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Кореневі бази для несиметричних цілих білінійних форм

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Root bases for nonsymmetrical integer bilinear forms

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У даній роботі розглядається задача класифікації кореневих баз для несиметричної білінійної форми з додатно визначеною квадратичною формою. Ми використовуємо теорію диференціальних градуйованих категорій та процедуру їх приведення. Градуйований сагайдак допускає тривіальну структуру диференціальної градуйованої категорії. Коренева база градуйованого сагайдака типу Динкіна, яка приведена зі стандартної кореневої бази, визначає структуру диференціальної градуйованої категорії та відповідну несиметричну білінійну форму. В роботі ми показуємо, що якщо на категорії шляхів зв'язного градуйованого графа деякої несиметричної білінійної форми можна визначити структуру диференціальної градуйованої категорії, яка задовольняє очевидним умовам коректності, то стандартна база цієї форми є редукованою з деякого градуйованого сагайдака типу Динкіна. Доведення основної теореми використовує теорему з [4] для білінійних та квадратичних форм відносно їх коренів та кореневих баз.
Ключові слова: дійсний корінь, білінійна форма, диференціальна градуйована категорія, діаграма Динкіна.

This work concerns with the problem of classification of root bases for the nonsymmetrical bilinear forms with positive definite quadratic form. We use the theory of differential graded categories and corresponding nonsymmetrical bilinear forms, and reduction algorithm for them. A graded quiver has the trivial structure of differential category. Any root base of graded quiver of Dynkin type which is reduced from the standard root base, defines the structure of differential graded category and the corresponding nonsymmetrical bilinear form. In this paper we show, if the path category of connected graded graph of some nonsymmetrical bilinear form allows the structure of the differential graded category which satisfies the obvious conditions of correctness, then a standard basis of this bilinear form is reduced from the standard root base of some graded quiver of Dynkin type. In this work we apply the non classical but modified concept of root base to the classical theory of bilinear and quadratic forms and their applications. The proof of the main theorem uses theorem from [4] for the bilinear and quadratic forms with respect to their roots and root bases.
Key words: real root, bilinear form, differential graded category, Dynkin diagram
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1 Preliminaries

The reduction algorithm of linear categories and other structures is widely used in the representation theory. This approach allows to study representations inductively, reducing the corresponding categories step by step ([1]). On the other hand, the important characteristic of represented structure is the induced quadratic form whose roots correspond to the indecomposable representations. The theory of quadratic forms is well known ([2], [3]). We give the simultaneous reduction algorithm of transformation of the differential graded category with special properties and the underlined bilinear and quadratic form to the canonical forms.

Root systems are just one example among a large number of mathematical objects of "finite type" which are classified by (some class of) Dynkin diagrams. This approach allows to establish a connection between forms theory and representation theory. We use non classical but modified concept of root base. We prove that under certain finiteness and correctness conditions a root base is reduced to some classic one by certain transformations.

2 Differential graded categories and directed graded graphs

2.1 Graphs and forms

Let $\Gamma = (\Gamma_0, \Gamma_1, \deg)$ be a finite directed graded graph with vertex set Γ_0, the arrows set $\Gamma_1 \subset \Gamma_0 \times \Gamma_0$. The graph Γ is called *graded* (or $\mathbb{Z}$-graded) if there is the map $\deg : \Gamma_1 \to \mathbb{Z}$, such that $\Gamma_1^q = \bigsqcup_{\mathtt{i},\mathtt{j}\in\Gamma_0} \Gamma_1^q(\mathtt{i},\mathtt{j}) = \deg^{-1}(q)$, $\Gamma_1 = \bigsqcup_{q\in\mathbb{Z}} \Gamma_1^q$. The graph Γ is assumed to be finite which means that $|\Gamma_0| \leqslant \infty$ and all vector spaces $\Bbbk\Gamma_1(i,j)$ are finite dimensional.

Let $\Bbbk$ be an algebraically closed field. Given a finite graded directed graph $\Gamma = (\Gamma_0, \Gamma_1, \deg)$, we consider $\Bbbk\Gamma$ the $\Bbbk$-linear path category of the graded graph Γ which is freely generated over $\Bbbk$ by all the pathes on Γ. The category $\Bbbk\Gamma$ inherits the graduation from Γ in a natural way.

We denote by $|a| = \deg a \pmod 2$ the parity of $\deg a$. We regard Γ_1 as a union $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^1$ with Γ_1^0 be a set of arrows of even degree, and Γ_1^1 be a set of arrows of odd degree, the arrows of degree 0 (resp, of degree 1) are called and pictured as solid (resp., dotted) arrows. The graph Γ is called *bigraded* or *bigraph* if $\mathtt{d} : \Gamma_1 \to \{0,1\}$. The graph Γ is called *classic* if $\deg(a) = 0$ for any $a \in \Gamma_1$.

For any $a \in \Gamma_1(i,j)$, we denote by $\bar{a}$ the directed edge from i to j having the degree $\deg \bar{a} = |a|$, and we put $\overline{\Gamma}_1 = \{\bar{a} \mid a \in \Gamma_1\}$. We denote by $\overline{\Gamma} = (\Gamma_0, \overline{\Gamma}_1)$ the directed bigraph obtained from Γ by taking degree modulo 2. Here $\bar{\Gamma}_1 = \bar{\Gamma}_1^0 \cup \bar{\Gamma}_1^1$.

