. Let $C(j, M_n)$ denotes the collection of all subsets consisting of j vectors chosen from the M_n vectors of $\mathcal{M}_n$. Let $\hat{S} \in C(j, M_n)$ be subspace generated by the element of $C(j, M_n)$. Let P_S be an orthogonal projection on S.

Consider the set

$$A = \left\{\omega : \| P_S z(\omega) \|_2 \geqslant \frac{\sqrt{\Lambda_n}\sqrt{N_{\mathcal{L}}(s^0) + N_{\mathcal{L}}(\hat{s}^*)}}{2\sqrt{\lambda}}\right\}$$

and

$$B_j \equiv \left\{\omega : \sup_{\hat{S}\in C(j,M_n)} \| P_{\hat{S}} z(\omega) \|_2 \geqslant \frac{\sqrt{\Lambda_n}\sqrt{j}}{2\sqrt{\lambda}}\right\}$$

Then $A \subseteq \bigcup_{j=j_0}^{M_n} B_j$. Let $a = \frac{\sqrt{\Lambda_n}\sqrt{j}}{2\sqrt{\lambda}}$ and fixed subspace $\hat{S}_1 \in C(j, M_n)$ be of dimension d, $d \leqslant j$.

Consider the set

$$D_j = \{\omega : \| P_{\hat{S}_1} z(\omega) \|_2 \geqslant a\}$$

Let $\{e_1, e_2, ..., e_d\}$ be an orthonormal basis of $\hat{S}_1$. Extend $\{e_1, e_2, ..., e_d\}$ to an orthonormal basis $\mathbb{R}^n$

$$\varepsilon = \{e_1, e_2, ..., e_d, e_{d+1}, ..., e_n\}.$$

Then

$$z = \sum_{k=1}^{n} \langle z, e_k\rangle e_k$$

and

$$\xi = P_{\hat{S}_1} z = \sum_{k=1}^{d} \langle z, e_k\rangle e_k.$$

Evidently, $\xi_k[\varepsilon] = z_k[\varepsilon]$ when $k = \overline{1,d}$ and $\xi_k[\varepsilon] = 0$ when $k = \overline{d+1,n}$

From Chebyshev inequality and (10) and the fact that the random variables are independent we obtain

$$P(D_j) = P\{\| \xi \|_2 > a\} = P\left\{\exp\left\{\frac{t\|\xi\|_2^2}{\tau^2}\right\} > \exp\left\{\frac{ta^2}{\tau^2}\right\}\right\} \leqslant$$
$$\leqslant \frac{E\exp\{\frac{t}{\tau^2}\sum_{k=1}^{d}\xi_k^2\}}{\exp\left\{\frac{ta^2}{\tau^2}\right\}} = \frac{E\prod_{k=1}^{d}\exp\left\{\frac{t}{\tau^2}\xi_k^2\right\}}{\exp\left\{\frac{ta^2}{\tau^2}\right\}} \leqslant$$
$$\frac{E\prod_{k=1}^{j}\exp\left\{\frac{t}{\tau^2}\xi_k^2\right\}}{\exp\left\{\frac{ta^2}{\tau^2}\right\}} = \frac{\prod_{k=1}^{j}E\exp\left\{\frac{t}{\tau^2}\xi_k^2\right\}}{\exp\left\{\frac{ta^2}{\tau^2}\right\}} \leqslant$$
$$\leqslant \left(1 + t2^{\frac{q+2}{q}} C_q \exp\left\{t^{\frac{q}{q-2}} C_{\frac{1}{2},q}\right\}\right)^j \exp\left\{-\frac{ta^2}{\tau^2}\right\}$$

Denote $C(t) = 1 + t2^{\frac{q+2}{q}} C_q \exp\left\{t^{\frac{q}{q-2}} C_{\frac{1}{2},q}\right\}$. Then

$$P(D_j) \leqslant C^j(t) \exp\left\{-\frac{ta^2}{\tau^2}\right\} = = C^j(t)\exp\left\{-\frac{\tau^2 tj(1+2\log M_n)}{\tau^2}\right\} = = C^j(t)e^{-jt}M_n^{-2jt}$$

With this common bound, i.e. independent of the particular $\hat{S}_1$, we have

$$P(B_j) \leqslant \#C(j, M_n)C^j(t)M_n^{-2jt}e^{-jt} = \binom{M_n}{j} C^j(t)M_n^{-2jt}e^{-jt}$$

