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Про структуру множини усіх скінченно-породжуваних напівгруп спеціальних унітарних операторів у просторі $\mathbf{C}^2$.

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On the structure of the set of all finitely generated semigroups of special unitary operators in the space $\mathbf{C}^2$.

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У статті досліджено структуру множини усіх скінченно породжуваних комутативних напівгруп спеціальних унітарних операторів (термін «спеціальний» означає, що визначник матриці, яка визначає унітарний оператор, дорівнює одиниці) у 2-вимірному комплексному просторі. Охарактеризовано комутативні скінченно породжувані напівгрупи обертів сфери Блоха навколо координатних вісей. У деталях досліджено структуру множини усіх комутативних напівгруп, які породжуються двома елементами. Виділено деякі комутативні моноїди, які породжуються двома елементами.

Ключові слова: унітарні оператори, скінченно породжувані напівгрупи, комутативність.

In the given paper it is investigated the structure of the set of all finitely generated commutative semigroups of special unitary operators (the term «special» means that the determinant of a matrix that defines unitary operator equals to unit) in 2-dimensional complex space. Commutative finitely generated semigroups of rotation of the Bloch sphere around coordinate axes are characterized. Also it is investigated in detail the structure of the set of all commutative semigroups that are generated by two elements. Some commutative monoids generated by two elements are extracted.

Key Words: unitary operators, finitely generated semigroups, commutability.

Статтю представив професор Буй Д.Б.

Introduction

One of basic problem of quantum finite automata (QFA) theory is characteristic of languages accepted by this or the other model of QFA under these or the other restrictions on the model of QFA, as well as on the set of associated unitary operators.

It is well known that there exist different models of QFA intended to recognize languages in the given alphabet (the state of the art for the theory of finite QA is presented in [1]). One of basic restriction on the model of QFA is the number of measurements of its state. Basic models for QFA with measurement of a state at finite instant only are MO-1QFA, L-QFA k QFA $(k \geq 2)$ (see [2-4]) and L-k QFA $(k \geq 2)$ (the last model was introduced in [5] as generalization of the model L-QFA for multi-letter case). These four models in the case of 1-qubit QFA were investigated in [5] under assumption that associated unitary operators are rotations of the Bloch sphere [6] around the y-axe. One of basic restriction on the set of unitary operators associated with QFA is the property "to commute each with the others". Thus, the problem of investigation of the structure of the set of all finitely generated commutative semigroups of unitary operators is actual one for the theory of semigroups, as well as for QFA theory. Unfortunately, this problem is not resolved in general case. Some general characteristics of such semigroups were established in [7].

In the given paper it is investigated the structure of the set of all commutative finitely generated semigroups of special unitary operators $V:\mathbf{C}^2 \to \mathbf{C}^2$ (unitary operator V is special if and only if $\det(V)=1$).

Remark 1. Our restriction to consider special unitary operators $V:\mathbf{C}^2 \to \mathbf{C}^2$ is justified only by the factor that any unitary operator $U:\mathbf{C}^2 \to \mathbf{C}^2$ can be presented in the form $U = e^{i\delta}V$, where $\delta \in \mathbf{R}$ and $V:\mathbf{C}^2 \to \mathbf{C}^2$ is some special unitary operator.

This case is the simplest non-trivial one for unitary operators, and it is intended to characterize the structure of some important class of languages accepted by 1-qubit QFA.

1. Basic notions

Let V be the set of all special unitary operators $V:\mathbf{C}^2\to\mathbf{C}^2$, and S be the set of all finitely generated commutative semigroups $\mathsf{G}=(G,\cdot)$, such that $G\subseteq\mathsf{V}$. The semigrop generated by elements $V_1,\ldots,V_k\in\mathsf{V}$ $(k\in\mathbf{N})$ is denoted by $(\langle V_1,\ldots,V_k\rangle,\cdot)$. Without loss of generality in what follows it is suggested that for any semigroup $(\langle V_1,\ldots,V_k\rangle,\cdot)\in\mathsf{S}$ $(k\ge 2)$ the following condition holds

$$(\forall r_1,r_2\in\mathbf{N}_k)(r_1\ne r_2\Rightarrow(\forall n\in\mathbf{N})(V_{r_1}^n\ne V_{r_2}). \quad (1)$$