Furthermore, for any $a \in \Gamma_1(i,j)$, we denote by $\tilde{a}$ the undirected edge between vertices i and j, and we put $\widetilde{\Gamma}_1 = \{\tilde{a} \mid a \in \Gamma_1\}$. We denote by $\widetilde{\Gamma} = (\Gamma_0, \widetilde{\Gamma}_1)$ the undirected bigraph obtained from $\overline{\Gamma}$ by deleting the orientation of the arrows. So defined graph $\widetilde{\Gamma}$ will be called the *scheme* of Γ.

All graphs to be considered are assumed to be finite.

We associate with the directed graded graph Γ the non symmetric bilinear form $\langle\, ,\, \rangle : \mathbb{Z}^{\Gamma_0} \times \mathbb{Z}^{\Gamma_0} \to \mathbb{Z}$ together with its quadratic form χ. We denote by $E = \{\mathtt{e}_i\}_{i\in\Gamma_0}$ a standard base (system of generators) of the lattice $\mathbb{Z}^{\Gamma_0}$. The elements of $\mathbb{Z}^{\Gamma_0}$ will be written as either $\mathtt{x} = (x_i)$ or $\mathtt{x} = \sum_{i\in\Gamma_0} x_i \mathtt{e}_i$. Define a non symmetric bilinear form $\langle -,- \rangle = \langle -,- \rangle_\Gamma : \mathbb{Z}^{\Gamma_0} \times \mathbb{Z}^{\Gamma_0} \to \mathbb{Z}$, called the *Euler form* of Γ: for $\mathtt{x} = (x_i)$, $\mathtt{y} = (y_i) \in \mathbb{Z}^{\Gamma_0}$,

$$\langle \mathtt{x}, \mathtt{y} \rangle = \sum_{i\in\Gamma_0} x_i y_i - \sum_{a\in\Gamma_1(i,j)} (-1)^{|a|} x_i y_j,$$

By the definition, the bilinear forms $\langle -,- \rangle_\Gamma$ and $\langle -,- \rangle_{\overline{\Gamma}}$ are identical. The symmetrization of Euler form $(\mathtt{x},\mathtt{y}) = \langle \mathtt{x},\mathtt{y} \rangle + \langle \mathtt{y},\mathtt{x} \rangle$ is independent of the orientation of Γ, so it is defined by $\widetilde{\Gamma}$ and it is called the *symmetric Euler form* of Γ. The integer unit quadratic form $\chi : \mathbb{Z}^{\Gamma_0} \to \mathbb{Z}$ such that

$$\chi(x) = \sum_{\mathtt{i}\in\Gamma_0} x_{\mathtt{i}}^2 - \sum_{a\in\Gamma_1(\mathtt{i},\mathtt{j})} (-1)^{|a|} x_{\mathtt{i}} x_{\mathtt{j}}$$

associated with the Euler form $(-,-)$ is called the *Tits form* on $\mathbb{Z}^{\Gamma_0}$, it only depends on the graph $\widetilde{\Gamma}$.

Recall that a quadratic form χ is called *positive definite* if $\chi(x) > 0$, for all $x \neq 0$. The classic connected undirected graph having the positive Tits form is one of Dynkin diagrams. The connected directed graded graph Γ is called a (graded) *Dynkin quiver* if its scheme $\widetilde{\Gamma}$ is a Dynkin diagram, and *classic Dynkin quiver* (or 0-quiver) if Γ is a classic graph and we assign to each arrow the degree 0.

In path category $\Bbbk\Gamma$, we denote $\operatorname{coeff}_{x_1\ldots x_k} x = \kappa$, $\kappa \in \Bbbk$ whenever $x = \kappa x_1 \ldots x_k + \ldots$ is a basis decomposition. In the category $\Bbbk\Gamma$, we have such that $\deg x_1 x_2 \ldots x_k = \sum_{i=1}^k \deg x_i$.

The full subgraph Γ_S, $S \subset \Gamma_0$ is called *closed contour* if there is an ordering $S = \{\mathtt{i}_1, \ldots, \mathtt{i}_k\}$ such that $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1}) \cup \Gamma_1(\mathtt{i}_{j+1}, \mathtt{i}_j)| > 0$, $j = 1, \ldots, k-1$, and $|\Gamma_1(\mathtt{i}_1, \mathtt{i}_k) \cup \Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S, $S = \{\mathtt{i}_1, \ldots, \mathtt{i}_k\} \subset \Gamma_0$ is called *clear* if $\Gamma_1(\mathtt{i}_s, \mathtt{i}_t) \cup \Gamma_1(\mathtt{i}_t, \mathtt{i}_s) = \varnothing$, $|s-t| > 1 \pmod k$. The closed contour Γ_S is called *cyclic paths* if $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1})| > 0$, $j = 1, \ldots, k-1$, and $|\Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S is called *detour graded contour* if $\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1}) = \{a_j\}$,

$j = 1, \dots, k-1$, and $\Gamma_1(\mathtt{i}_1, \mathtt{i}_k) = \{a\}$, besides $\deg a = \sum_{j=1}^{k-1} \deg a_j$.