Therefore,

$$P(A) \leqslant \sum_{j=j_0}^{M_n} \frac{M_n^j}{j!} C^j(t)e^{-jt}M_n^{-2jt} \leqslant$$
$$\leqslant \sum_{j=1}^{M_n} \frac{C^j(t)}{j!} e^{-jt} M_n^{(1-2t)j} \leqslant$$
$$\leqslant M_n^{(1-2t)j_0} e^{-tj_0} \sum_{j=1}^{M_n} \frac{C^j(t)}{j!} \leqslant \frac{\exp\{C(t)-tj_0\}}{M_n^{(2t-1)j_0}}.$$

Choosing $t = 1$ we obtain required inequality. □

Theorem 4. *Let $\varphi(x)$ be defined as in (5) namely*

$$\varphi^*(x) = \begin{cases} \frac{p}{4}x^2, & 0 < x \leqslant \frac{2}{p}, \\ x - \frac{1}{p}, & \frac{2}{p} \leqslant x \leqslant 1, \\ \frac{p-1}{p}x^{\frac{p}{p-1}}, & x > 1. \end{cases}$$

data is presented as $y = s + z$ where $z \in Sub_\varphi(\Omega)$ (see (12)) and let $\delta_n(\lambda) = 4\lambda\left(q(1 + (2+\beta)\log M_n)\right)^{\frac{2}{q}} j_0^{1-\frac{2}{q}}$ then for any $\beta > 0$

$$P\left\{\| \hat{s}^* - s \|_2^2 \geqslant \frac{2\lambda\delta_n(\lambda)}{\lambda-2}\mathcal{R}^*(s,\mathcal{L})\right\} \leqslant \frac{e^2}{M_n^{j_0}}, \quad (15)$$

where $\mathcal{R}^*(s,\mathcal{L}) = \min_{\mathcal{B}} \sum_i \min\left(s_i^2\left[\mathcal{B}\right],\tau^2\right)$.

Proof. In the proof we will use the same notation and set as in the proof of Theorem 3.

Let $T(\lambda) = 4\lambda\left(q(1+(2+\beta)\log M_n)\right)^{2/q}$ and $\delta_n(\lambda) = T(\lambda)j^{1-2/q}$, where $1 \leqslant j_0 \leqslant j \leqslant M_n$. It follows from Chebyshev inequality that

$$P(D_j) = P\{\| \xi \|_2 > a\} =$$
$$= P\left\{\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\sqrt{\sum_{k=1}^{d}\xi_k^2}\right)^q\right\} > \exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}\right\}$$
$$\leqslant E\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\sqrt{\sum_{k=1}^{d}\xi_k^2}\right)^q\right\}\exp\left\{-\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}$$

Since $\varphi^*(x)$ is even and $\varphi^*(\sqrt{x})$ in this case is concave then

$$P(D_j) \leqslant \frac{E\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\right)^q\sum_{k=1}^{d}|\xi_k|^q\right\}}{\exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}} =$$
$$= \frac{E\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\right)^q\sum_{k=1}^{d}\xi_k^q\right\}}{\exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}} = \frac{E\prod_{k=1}^{d}\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\right)\xi_k^q\right\}}{\exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}}$$
$$\leqslant \frac{E\prod_{k=1}^{j}\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\right)^q\xi_k^q\right\}}{\exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}} = \frac{\prod_{k=1}^{j}E\exp\left\{\frac{1}{q}\left(\frac{t}{\tau}\right)^q\xi_k^q\right\}}{\exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}}$$

From (10) it follows that

$$P(D_j) \leqslant \frac{\prod_{k=1}^{j}\frac{2}{1-t^q}}{\exp\left\{\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}} =$$
$$= \left(\frac{2}{1-t^q}\right)^j\exp\left\{-\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}$$

Function $V(t) = \left(\frac{2}{1-t^q}\right)^j\exp\left\{-\frac{1}{q}\left(\frac{at}{\tau}\right)^q\right\}$ reaches its minimum at the point $\acute{t} = \left(1-\frac{q\tau^q j}{a^q}\right)$.