It is well known (see [6], for example) that any special unitary operator $V\in\mathsf{V}$ can be presented in the form

$$V=\begin{pmatrix} e^{i\alpha}\cos\frac{\gamma}{2} & -e^{-i\beta}\sin\frac{\gamma}{2} \\ e^{-i\beta}\sin\frac{\gamma}{2} & e^{-i\alpha}\cos\frac{\gamma}{2}\end{pmatrix}\quad(\alpha,\beta,\gamma\in\mathbf{R}). \quad (2)$$

Thus without loss of generality we can assume in what follows that V is the set of all unitary operators V defined by identity (2), such that $\alpha,\beta\in[0,2\pi)$ and $\gamma\in[0,4\pi)$.

Important special unitary operators $U:\mathbf{C}^2\to\mathbf{C}^2$ are $R_\gamma^{(1)}$, $R_\gamma^{(2)}$ and $R_\gamma^{(3)}$ are defined by formulae

$$R_\gamma^{(1)}=\begin{pmatrix}\cos 0.5\frac{\gamma}{2} & -i\sin 0.5\frac{\gamma}{2}\\ i\sin 0.5\frac{\gamma}{2} & \cos 0.5\frac{\gamma}{2}\end{pmatrix},$$

$$R_\gamma^{(2)}=\begin{pmatrix}\cos 0.5\frac{\gamma}{2} & -\sin 0.5\frac{\gamma}{2}\\ \sin 0.5\frac{\gamma}{2} & \cos 0.5\frac{\gamma}{2}\end{pmatrix},$$

$$R_\gamma^{(3)}=\begin{pmatrix}e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}.$$

Unitary operators $R_\gamma^{(1)}$, $R_\gamma^{(2)}$ and $R_\gamma^{(3)}$ are rotations of the Bloch sphere through the angle $\frac{\gamma}{2}$ around, correspondingly, the x-axe, the y-axe and the z-axe. It is well known that any special unitary operator $V:\mathbf{C}^2\to\mathbf{C}^2$ can be presented as superposition $V=R_{\gamma_1}^{(3)}R_{\gamma_2}^{(2)}R_{\gamma_3}^{(3)}$.

2. The structure of the set S

Let

$$\mathsf{S}_1=\{(\langle V\rangle,\cdot)\,|\,V\in\mathsf{V}\}.$$

It is evident that the following lemma holds.

Lemma 1. The set S_1 consists of all commutative cyclic semigroups.

Let

$$\mathsf{V}^{(l)}=\{R_\gamma^{(l)}\,|\,\gamma\in[0,4\pi)\}\quad(l=1,2,3).$$

Setting

$$\mathsf{S}_1^{(l)}=\{(\langle V\rangle,\cdot)\,|\,V\in\mathsf{V}^{(l)}\}\quad(l=1,2,3),$$

we extract in the set S_1 the sets of all commutative cyclic semigroups of rotation of the Bloch sphere through fixed angle around fixed coordinate axe.

Since for any $\gamma_1,\gamma_2\in[0,4\pi)$ holds identity

$$R_{\gamma_1}^{(l)}R_{\gamma_2}^{(l)}=R_{\gamma_2}^{(l)}R_{\gamma_1}^{(l)}=R_{(\gamma_1+\gamma_2)(\mathrm{mod}\,4\pi)}^{(l)}\quad(l=1,2,3), \quad (3)$$

then $(\langle R_\gamma^{(l)}\rangle,\cdot)\in\mathsf{S}_1^{(l)}$ $(\gamma\in[0,4\pi),l=1,2,3)$ is finite semigroup if and only if $\gamma(\mathrm{mod}\,\pi)\in\mathbf{Q}_+$.

Identity (3) implies that it can be extracted the following important subset of the set of all finitely generated commutative non-cyclic semigroups $(\langle V_1,\ldots,V_k\rangle,\cdot)\in\mathsf{S}$ $(k\ge 2)$.