In further considerations, we will consider the class of graphs with some restrictions. The finite directed graded graph Γ is called *correctly defined graph* if its path category $\Bbbk\Gamma$ does not have cyclic paths and multiple edges, and, moreover, any clear closed contour is detour graded. Then we have $\Gamma_1(i,i) = \varnothing$, $|\Gamma_1(i,j)| \leqslant 1$, and $|\Gamma_1(i,j)| \cdot |\Gamma_1(j,i)| = 0$ for any $1 \leqslant i,j \leqslant |\Gamma_0|$. The Tits form χ_Γ in such case is unit which means that coefficient on x_i^2 equals 1.

Let Γ be a correctly defined graph. For the elements of standard base E holds $(\mathtt{e}_i, \mathtt{e}_i) = \chi(\mathtt{e}_i) = 1$, and $\Gamma_1(\mathtt{i},\mathtt{j}) = \varnothing$. Moreover, $|\bar\Gamma_1^0(\mathtt{i},\mathtt{j})| = \max\{-\langle \mathtt{e}_i, \mathtt{e}_j\rangle, 0\}$, and $|\bar\Gamma_1^1(\mathtt{i},\mathtt{j})| = \max\{\langle \mathtt{e}_i, \mathtt{e}_j\rangle, 0\}$.

2.2 Real roots and Weyl group

We consider a correctly defined finite directed graded graphs $\Gamma = (\Gamma_0, \Gamma_1, \det)$ together with underlined bigraph $\overline{\Gamma}$, its scheme $\widetilde{\Gamma}$, quadratic forms χ and correspondent bilinear form $\langle , \rangle$.

For any $x \in \mathbb{Z}^{\Gamma_0}$ such that $(x,x) = 1$, we denote by σ_x the *reflection* of $\mathbb{Z}^{\Gamma_0}$ on the hyperplane orthogonal to x, then $\sigma_x(y) = y - 2(x,y)x$, $y \in \mathbb{Z}^{\Gamma_0}$. For simplicity, we assume $\Gamma_0 = \{1, 2, \dots, n\}$, $n = |\Gamma_0|$, then $\mathbb{Z}^{\Gamma_0} \simeq \mathbb{Z}^n$. We give a widely knows description of the basic properties of reflections.

Lemma 1. *For any $x, y, z \in \mathbb{Z}^n$, $(x,x) = 1$ there are: 1) $\sigma_x^2 = id_{\mathbb{Z}^n}$, so $\sigma_x : \mathbb{Z}^n \to \mathbb{Z}^n$ is an involution map; 2) $\sigma_{-x}(y) = \sigma_x(y)$; 3) $\sigma_x(-y) = -\sigma_x(y)$; 4) $\sigma_x(x) = -x$; 5) $(\sigma_x(y), \sigma_x(z)) = (y,z)$; 6) $\sigma_{\sigma_x(y)}(z) = \sigma_x \sigma_y \sigma_x^{-1}(z)$.*

The reflection with respect to i-th element of standard base E is defined by $\sigma_i(y) = y - (y, \mathtt{e}_i)\mathtt{e}_i$. In this case, we call σ_i a *simple reflection* on $\mathbb{Z}^{\Gamma_0}$, we call $\mathtt{e}_i$ a *simple root*, and denote by Π_χ the set of all simple roots.

Define the *Weyl group* W of the graph Γ to be the subgroup of $Aut(\mathbb{Z}^{\Gamma_0})$ generated by the simple reflections σ_i respectively to the elements of standard base E. Because the symmetric bilinear form $(-,-)$ is independent of the orientation of Γ, the group W is independent of the orientation of Γ, so is defined by $\widetilde{\Gamma}$ (or, equivalently, by the quadratic form χ). An element of $\mathbb{Z}^{\Gamma_0}$ is called a *real root* provided it belongs to the W-orbits of some $\mathtt{e}_i$, The set $\Phi_\chi = \bigcup_{w \in W} w(\Pi_\chi) \subset \mathbb{Z}^{\Gamma_0}$ is called the set of real roots of χ (obviously, each $x \in \Phi_\chi$ satisfies $\chi(x) = \langle x, x\rangle = 1$). The root $x \in \mathbb{Z}^{\Gamma_0}$ is called *sincere* if $x_i \neq 0$ for all $i \in \Gamma_0$. It is well-known that all real roots of the classic Dynkin quivers are positive or negative.

Any generator set $R = \{\alpha_i\}_{i \in \overline{1,n}}$ of the lattice $\mathbb{Z}^{\Gamma_0} \simeq \mathbb{Z}^n$ is called a *root base* with respect to the graph Γ (and to the bilinear form $\langle , \rangle_\Gamma$), if $\alpha_i \in \Phi_\chi$ for any $i \in \overline{1,n}$. We denote by $\Gamma_R = \Gamma' = (\Gamma_0', \Gamma_1')$ the directed bigraph constructed in the following way: (i) $\Gamma_0' = R$; (ii) for any $i, j \in \overline{1,n}$, $i \neq j$ there are $|\langle \alpha_i, \alpha_j \rangle|$ edges from $\Gamma_1'(\alpha_i, \alpha_j)$ either of degree 0 if $\langle \alpha_i, \alpha_j\rangle < 0$, and of degree 1 otherwise; (iii) $|\Gamma_1'(\alpha_i, \alpha_i)| = \varnothing$. Note that for the positive definite form, $|\langle \alpha_i, \alpha_j\rangle| \leqslant 1$ and $\langle \alpha_i, \alpha_i\rangle = 1$. We call Γ_R the directed *bigraph associated with a root base* R and non symmetric bilinear form $\langle , \rangle$.