Taking into account the inequality $1+(2+\beta)\log M_n \leqslant M_n^\beta$ we obtain

$$P(D_j) \leqslant V(\acute{t}) = \left(\frac{2a^q}{qj\tau^q}\right)^j\exp\left\{-\frac{a^q}{\tau^q q}+j\right\} =$$
$$= 2^j\left(1+(2+\beta)\log M_n\right)^j \times$$
$$\times\exp\{-(2+\beta)j\log M_n\}$$
$$\leqslant 2^j M_n^{\beta j}M_n^{-(2+\beta)j} = 2^j M_n^{-2j}$$

Then for the common bound the following inequality holds

$$P(B_j) \leqslant \#C(j, M_n)2^j M_n^{-2j} = \begin{pmatrix} M_n \\ j \end{pmatrix} 2^j M_n^{-2j}$$

$$P(A) \leqslant \sum_{j=j_0}^{M_n}\frac{M_n^j}{j!}M_n^{-2j} \leqslant \sum_{j=1}^{M_n}\frac{2^j}{j!}M_n^{-j} \leqslant$$
$$\leqslant \frac{1}{M_n^{j_0}}\sum_{j=1}^{M_n}\frac{2^j}{j!} \leqslant \frac{e^2}{M_n^{j_0}}$$

Since $j_0 \leqslant j$ and $q < 2$ then $T(\lambda)j^{1-2/q} \leqslant T(\lambda)j_0^{1-2/q}$.

Therefore

$$\frac{e^2}{M_n^{j_0}} \geqslant P\left\{\| \hat{s}^* - s \|_2^2 \geqslant \frac{2\lambda T(\lambda)j^{1-\frac{2}{q}}}{\lambda-2}\mathcal{R}^*(s,\mathcal{L})\right\}$$
$$\geqslant P\left\{\| \hat{s}^* - s \|_2^2 \geqslant \frac{2\lambda T(\lambda)j_0^{1-\frac{2}{q}}}{\lambda-2}\mathcal{R}^*(s,\mathcal{L})\right\} =$$
$$= P\left\{\| \hat{s}^* - s \|_2^2 \geqslant \frac{2\lambda\delta_n(\lambda)}{\lambda-2}\mathcal{R}^*(s,\mathcal{L})\right\}.$$

□

Example 3. ([5]) The random variable ξ has centered Weibull distribution, namely

$$P\{\xi > x\} = \frac{1}{2}\exp\{-cx^\alpha\} \;\; x > 0$$

$$P\{\xi < x\} = \frac{1}{2}\exp\{-c \mid x \mid^\alpha\} \;\; x < 0$$

when $\alpha > 2$ belongs to the space $Sub_\varphi(\Omega)$ and $\tau_\varphi(\xi) \leqslant 2S_\varphi e^{\frac{49}{48}}$, where S_φ is defined in Theorem 2.

Example 4. Let $\alpha < 2$. For simplicity, consider the particular case of Weibull distribution and put $c = 1/\alpha$ then

$$P\{\mid \xi \mid > x\} = \exp\left\{-\frac{1}{\alpha}x^\alpha\right\}$$

Consider $Sub_\varphi(\Omega)$ where

$$\varphi(x) = \begin{cases} \frac{x^2}{p}, & \mid x \mid \leqslant 1, \\ \frac{x^p}{p}, & \mid x \mid > 1 \end{cases} \quad \text{when } p > 2, \quad (16)$$

Then

$$\varphi^*(x) = \begin{cases} \frac{p}{4}x^2, & 0 < x \leqslant \frac{2}{p}, \\ x - \frac{1}{p}, & \frac{2}{p} \leqslant x \leqslant 1, \\ \frac{p-1}{p}x^{\frac{p}{p-1}}, & x > 1, \end{cases}$$

Put $\frac{p}{p-1} = \alpha$ where $\frac{1}{\alpha} + \frac{1}{p} = 1$. Then $p = \frac{\alpha}{\alpha-1}$. Namely

$$\varphi^*(x) = \begin{cases} \frac{x^2}{4}\frac{\alpha}{\alpha-1}, & 0 < x \leqslant \frac{2}{\alpha}(\alpha-1), \\ x - \frac{\alpha-1}{\alpha}, & \frac{2}{\alpha}(\alpha-1) \leqslant x \leqslant 1, \\ \frac{1}{\alpha}x^\alpha, & x > 1, \end{cases}$$

From Lemma 2 it follows that $x^\alpha\frac{1}{\alpha} \geqslant \varphi^*(x)$. Then

$$P\{\mid \xi \mid > x\} = \exp\left\{-\frac{1}{\alpha}x^\alpha\right\} \leqslant \exp\{-\varphi^*(x)\}.$$

Consequently, $\xi \in Sub_\varphi(\Omega)$ and $\tau_\varphi(\xi) \leqslant 2S_\varphi e^{\frac{49}{48}}$, where S_φ is defined in Theorem 2.