For all $k\ge 2$ $(k\in\mathbf{N})$ we set

$$\mathsf{S}_{2l}^{(k)}=(\langle R_{\gamma_1}^{(l)},\ldots,R_{\gamma_k}^{(l)}\rangle,\cdot)\,|\,R_{\gamma_1}^{(l)},\ldots,R_{\gamma_k}^{(l)}\in\mathsf{V}^{(l)}\ \&$$
$$\&\ (\forall r_1,r_2\in\mathbf{N}_k)(\forall n\in\mathbf{N})(r_1\ne r_2\Rightarrow$$
$$\Rightarrow(R_{\gamma_{r_1}}^{(l)})^n\ne R_{\gamma_{r_2}}^{(l)})\}\quad(l=1,2,3).$$

It is evident that for any two fixed numbers $\gamma_{r_1},\gamma_{r_2}\in[0,4\pi)$ $(\gamma_{r_1}\ne\gamma_{r_2})$ and any integer $l\in\mathbf{N}_3$ identity $(R_{\gamma_1}^{(l)})^n=R_{\gamma_2}^{(l)}$ holds for some integer $n\in\mathbf{N}$ if and only if $n\gamma_{r_1}-\gamma_{r_2}\equiv 0(\mathrm{mod}\,4\pi)$.

Let

$$\mathsf{S}_{2l}=\bigcup_{k=2}^{\infty}\mathsf{S}_{2l}^{(k)}\quad(l=1,2,3),$$

and

$$\mathsf{S}_2=\bigcup_{l=1}^{3}\mathsf{S}_{2l}.$$

Thus, the following lemma holds.

Lemma 2. The set S_2 consists of all finitely generated commutative non-cyclic semigroups of rotation of the Bloch sphere around fixed coordinate axe.

It is evident that any commutative semigroup $(\langle R_{\gamma_1}^{(j)},\ldots,R_{\gamma_k}^{(j)}\rangle,\cdot)\in\mathsf{S}_{2j}^{(k)}$ (where $k\in\mathbf{N}$ $(k\ge 2)$ and $j=1,2,3$) is finite one if and only if $\gamma_r(\mathrm{mod}\,\pi)\in\mathbf{Q}$ for all integers $r\in\mathbf{N}_k$.

Now we investigate the structure of the set of all finitely generated non-cyclic commutative semigroups $(\langle V_1,\ldots,V_k\rangle,\cdot)\in\mathsf{S}$ $(k\ge 2)$.

To achieve our aim it is sufficient to investigate conditions under which commute two different special unitary operators

$$V_j=\begin{pmatrix} e^{i\alpha_j}\cos\frac{\gamma_j}{2} & -e^{-i\beta_j}\sin\frac{\gamma_j}{2} \\ e^{i\beta_j}\sin\frac{\gamma_j}{2} & e^{-i\alpha_j}\cos\frac{\gamma_j}{2}\end{pmatrix}\in\mathcal{V}\quad (j=1,2), \quad (4)$$

such that $\alpha_j,\beta_j\in[0,2\pi)\ \ (j=1,2)$ and $\gamma_j\in[0,4\pi)$ $(j=1,2)$.

Remark 2. We would set these or the others restrictions on the structure of special unitary operator V_1 and determine corresponding restrictions on the structure of special unitary operator V_2.

Formula (4) implies that

$$V_1V_2=V_2V_1\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}(e^{i(-\beta_1+\beta_2)}-e^{i(-\beta_2+\beta_1)})\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}=0\\ e^{i\beta_1}(e^{i\alpha_1}-e^{-i\alpha_1})\sin\frac{\gamma_2}{2}\cos\frac{\gamma_1}{2}=\\ \qquad =e^{i\beta_2}(e^{i\alpha_2}-e^{-i\alpha_2})\sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}(e^{i2(-\beta_1+\beta_2)}-1)\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}=0\\ e^{i\beta_1}\sin\alpha_1\sin\frac{\gamma_2}{2}\cos\frac{\gamma_1}{2}=\\ \qquad =e^{i\beta_2}\sin\alpha_2\sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}\end{cases}. \quad (5)$$

The first identity in (5) implies that we can analyze the following cases.

Case 1. Let

$$\sin\frac{\gamma_1}{2}=0\quad (\gamma_1\in[0,4\pi)).$$

Thus, $\gamma_1\in\{0,2\pi\}$.