The root base R is *correctly defined* if its underlined graph Γ_R is such. In this case $\Gamma_1'(\alpha_i, \alpha_i) = \varnothing$, $|\Gamma_1'(\alpha_i, \alpha_j)| \leqslant 1$, and $|\Gamma_1'(\alpha_i, \alpha_j)| \cdot |\Gamma_1'(\alpha_j, \alpha_i)| = 0$ for any $i, j \in \overline{1,n}$. We say that the correctly defined graph $\Gamma' = \Gamma_R$ is graded correctly if $\deg(a) \equiv \langle \alpha_i, \alpha_j \rangle \pmod 2$ for any $a \in \Gamma_1'(\alpha_i, \alpha_j)$, $i, j \in \overline{1,n}$, $i \neq j$.

Let $\{\mathtt{e}_i\}_{i \in \overline{1,n}}$ be a standard root base of the lattice $\mathbb{Z}^{\Gamma_0} \simeq \mathbb{Z}^n$. For $\mathtt{i} \in \Gamma_0$, we denote by $T_\mathtt{i} : \mathbb{Z}^n \to \mathbb{Z}^n$ the $\mathbb{Z}$-linear transformation: $T_\mathtt{i}(\mathtt{e}^t) = \begin{cases} \mathtt{e}^t, & \text{if } t \neq i; \\ -\mathtt{e}^i, & \text{if } t = i. \end{cases}$ We call $T_\mathtt{i}$ a *sign change* for χ in the vertex $\mathtt{i}$. For $\mathtt{i}, \mathtt{j} \in \Gamma_0$, we denote by $T_{\mathtt{ij}}^\varepsilon : \mathbb{Z}^n \to \mathbb{Z}^n$ the $\mathbb{Z}$-linear transformation ([3], [4]): $T_{\mathtt{ij}}^\varepsilon(\mathtt{e}^t) = \begin{cases} \mathtt{e}^t, & \text{if } t \neq i; \\ \mathtt{e}^i + (-1)^{|\{\mathtt{i},\mathtt{j}\}|}\mathtt{e}^j, & \text{if } t = i. \end{cases}$ with $\varepsilon = (-1)^{|\{\mathtt{i},\mathtt{j}\}|} \in \{+,-\}$. If a degree $|\{\mathtt{i},\mathtt{j}\}|$ is even then we call $T_{\mathtt{ij}}^+$ an *inflation* for χ, if $|\{\mathtt{i},\mathtt{j}\}|$ is odd, we call $T_{\mathtt{ij}}^-$ a *deflation* for χ.

We denote the corresponding transformations of quadratic form and an integral lattice $\mathbb{Z}^n$ by the same letter. So there are $T : \chi \to \chi' = \chi T$ for the quadratic form and $T : \mathtt{r} \to \mathtt{r}' = \mathtt{r}T$ for vector $r = \sum_{\mathtt{j} \in \Gamma_0} r_\mathtt{j} \mathtt{e}_\mathtt{j}$, such that $\sum_{\mathtt{j} \in \Gamma_0} r_\mathtt{j} \mathtt{e}_\mathtt{j} = \sum_{\mathtt{j} \in \Gamma_0} r_\mathtt{j}' \mathtt{e}_\mathtt{j}'$ or $\chi(\mathtt{r}) = \chi'(\mathtt{r}')$.

Two integral forms $\chi, \chi' : \mathbb{Z}^n \to \mathbb{Z}$ are called *equivalent* (or *$\mathbb{Z}$-equivalent*) if they describe the same maps up to above changes of basis, that is, if there exists a linear $\mathbb{Z}$-invertible transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ which is a composition of admitted transformations such that $\chi' = \chi T$. The next simple lemma holds.

Lemma 2. *Let $T : \chi \to \chi T$ be an equivalence of the quadratic forms. If χ is an integral unit form, then χT is an integral unit form as well, and χT is positive (non negative, critical) if and only if χ is positive (non negative, critical).*

Theorem 2.1. *([3])Let χ be a connected integral positive unit form, $\overline{\Gamma}$ its directed bigraph. Then there is a sequence of sign changes and deflations with the composition T such that the bigraph $\overline{\Gamma}T$ of the form χT is a 0-forrest of Dynkin type. For the positive form, $\overline{\Gamma}T$ is a disjoint union of some of the following Dynkin diagrams: A_n $(n > 1)$, D_n $(n \geqslant 4)$, or E_n $(n = 6, 7, 8)$. The Dynkin type is uniquely defined by χ.*

2.3 DGC structure

The $\Bbbk$-linear category $\mathcal{U}$ over an algebraically closed field $\Bbbk$ is called *graded* if it is endowed with a multiplicative mapping $\deg : \mathcal{U} \to \mathbb{Z}$ and $\mathcal{U}(\mathtt{i},\mathtt{j}) = \oplus_{q\in\mathbb{Z}}\mathcal{U}_q(\mathtt{i},\mathtt{j})$ is a sum of finite dimensional vector spaces $\mathcal{U}_q(\mathtt{i},\mathtt{j}) = \deg^{-1}(q)$, $\mathtt{i},\mathtt{j} \in \mathrm{Ob}\,\mathcal{U}$. For $x \in \mathcal{U}$, we denote $|x| = \deg x \pmod 2$ and $\hat{x} = (-1)^{|x|}x$. The graded $\Bbbk$-category $\mathcal{U}$ is called the *differential graded category* or *dgc* if there is the differential $\mathtt{d} : \mathcal{U} \to \mathcal{U}$ which maps $\mathtt{d} : \mathcal{U}_q(\mathtt{i},\mathtt{j}) \to \mathcal{U}_{q+1}(\mathtt{i},\mathtt{j})$, $q \in \mathbb{Z}$, $\mathtt{i},\mathtt{j} \in \mathrm{Ob}\,\mathcal{U}$, and the following properties hold:

1) $\mathtt{d}(1_{\mathtt{i}}) = 0$, $\mathtt{i} \in \mathrm{Ob}\,\mathcal{U}$;
2) Leibnitz rule: $\mathtt{d}(x_1 \dots x_{i-1}x_i \dots x_k) =$
$= \sum_{i=1}^{k}\hat{x}_1 \dots \hat{x}_{i-1}\mathtt{d}(x_i)x_{i+1} \dots x_k =$
$= \sum_{i=1}^{k}(-1)^{|x_1|}x_1 \dots (-1)^{|x_i|}x_i x_{i+1} \dots x_k$;
3) $\mathtt{d}^2 = 0$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \deg)$ be a graded directed graph, and let $\Bbbk\Gamma$ be its path category. We denote $\mathrm{coeff}_{x_1\dots x_k}x = \kappa$, $\kappa \in \Bbbk$ whenever $x = \kappa x_1 \dots x_k + \dots$ is a basis decomposition. The category $\Bbbk\Gamma$ inherits the degree (graduation) from Γ such that $\deg x_1 x_2 \dots x_k = \sum_{i=1}^{k}\deg x_i$.

The full subgraph Γ_S, $S \subset \Gamma_0$ is called *closed contour* if there is an ordering $S = \{\mathtt{i}_1, \dots, \mathtt{i}_k\}$ such that $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1}) \cup \Gamma_1(\mathtt{i}_{j+1}, \mathtt{i}_j)| > 0$, $\quad j = 1, \dots, k-1$, and $|\Gamma_1(\mathtt{i}_1, \mathtt{i}_k) \cup \Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S, $S = \{\mathtt{i}_1, \dots, \mathtt{i}_k\} \subset \Gamma_0$ is called *clear* if $\Gamma_1(\mathtt{i}_s, \mathtt{i}_t) \cup \Gamma_1(\mathtt{i}_t, \mathtt{i}_s) = \varnothing$, $\quad |s - t| > 1 \pmod k$. The closed contour Γ_S is called *oriented cycle* if $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1})| > 0$, $\quad j = 1, \dots, k-1$, and $|\Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S is called *detour graded contour* if $\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1}) = \{a_j\}$, $j = 1, \dots, k-1$, and $\Gamma_1(\mathtt{i}_1, \mathtt{i}_k) = \{a\}$, besides $\deg a = \sum_{j=1}^{k-1}\deg a_j$. Denote $x_{\mathtt{ij}}$ the edge from the vertice starting in $\mathtt{i}$ and ending in $\mathtt{j}$. Detour contour Γ_S is called *active* (or contour of differenrial type) if $\kappa x_{\mathtt{i}_1\mathtt{i}_2} \dots x_{\mathtt{i}_{k-1}\mathtt{i}_k}$ is a summand of differential of the edge $x_{\mathtt{i}_1\mathtt{i}_k}$. The edge $a \in \Gamma_1(\mathtt{i},\mathtt{j})$ is called *deep* if there are no other pathes on Γ from $\mathtt{i}$ to $\mathtt{j}$. The edge $a \in \Gamma_1(\mathtt{i},\mathtt{j})$ is called *minimal* if $\mathtt{d}(a) = 0$.

We consider a *dgc* $\mathcal{U}$ with $|\mathrm{Ob}\,\mathcal{U}| < \infty$. Define the underlined directed graded graph $\Gamma = \Gamma(\mathcal{U})$ such that $\Gamma_0 = \mathrm{Ob}\,\mathcal{U}$, and $\Gamma_1(\mathtt{i},\mathtt{j})$ is a basis of $(\mathcal{U}/\mathcal{U}^{\otimes 2})(\mathtt{i},\mathtt{j})$, $\mathtt{i},\mathtt{j} \in \Gamma_0$ with the induced graduation. The differential $\mathtt{d}$ induces the map $\mathtt{d} : \Gamma_1^q \to \Bbbk\Gamma_{q+1}(\mathtt{i},\mathtt{j})$, $\quad \mathtt{i},\mathtt{j} \in \Gamma_0$, $\ q \in \mathbb{Z}$ which is extended on the whole $\Bbbk\Gamma$ by Leibnitz rule.