3 Conclusions

Regardless the fact that in Theorem 3 due to complexity of calculations the smallest value of the probability could not be found exactly. However, the result for $j_0 \geqslant C$ is much better than the result obtained for Gaussian and strictly sub-Gaussian noise. This is natural, since the random variables that belong to the space $Sub_\varphi(\Omega)$ with the function $\varphi(x)$ which is defined in (4) have "lighter tails" of distribution than the random variables from the space $SSub(\Omega)$. Note also that the result of Theorem 4 is worse than the results obtained in Theorem 3 and for Gaussian and strictly sub-Gaussian noise. This is because the random variables from the space $Sub_\varphi(\Omega)$ where $\varphi(x)$ is defined in (5) have "heavier tails" of distributions. The importance of Theorem 4 is that either the results of Theorem 3, or the results from the paper [4] for these random variables cannot be used.

Список використаних джерел

1. *Buldygin V.V.* Metric characterization of random variables and random processes/Buldygin V.V., Kozachenko Yu.V. Amer. Math. Soc.— 2000.
2. *Krasnoselsky M. A.* Convex functions and Orlicz spaces./ Krasnoselsky M. A., Rutitsky Y. B. Noordhof, Groningen. — 1961.
3. *D.L. Donoho* Ideal denoising in an orthonormal basis chosen from a library of bases./Donoho D.L. , Johnstone I.M. C.R. Acad. Sci. Paris A,— 1994. — P. 1317-1322.
4. *Ferrando Sebastian E.* Ideal denoising for signals in sub-gaussian noise,/ Ferrando Sebastian E., Pyke Randall Applied and Computational Harmonic Analisis 2008. — V. 24. no. 1— P. 1-13.
5. *Guiliano Antonini R.* Spaces of φsubgaussian random variables./Guiliano Antonini R.,. Kozachenko Yu, and Nikitina T. Memorie di Mathematica e Applicazioni XXVII. — 2003. — no. 5. — P. 95-124.
6. *Kozachenko Yu.* Banach space of random variables of sub-Gaussian type./ Kozachenko Yu. and Ostrovskij I. E. Theory. Probab. Math. Statist — 1985. — V. 32.— P. 45–56.
7. *Krim H.,* On denoising and best signal representation./Krim, H., Tucker, D., Mallat, S., Donoho, D. IEEE Trans. Inform. Theory — V. 45. no. 7— PP. 2225-2238. 1999.

References

1. BULDYGIN, V., KOZACHENKO, YU. (2000) *Metric characterization of random variables and random processes* Amer. Math. Soc.
2. KRASNOSELSKY, M., RUTITSKY, Y. *Convex functions and Orlicz spaces.* Noordhof, Groningen. — 1961.
3. DONOHO, D. , JONSTONE, I.(1994) *Ideal denoising in an orthonormal basis chosen from a library of bases.* C.R. Acad. Sci. Paris A,— P. 1317-1322.
4. FERRANDO, S, PYKE, R (2008) *Ideal denoising for signals in sub-gaussian noise,* Applied and Computational Harmonic Analisis V. 24. no. 1— P. 1-13.
5. GUILIANO ANTONINI, R., KOZACHENKO, YU. and NIKITINA,T. (2003)*Spaces of φsubgaussian random variables.* Memorie di Mathematica e Applicazioni XXVII. — no. 5. — P. 95-124.
6. KOZACHENKO, YU. and OSTROVSKIJ, I.(1985) *Banach space of random variables of sub-Gaussian type.*Theory. Probab. Math. Statist — V. 32.— P. 45–56.
7. KRIM, H., TUCKER, D., MALLAT, S., DONOHO, D.(1999) *On denoising and best signal representation.* IEEE Trans. Inform. Theory — V. 45. no. 7— PP. 2225-2238.

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