Setting $\sin\frac{\gamma_1}{2}=0$ and $\cos\frac{\gamma_1}{2}=\pm1$ in (4) we get that $V_1\in\mathsf{V}_1(\alpha_1)$, where

$$\mathsf{V}_1(\alpha)=\{\widetilde{R}^{(3)}_\alpha,-\widetilde{R}^{(3)}_\alpha\}\quad (\alpha\in[0,2\pi)), \quad (6)$$

and

$$\widetilde{R}^{(3)}_\alpha=\begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{-i\alpha}\end{pmatrix}\quad (\alpha\in[0,2\pi)).$$

It is evident that special unitary operator $\widetilde{R}^{(3)}_\alpha$ $(\alpha\in[0,2\pi))$ is rotation of the Bloch sphere through the angle $-\alpha$ around the z-axe.

Thus, identity

$$\widetilde{R}^{(3)}_{\alpha_1}\widetilde{R}^{(3)}_{\alpha_2}=\widetilde{R}^{(3)}_{\alpha_2}\widetilde{R}^{(3)}_{\alpha_1}=\widetilde{R}^{(3)}_{(\alpha_1+\alpha_2)(\mathrm{mod}\,2\pi)}$$

holds for any $\alpha_1,\alpha_2\in[0,2\pi)$.

The second identity in (5) takes the form

$$\sin\alpha_1\sin\frac{\gamma_2}{2}=0\quad (\alpha_1\in[0,2\pi),\gamma_2\in[0,4\pi)). \quad (7)$$

The following cases can take the place.

Case 1.1. Let

$$\sin\frac{\gamma_2}{2}=0\quad (\gamma_2\in[0,4\pi)).$$

Thus, $\gamma_2\in\{0,2\pi\}$.

Setting $\sin\frac{\gamma_2}{2}=0$ and $\cos\frac{\gamma_2}{2}=\pm1$ in (4) we get that that $V_2\in\mathsf{V}_1(\alpha_2)$.

Let

$$\mathsf{S}_3=\bigcup_{k=2}^{\infty}\mathsf{S}_3^{(k)},$$

where

$$\mathsf{S}_3^{(k)}=(\langle V_1,\ldots,V_k\rangle,\cdot)\,|\,V_1,\ldots,V_k\in\bigcup_{\omega\in[0,2\pi)}\mathsf{V}_1(\omega)\ \&$$

$$\&\ (\forall r_1,r_2\in\mathbf{N}_k)(\forall n\in\mathbf{N})(r_1\neq r_2\Rightarrow V_{r_1}^n\neq V_{r_2})\}.$$

It is evident that for any two fixed numbers $\alpha_{r_1},\alpha_{r_2}\in[0,2\pi)$:

1) if $V_{r_1}=\widetilde{R}^{(3)}_{\alpha_{r_1}}$ and $V_{r_2}=\widetilde{R}^{(3)}_{\alpha_{r_2}}$, or $V_{r_1}=-\widetilde{R}^{(3)}_{\alpha_{r_1}}$ and $V_{r_2}=-\widetilde{R}^{(3)}_{\alpha_{r_2}}$ then identity $V_{r_1}^n=V_{r_2}$ holds for some integer $n\in\mathbf{N}$ if and only if relation $n\alpha_{r_1}-\alpha_{r_2}\equiv0\,(\mathrm{mod}\,2\pi)$ holds;

2) if $V_{r_1}=\widetilde{R}^{(3)}_{\alpha_{r_1}}$ and $V_{r_2}=-\widetilde{R}^{(3)}_{\alpha_{r_2}}$, then identity $V_{r_1}^n=V_{r_2}$ holds for some integer $n\in\mathbf{N}$ if and only if $\pi^{-1}(n\alpha_{r_1}-\alpha_{r_2})$ is odd integer;

3) if $V_{r_1}=-\widetilde{R}^{(3)}_{\alpha_{r_1}}$ and $V_{r_2}=\widetilde{R}^{(3)}_{\alpha_{r_2}}$, then identity $V_{r_1}^n=V_{r_2}$ holds for some integer $n\in\mathbf{N}$ if and only if either n and $\pi^{-1}(n\alpha_{r_1}-\alpha_{r_2})$ are odd integers, or n is even integer and relation $n\alpha_{r_1}-\alpha_{r_2}\equiv0\,(\mathrm{mod}\,2\pi)$ holds.

Thus, the following lemma holds.

Lemma 3. The set S_3 consists of finitely generated non-cyclic commutative semigroups.