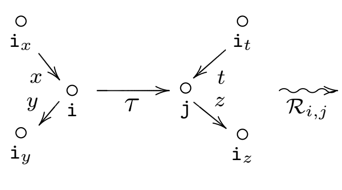
The graph Γ correspondent to the finite dimensional differential graded category is finite. The finitely generated graph Γ is called *correctly defined graph* if its path category $\Bbbk\Gamma$ has no cyclic paths and multiple edges, and, any clear closed contour is detour graded. The *dgc* $\mathcal{U}$ is called *correctly defined* if the underlined directed graded graph Γ is correctly defined, and any clear closed contour is active. In such a case, we call $(\mathcal{U}, \Gamma)$ a *correctly defined* dgc *problem.*

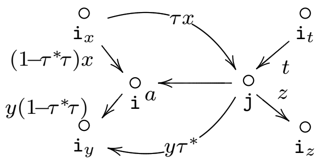
3 Reductions and forms of quiver type

3.1 Reduction of a *dgc*

We consider the *dgc* $\mathcal{U}$ together with underlined graded graph Γ with $\Gamma_0 = Ob\mathcal{U}$. For $\mathtt{i},\mathtt{j} \in \Gamma_0$ we have defined in detail in [4] the reduction $\mathcal{R}_{\mathtt{ij}} : (\mathcal{U}, \Gamma) \to (\mathcal{U}', \Gamma')$. Here we briefly describe the resulting *dgc* and graph. If the vertices $\mathtt{i}$ and $\mathtt{j}$ are not incident on Γ then the reduction $\mathcal{R}_{\mathtt{ij}}$ is trivial, hence $\Gamma' = \Gamma$, $\mathcal{U}' = \mathcal{U}$. We apply the reduction for the case in which there is single deep arrow τ between $\mathtt{i}$ and $\mathtt{j}$ for two possible directions.

We draw all edges as solid arrows but they can have different degrees, we depict the direction of the arrow, if it does not matter. Suppose that $\tau \in \Gamma_1(\mathtt{i},\mathtt{j})$ is a single deep arrow with degree $\deg\tau = d$. The general case is:





Resulting the construction from [4] we conclude, there is $\tau^* : \mathtt{j} \to \mathtt{i}$ such that $\tau\tau^* = \mathtt{1}_\mathtt{j}$, and $\mathtt{1}_\mathtt{i} = \mathtt{1}_{\mathtt{i}_1} + \mathtt{1}_{\mathtt{i}_2} = (1-\tau^*\tau) + \tau^*\tau$ is a decomposition on the sum of mutually commuting idempotents. Then $\deg\tau^* = -\deg\tau = -d$. There is a new arrow $a = \varphi_{21}\tau^* : \mathtt{j} \to \mathtt{i}$ on Γ' with $\deg(a) = \deg\tau^* + 1 = 1 - d$, and $\mathtt{d}(a) = 0$. Each arrow $x \in \Gamma_1(i_x, i)$ splits into two arrows τx and $(1-\tau^*\tau)x$ on Γ', and $a\,(\tau x) \in \partial\big((1-\tau^*\tau)x\big)$. Each arrow $y \in \Gamma_1(i, i_y)$ splits on two arrows $y\tau^*$ and $y(1-\tau^*\tau)$, and besides $y(1-\tau^*\tau)\,a \in \partial\big(y\tau^*\big)$. All other arrows do not change. The graduation and differential are defined in algorithmic way from the structure of $\mathcal{U}$. The differential on is obtained by substitution $\mathtt{1}_\mathtt{i} = (1-\tau^*\tau) + \tau^*\tau$. Any path crossing on $\mathtt{i}$ is a combination of pathes:
$y_1 \dots y_q\, yx\, x_p \dots x_1 \iff$
$$y_1 \dots y_q\ \big(y(1-\tau^*\tau)\ \ y\tau^*\tau\big)\begin{pmatrix} (1-\tau^*\tau)x \\ \tau^*\tau x \end{pmatrix} x_p \dots x_1.$$

In the second case there is a single deep arrow $\tau \in \Gamma_1(\mathtt{j}, \mathtt{i})$. It can be considered similarly by adding $\tau^* : \mathtt{i} \to \mathtt{j}$ such that $\tau^*\tau = \mathtt{1}_\mathtt{j}$ and using the decomposition $\mathtt{1}_\mathtt{i} = \mathtt{1}_{\mathtt{i}_1} + \mathtt{1}_{\mathtt{i}_2} = (1-\tau\tau^*) + \tau\tau^*$ to the sum of mutually commuting idempotents.

We obtain the directed graded $\Gamma' = (\Gamma_0, \Gamma_1')$ and the *dgc* $\mathcal{U}'$, the correspondent transformation is denoted by $\mathcal{R}_{ij} : (\Gamma, \mathcal{U}) \to (\Gamma', \mathcal{U}')$ and it is called *reduction* from $\mathtt{i}$ to $\mathtt{j}$ (along the arrow τ).

Let $\Gamma_1(\mathtt{i}, \mathtt{j}) = \{\tau\}$. The reduction $\mathcal{R}_{\mathtt{ij}} : \mathcal{U} \to \mathcal{U}'$ is transferred to the transformation of lattices $\mathcal{R}_{\mathtt{ij}} : \mathbb{Z}^{\Gamma_0} \to \mathbb{Z}^{\Gamma_0}$. For any $x \in \mathbb{Z}^{\Gamma_0}$ we obtain: $\mathcal{R}_{\mathtt{ij}} : \mathtt{x} \to \mathtt{x}'$ where $x_i' = x_i - (-1)^{|\tau|}x_j$ and $x_k' = x_k$ otherwise. Transformation $\mathcal{R}_{\mathtt{ij}} : (\mathcal{U}, \Gamma) \to (\mathcal{U}', \Gamma')$ is defined. Sometimes we denote it by $\mathcal{R}^+_{\mathtt{ij}}$ if $|\tau|$ is even, and by $\mathcal{R}^-_{\mathtt{ij}}$ otherwise.