It is evident that it holds inclusion

$$\mathsf{S}_3\subset\mathsf{S}_{23}^{(2)}.$$

Case 1.2. Let

$$\sin\frac{\gamma_2}{2}\neq0\quad (\gamma_2\in[0,4\pi)).$$

Thus, $\gamma_2\in[0,4\pi)\setminus\{0,2\pi\}$.

Identity (7) takes the form

$$\sin\alpha_1=0\quad (\alpha_1\in[0,2\pi)).$$

Thus, $\alpha_1 \in \{0, \pi\}$ and identity (6) takes the form $V_1 \in \{I, -I\}$, where I is the unit 2×2 -matrix.

Let

$$V_2 = \{I, -I\}$$

and V_3 be the set of all special unitary operators $V \in V$ defined by formula (1), such that $\gamma \in [0,4\pi) \setminus \{0,2\pi\}$ and $V^n \notin V_2$ for all $n \in \mathbf{N}$.

We set

$$S_4 = \{(\langle V_1, V_2\rangle, \cdot)\} \mid V_1 \in V_2, V_2 \in V_3\}.$$

Thus, the following lemma holds.

Lemma 4. The set S_4 consists of finitely generated non-cyclic commutative semigroups.

Remark 3. It is evident that if in the case 1 we set $\sin\frac{\gamma_2}{2} = 0 \quad (\gamma_2 \in [0,4\pi))$ instead of $\sin\frac{\gamma_1}{2} = 0$ $(\gamma_1 \in [0,4\pi))$, we get the same lemmas 2,3 and 4.

Case 2. Let

$$\begin{cases} \sin\frac{\gamma_1}{2} \neq 0 \ (\gamma_1 \in [0,4\pi)) \\ \sin\frac{\gamma_2}{2} \neq 0 \ (\gamma_2 \in [0,4\pi)) \end{cases}. \tag{8}$$

Thus, $\gamma_j \in [0,4\pi) \setminus \{0,2\pi\} \quad (j = 1,2)$.

The first identity in (5) takes the form

$$e^{i2(-\beta_1+\beta_2)} = 1 \quad (\beta_1, \beta_2 \in [0,2\pi)). \tag{9}$$

Without loss of generality we can assume that it holds inequality $\beta_1 \le \beta_2$ (since if $\beta_2 < \beta_1$ we can set $V_1 := V_2$ and $V_2 := V_1$).

Since $\beta_j \in [0,2\pi)$ and $\beta_1 \le \beta_2$ then identity (9) implies that either $\beta_2 = \beta_1$, or $\beta_2 = \beta_1 + \pi$.

Formula (4) implies that:

1) if $\beta_2 = \beta_1$ then

$$V_j = \begin{pmatrix} e^{i\alpha_j}\cos\frac{\gamma_j}{2} & -e^{-i\beta_1}\sin\frac{\gamma_j}{2} \\ e^{i\beta_1}\sin\frac{\gamma_j}{2} & e^{-i\alpha_j}\cos\frac{\gamma_j}{2} \end{pmatrix} \quad (j = 1,2); \tag{10}$$

2) if $\beta_2 = \beta_1 + \pi$ then

$$V_1 = \begin{pmatrix} e^{i\alpha_1}\cos\frac{\gamma_1}{2} & -e^{-i\beta_1}\sin\frac{\gamma_1}{2} \\ e^{i\beta_1}\sin\frac{\gamma_1}{2} & e^{-i\alpha_1}\cos\frac{\gamma_1}{2} \end{pmatrix}, \tag{11}$$

$$V_2 = \begin{pmatrix} e^{i\alpha_2}\cos\frac{\gamma_2}{2} & e^{-i\beta_1}\sin\frac{\gamma_2}{2} \\ -e^{i\beta_1}\sin\frac{\gamma_2}{2} & e^{-i\alpha_2}\cos\frac{\gamma_2}{2} \end{pmatrix}. \tag{12}$$

The second identity in (5) takes the form

$$\sin\alpha_1 \sin\frac{\gamma_2}{2}\cos\frac{\gamma_1}{2} = \pm\sin\alpha_2 \sin\frac{\gamma_1}{2}\cos\frac{\gamma_2}{2}, \tag{13}$$

where $\gamma_1, \gamma_2 \in [0,4\pi) \setminus \{0,2\pi\}$.

The following cases can take the place.