Note that $\mathcal{R}_{\mathtt{ij}}\mathcal{U}$ is not augmented *dgc* and it is directed cycle-free, but it is not necessarily regular. Namely, it is possible that $\Gamma_1(\mathtt{k}, \mathtt{j}) = \{x, y\}$ for some $\mathtt{k} \in \Gamma_0$. By the construction, in this case $\mathtt{d}(x) = \kappa y + l$ where $l \in \mathcal{P}^2$ and $|y| = |x| + 1$. Then we put: $x = 0$, $\mathtt{d}(x) = 0$, $y = -\kappa^{-1}l$, and obtain the new dgc $\mathcal{U}'$ with the graph Γ'. We say that $\mathcal{U}'$ is obtained from $\mathcal{U}$ by *regularization* on x, y. The quadratic form χ and the attached vector $r \in \mathbb{Z}^{\Gamma_0}$ do not change after regularization operation. The case $|\Gamma_1(\mathtt{j}, \mathtt{k})| = 2$ is analogous. Given a reduced *dgc* $\mathcal{R}_{\mathtt{ij}}\mathcal{U}$, we can do some number of regularization procedures to obtain the regular *dgc* problem. We call this transformation a *complete reduction* and denote it with the same letter $\mathcal{R}_{\mathtt{ij}}$.

Let $\Gamma_1(\mathtt{i}, \mathtt{j}) \cup \Gamma_1(\mathtt{j}, \mathtt{i}) = \{\tau\}$. The standard root base $E = \{\mathsf{e}_i\}$ of $\mathbb{Z}^{\Gamma_0}$ transforms by the reduction $\mathcal{R}_{\mathtt{ij}}$ to the root base $E' = \{\mathsf{e}_i'\}$ where $\mathsf{e}_j' = \mathsf{e}_j + (-1)^{|\tau|}\mathsf{e}_i = \omega_i(\mathsf{e}_j)$ and $\mathsf{e}_k' = \mathsf{e}_k$ for $k \neq j$. Then E' is a root base as well. We denote by $\mathcal{R}_{\mathtt{ij}}\Gamma$ the directed bigraph associated with the root base E' and with the bilinear form $\langle\,,\,\rangle_\Gamma$. The reduction $\mathcal{R}_{\mathtt{ij}}$ of correctly defined *dgc* $\mathcal{U}$ is called *admitted* if $\mathtt{i}$, $\mathtt{j}$ are 1-*connected points*, it means, an edge $\tau \in \Gamma_1(\mathtt{i}, \mathtt{j}) \cup \Gamma_1(\mathtt{j}, \mathtt{i})$ is deep.

Lemma 3. *Let $(\mathcal{U}, \Gamma)$ be a correctly defined* dgc *problem having positive definite Tits form, let $\tau \in \Gamma_1(\mathtt{i}, \mathtt{j})$ be a deep regular edge, and let $\mathcal{R}_{\mathtt{ij}} : (\mathcal{U}, \Gamma) \to (\mathcal{U}', \Gamma')$ be an admitted complete reduction. Then $\big(\mathcal{R}_{\mathtt{ij}}\mathcal{U}, \mathcal{R}_{\mathtt{ij}}\Gamma\big)$ is correctly defined* dgc *problem and the bigraphs $\mathcal{R}_{\mathtt{ij}}\Gamma$, $\overline{\Gamma}'$ coincide.*

The proof follows from the structure of *dgc* and of the positivity of quadratic form.

Given a correctly defined *dgc* problem $(\mathcal{U}, \Gamma)$ we construct the *compositions of admitted reductions* of a type $\mathcal{R}_{\mathtt{ij}}$. We begin with $(\mathcal{U}, \Gamma)$ and standard root base E of $\mathbb{Z}^{\Gamma_0}$. Assume that $\mathcal{R}' : (\mathcal{U}, \Gamma) \to (\mathcal{U}', \Gamma')$ is a composition of $r > 0$ simple reductions, and let $E' = \{\alpha_i'\}_{i \in \Gamma_0}$ be obtained root base of $(\mathcal{U}', \Gamma')$. For any $i, j \in \Gamma_0$, with 1-connected on Γ' roots α_i', α_j', we fulfill reduction $\mathcal{R}'' : (\mathcal{U}', \Gamma') \to (\mathcal{U}'', \Gamma'')$, and we construct root base $E'' = \{\alpha_i''\}_{i \in \Gamma_0}$ such that $\alpha_j'' = \omega_{\alpha_i'}(\alpha_j')$ and $\alpha_k'' = \alpha_k$ for $k \neq j$. Reduction $\mathcal{R} = \mathcal{R}''\mathcal{R}' : (\mathcal{U}, \Gamma, E) \to (\mathcal{U}'', \Gamma'', E'')$ is a composition of $r+1$ simple reductions, it is called admitted reduction.

For the correctly defined *dgc* problem $(\mathcal{U}, \Gamma)$, the root base R is called *reduced root base* (from the standard base E) if there exists an admitted reduction $(\mathcal{U}, \Gamma, E) \to (\mathcal{U}', \Gamma', R)$.

Let $(\mathcal{U}, \Gamma)$ be a *dgc* problem, $\Gamma = (\Gamma_0, \Gamma_1, \deg)$ and $\mathtt{i} \in \Gamma_0$. We construct a new graph $\Gamma' = \mathcal{T}_\mathtt{i}(\Gamma) = (\Gamma_0, \Gamma_1, \deg')$ by changing the grading of Γ in the following way. For any $a \in \Gamma_1(\mathtt{j}, \mathtt{i})$ (resp., $b \in \Gamma_1(\mathtt{i}, \mathtt{j})$) we put $\deg'(a) = \deg(a) + 1$ (resp., $\deg'(b) = \deg(b) - 1$). The structure of *dgc* problem $(\mathcal{U}, \Gamma, \deg)$ is transferred to the problem $(\mathcal{U}, \Gamma, \deg') = \mathcal{T}_\mathtt{i}(\mathcal{U}, \Gamma, \deg)$ because the condition $\deg'\big(\mathtt{d}(a)\big) = \deg\big(\mathtt{d}(a)\big)$ trivially holds for any $a \in \Gamma_1$. The transposition $\mathcal{T}$ of *dgc* problem is called *shift of grading* if it is a composition of transformations of a type $\mathcal{T}_\mathtt{i}$.

3.2 Forms of quiver type

We say that the *dgc* problem $(\mathcal{U}, \Gamma)$ is of *quiver type* (resp., of *Dynkin quiver type*) if there exists $Q = (Q_0, Q_1, \deg)$ a graded quiver (resp., a graded quiver of Dynkin type), a reduced root base R of the quiver, and an admitted reduction $(\mathcal{U}_Q, Q, E) \to (\mathcal{U}_R, \Gamma_R, R)$ such that problems $(\mathcal{U}, \Gamma)$ and $(\mathcal{U}_R, \Gamma_R)$ coincide. In this case bilinear form is an Euler form of quiver, and the quadratic form is Tits form.

Any graded quiver $Q = (Q_0, Q_1)$ is inherent in the structure of trivial correctly defined differential graded problem $(\mathcal{U}_Q, Q, E^{Q_0})$ where E^{Q_0} is a standard base of $\mathbb{Z}^{|Q_0|}$. Then, by the definition, any problem $(\mathcal{U}, \Gamma, E^{\Gamma_0})$ is a correctly defined whenever it is reduced from the problem $(\mathcal{U}_Q, Q, E^{Q_0})$ for some Dynkin quiver Q.

Theorem 3.1. *Let $\mathcal{U}$ be a correctly defined differential graded category, $\Gamma = (\Gamma_0, \Gamma_1)$ be its connected graded graph having a positive definite quadratic form $\chi = \chi_\Gamma$, and E be a standard base of $\mathbb{Z}^{\Gamma_0}$. Then the* dgc *problem $(\mathcal{U}, \Gamma, E)$ is of Dynkin graded quiver type.*

The proof of Theorem consists in modification and application of Theorem from [4] for the bilinear and quadratic forms with respect to their roots and root bases.

Corollary 1. *Under the assumptions of Theorem* 3.1*, there is a shift of grading $\mathcal{T} : (\mathcal{U}, \Gamma, E) \to (\mathcal{U}', \Gamma', E)$ such that the* dgc *problem $(\mathcal{U}', \Gamma', E)$ is of classic Dynkin type.*

Theorem 3.2. *The directed graded graph Γ with positive definite quadratic form χ_Γ is reduced to a Dynkin quiver if and only if Γ can be endowed with the the structure of correctly defined* dgc *problem.*

Corollary 2. *Let $\langle\, , \rangle$ be a non symmetric integer bilinear form with associated connected bigraph $B = (B_0, B_1)$, and the quadratic form χ_B is positive definite. Then there exists a graded Dynkin quiver Q and a reduced real root base R such that $\langle\, , \rangle = \langle\, , \rangle_R$ if and only if there is a correctly defined* dgc *problem $(\mathcal{U}, \Gamma, E)$ with E be a standard base of $\mathbb{Z}^{|B_0|}$ such that bigraphs $\overline{\Gamma}$ and B coincide.*

Example 1. For the 0-graded quiver $\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet}$ we consider the following root base: $R = \{f_1, f_2, f_3\}$ where $f_1 = -e_2$, $f_2 = e_1 + e_2$, $f_3 = e_3$. The constructed bigraph Γ_R is a clear closed contour contains 3 vertices, of a type

$$\begin{tikzcd} & \overset{f_2}{\bullet} \arrow[dr] & \\ f_1\bullet \arrow[ur] \arrow[rr, dashed] & & \bullet f_3 \end{tikzcd}$$

where $\deg a_{12} = \deg a_{23} = 0$, and $\deg a_{13} = -1$, and the differential is given by $d(a_{13}) = a_{23}a_{12}$. It is obtained by the sequence of reductions: $\mathcal{R}_{21}\,\mathcal{R}_{12}$. $\{e_1, e_2, e_3\} \mapsto \{e_1, w_{e_1}(e_2), e_3\} = \{e_1, e_1 + e_2, e_3\} \mapsto \{w_{e_1+e_2}(e_1), e_1 + e_2, e_3\} = \{-e_2, e_1 + e_2, e_3\}$ Graded graph

$$\begin{tikzcd} \bullet & \bullet \arrow[l, dashed] \arrow[d] \\ \bullet \arrow[u] \arrow[r] & \bullet \end{tikzcd}$$

can't be reduced.

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