Case 2.1. Let

$$\cos\frac{\gamma_1}{2} = 0 \quad (\gamma_1 \in [0,4\pi) \setminus \{0,2\pi\}).$$

Thus, $\gamma_1 \in \{\pi, 3\pi\}$.

Since $\cos\frac{\gamma_1}{2} = 0$ and $\sin\frac{\gamma_1}{2} = \pm 1$ then formulae (10) and (11) imply that $V_1 \in V_4(\beta_1)$, where the set $V_4(\beta) \ (\beta \in [0,2\pi))$ is defined by identity

$$V_4(\beta) = \{J_\beta, -J_\beta\} \quad (\beta \in [0,2\pi)), \tag{14}$$

and

$$J_\beta = \begin{pmatrix} 0 & -e^{-i\beta} \\ e^{i\beta} & 0 \end{pmatrix} \quad (\beta \in [0,2\pi)).$$

Identity (13) takes the form

$$\sin\alpha_2 \cos\frac{\gamma_2}{2} = 0, \tag{15}$$

where $\gamma_2 \in [0,4\pi) \setminus \{0,2\pi\}$ and $\alpha_2 \in [0,2\pi)$.

The following cases can take the place.

Case 2.1.1. Let

$$\cos\frac{\gamma_2}{2} = 0 \quad (\gamma_2 \in [0,4\pi) \setminus \{0,2\pi\}).$$

Thus, $\gamma_2 \in \{\pi, 3\pi\}$.

Since $\cos\frac{\gamma_2}{2} = 0$ and $\sin\frac{\gamma_2}{2} = \pm 1$, we get that $V_2 \in V_4(\beta_1)$.

Disequality $V_2 \neq V_1$ and formulae (14) imply that it holds identity $V_2 = -V_1$, where $V_1 \in V_4(\beta_1)$.

Let

$$S_4 = \{(\langle V, -V\rangle, \cdot) \mid V \in \bigcup_{\beta\in[0,2\pi)} V_4(\beta)\}.$$

Since for any number $\beta \in [0,2\pi)$ holds identity $J_\beta^2 = -I \quad (\beta \in [0,2\pi))$, the following lemma holds.

Lemma 5. The set S_4 consists of finite finitely generated non-cyclic commutative semigroups.

Case 2.1.2. Let $\cos\frac{\gamma_2}{2} \neq 0 \quad (\gamma_2 \in [0,4\pi))$.

Thus, $\gamma_2 \in [0,4\pi) \setminus \{0, \pi, 2\pi, 3\pi\}$.

Identity (15) takes the form

$$\sin\alpha_2 = 0 \quad (\alpha_2 \in [0,2\pi)).$$

Thus, $\alpha_2 \in \{0, \pi\}$.

Since $e^{i\alpha_2} = \pm 1$ then identities (10) and (11) imply that $V_2 \in V_5(\beta_1)$, where the set $V_5(\beta)$ $(\beta \in [0,2\pi))$ is determined by identity

$$\mathsf{V}_5(\beta) = \bigcup_{\gamma\in[0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}} \mathsf{V}_5(\gamma,\beta) \quad (\beta\in[0,2\pi)),$$

where

$$\mathsf{V}_5(\gamma,\beta) = \{U_j(\gamma,\beta) \mid j=1,\ldots,4\},$$

and

$$U_1(\gamma,\beta) = \begin{pmatrix} \cos\frac{\gamma}{2} & -e^{-i\beta}\sin\frac{\gamma}{2} \\ e^{i\beta}\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix},$$

$$U_2(\gamma,\beta) = \begin{pmatrix} \cos\frac{\gamma}{2} & e^{-i\beta}\sin\frac{\gamma}{2} \\ -e^{i\beta}\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix},$$

$$U_3(\gamma,\beta) = -U_2(\gamma,\beta),$$

$$U_4(\gamma,\beta) = -U_1(\gamma,\beta).$$

It is evident that:

1) if $\beta = 0$ then $U_1(\gamma,\beta)$ is rotation of the Bloch sphere through the angle $\frac{\gamma}{2}$ around the y-axe;

2) if $\beta = \frac{3\pi}{2}$ then $U_2(\gamma,\beta)$ is rotation of the Bloch sphere through the angle $\frac{\gamma}{2}$ around the x-axe.

Let

$$\mathsf{S}_5 = \bigcup_{\beta\in[0,2\pi)} \bigcup_{\gamma\in[0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}} \mathsf{S}_5(\gamma,\beta),$$

where

$$\mathsf{S}_5(\gamma,\beta) = \{\langle V_1, V_2\rangle \mid V_1 \in \mathsf{V}_4(\beta)\ \& \\ \& V_2 \in \mathsf{V}_5(\gamma,\beta)\ \&\ (\forall n\in\mathbf{N})(V_2^n \neq V_1)\}.$$

Thus, the following lemma holds.

Lemma 6. The set S_5 consists of finitely generated non-cyclic commutative semigroups.

Remark 4. It is evident that if in the case 2.1 we set $\cos\frac{\gamma_2}{2} = 0 \quad (\gamma_2 \in [0,4\pi)\setminus\{0,2\pi\})$ instead of $\cos\frac{\gamma_1}{2} = 0 \quad (\gamma_1 \in [0,4\pi)\setminus\{0,2\pi\})$, we get the same lemmas 5 and 6.

Case 2.2. Let

$$\begin{cases} \cos\frac{\gamma_1}{2} \neq 0 \ \ (\gamma_1 \in [0,4\pi)) \\ \cos\frac{\gamma_2}{2} \neq 0 \ \ (\gamma_2 \in [0,4\pi)) \end{cases}. \tag{16}$$

Thus, we get that $\gamma_1, \gamma_2 \in [0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}$.

The following cases can take the place.

Case 2.2.1. Let

$$\sin\alpha_1 = 0 \quad (\alpha_1 \in [0,2\pi)).$$

Thus, we get that $\alpha_1 \in \{0,\pi\}$.

Identity (13) takes the form

$$\sin\alpha_2 = 0 \quad (\alpha_2 \in [0,2\pi)).$$

Thus, we get that $\alpha_2 \in \{0,\pi\}$.

Formulae (10)-(12) imply that:

1) if $\beta_2 = \beta_1$ then

$$V_j = \begin{pmatrix} \pm\cos\frac{\gamma_j}{2} & -e^{-i\beta_1}\sin\frac{\gamma_j}{2} \\ e^{i\beta_1}\sin\frac{\gamma_j}{2} & \pm\cos\frac{\gamma_j}{2} \end{pmatrix} \quad (j=1,2); \tag{17}$$

2) if $\beta_2 = \beta_1 + \pi$ then

$$V_1 = \begin{pmatrix} \pm\cos\frac{\gamma_1}{2} & -e^{-i\beta_1}\sin\frac{\gamma_1}{2} \\ e^{i\beta_1}\sin\frac{\gamma_1}{2} & \pm\cos\frac{\gamma_1}{2} \end{pmatrix}, \tag{18}$$

$$V_2 = \begin{pmatrix} \pm\cos\frac{\gamma_2}{2} & e^{-i\beta_1}\sin\frac{\gamma_2}{2} \\ -e^{i\beta_1}\sin\frac{\gamma_2}{2} & \pm\cos\frac{\gamma_2}{2} \end{pmatrix}. \tag{19}$$

Formulae (17)-(19) imply that $V_2 \in \mathsf{V}_5(\gamma_2,\beta_1)$ and $V_1 \in \mathsf{V}_6(\gamma_1,\beta_1)$, where the set $\mathsf{V}_6(\gamma,\beta)$ is defined by identity

$$\mathsf{V}_6(\gamma,\beta) = \{U_1(\gamma,\beta), U_3(\gamma,\beta,)\}.$$

Let

$$\mathsf{S}_6 = \bigcup_{\beta\in[0,2\pi)} \mathsf{S}_6(\beta),$$

where

$$\mathsf{S}_6(\beta) = \bigcup_{\gamma_1,\gamma_2\in[0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}} \{\langle V_1, V_2\rangle \mid V_1 \in \mathsf{V}_6(\gamma_1,\beta)\ \& \\ \& V_2 \in \mathsf{V}_5(\gamma_2,\beta)\ \&\ (\forall n\in\mathbf{N})(V_1^n \neq V_2\ \&\ V_2^n \neq V_1)\},$$

Thus, the following lemma holds.

Lemma 7. The set S_6 consists of finitely generated non-cyclic commutative semigroups.

Remark 5. It is evident that if in the case 2.2.1 we set $\sin\alpha_2 = 0 \quad (\alpha_2 \in [0,2\pi))$ instead of setting $\sin\alpha_1 = 0 \quad (\alpha_1 \in [0,2\pi))$, we get the same lemma 7.

Case 2.2.2. Let

$$\begin{cases} \sin\alpha_1 \neq 0 \ \ (\alpha_1 \in [0,2\pi)) \\ \sin\alpha_2 \neq 0 \ \ (\alpha_2 \in [0,2\pi)) \end{cases}.$$

Thus, $\alpha_1, \alpha_2 \in [0,2\pi)\setminus\{0,\pi\}$.

Identity (13) takes the form

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \pm tg\frac{\gamma_1}{2}\, ctg\frac{\gamma_2}{2}, \tag{19}$$

under conditions that $\gamma_1, \gamma_2 \in [0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}$ and $\alpha_1, \alpha_2 \in [0,2\pi)\setminus\{0,\pi\}$.

Let $S'(\beta_1)$ be the set of all unordered pairs $\{V_1,V_2\}$ of special unitary operators, such that the following three conditions hold:

1) unitary operators V_j $(j=1,2)$ are defined by formula (10), where $\gamma_1,\gamma_2 \in [0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}$ and $\alpha_1,\alpha_2 \in [0,2\pi)\setminus\{0,\pi\}$;

2) identity $\dfrac{\sin\alpha_1}{\sin\alpha_2} = tg\dfrac{\gamma_1}{2}ctg\dfrac{\gamma_2}{2}$ holds;

3) disequalities $V_1^n \neq V_2$ $(n\in\mathbf{N})$ and $V_2^n \neq V_1$ $(n\in\mathbf{N})$ hold.

Similarly, let $S''(\beta_1)$ be the set of all ordered pairs (V_1,V_2) of special unitary operators, such that the following three conditions hold:

1) unitary operators V_1 is defined by formula (10) and unitary operators V_2 is defined by formula (12), under supposition that $\gamma_1,\gamma_2 \in [0,4\pi)\setminus\{0,\pi,2\pi,3\pi\}$ and $\alpha_1,\alpha_2 \in [0,2\pi)\setminus\{0,\pi\}$;

2) identity $\dfrac{\sin\alpha_1}{\sin\alpha_2} = -tg\dfrac{\gamma_1}{2}ctg\dfrac{\gamma_2}{2}$ holds;

3) disequalities $V_1^n \neq V_2$ $(n\in\mathbf{N})$ and $V_2^n \neq V_1$ $(n\in\mathbf{N})$ hold.

Let

$$\mathsf{S}_7 = \bigcup_{\beta_1\in[0,2\pi)}\{\langle V_1,V_2\rangle \mid \{V_1,V_2\}\in S'(\beta_1)\} \cup$$
$$\cup \bigcup_{\beta_1\in[0,2\pi)}\{\langle V_1,V_2\rangle \mid (V_1,V_2)\in S''(\beta_1)\}.$$

Thus, the following lemma holds.

Lemma 8. The set S_7 consists of finitely generated non-cyclic commutative semigroups.

Lemmas 1-8 and inclusion $\mathsf{S}_3 \subset \mathsf{S}_{23}^{(2)}$ imply that the following theorem holds.

Theorem. It holds inclusion $\mathsf{S} \supseteq \bigcup_{\substack{j=1\\ j\neq 3}}^{7}\mathsf{S}_j$.

Conclusions

In the given paper the structure of the set of all commutative finitely generated semigroups of special unitary operators $V:\mathbf{C}^2\to\mathbf{C}^2$ is investigated. Obtained results form strong base for investigation of subclasses of languages accepted by models of 1-qubit QFA with measurement of a state at finite instant only.

Solving the problem "$\mathsf{S} = \bigcup_{\substack{j=1\\ j\neq 3}}^{7}\mathsf{S}_j$?" forms some possible trend for future research. Analysis in detail the structure of the subset of all finite (also of all infinite) semigroups $\mathsf{G}\in\mathsf{S}$. Characterization the structure of the subset of all monoids $\mathsf{G}\in\mathsf{S}$, as well as of the set of all groups $\mathsf{G}\in\mathsf{S}$ determines the third trend for future research.

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