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Декомпозиції вільних тріоїдів

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Decompositions of free trioids

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Одним із найкращих методів вивчення будови різних алгебр є метод декомпозиції. Основна ідея цього методу полягає в розкладі алгебри на компоненти, можливо, більш простої структури, детальному вивченні компонент та встановленні взаємозв'язків між компонентами в межах цієї алгебри. Вищевказаний метод має застосування в теорії групоїдів, теорії напівгруп, теорії дімоноїдів. У цій статті охарактеризовано декомпозиції вільних тріоїдів у трисполуки підтріоїдів та представлено деякі найменші конгруенції на вільному тріоїді.

Ключові слова: тріоїд, вільний тріоїд, трисполука підтріоїдів, дімоноїд, напівгрупа, конгруенція.

One of the best methods used in studying the structure of different algebras is the decomposition method. The main idea of this method is to decompose an algebra into components, perhaps of simpler structure, to study components in details and to establish mutual relationships between components within the entire algebra. The mentioned method has applications in groupoid theory, semigroup theory, dimonoid theory. Jean-Louis Loday and Mari ́a O. Ronco constructed operads associated to the chain modules of simplexes and of Stasheff polytopes. The corresponding algebras have three operations and they are called associative trialgebras and dendriform trialgebras. A trioid is a set equipped with three binary associative operations satisfying some axioms. A trialgebra is just a linear analog of a trioid. Therefore, all results obtained for trioids can be applied to trialgebras. In this paper we characterize decompositions of free trioids into tribands and bands of subtrioids and present the least rectangular band congruence, the least left zero congruence and the least right zero congruence on a free trioid.

Key words: trioid, free trioid, triband of subtrioids, dimonoid, semigroup, congruence.
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1 Introduction

One of the best methods used in studying the structure of different algebras is the decomposition method. The main idea of this method is to decompose an algebra into components, perhaps of simpler structure, to study components in details and to establish mutual relationships between components within the entire algebra. The mentioned method has applications in groupoid theory, semigroup theory, dimonoid theory (see, e.g., [1], respectively, [2], [3]).

During the study of planar trees J.-L. Loday and M.O. Ronco [4] introduced a type of algebras, called trialgebras, which are vector spaces endowed with three binary associative operations satisfying eight axioms. A trialgebra is just a linear analog of a trioid [4] and therefore all results obtained for trioids can be applied to trialgebras. A free trioid of rank 1 was given in [4]. A trioid which is isomorphic to the free trioid of rank 1 was considered in [5]. Free trioids play an important role in constructing free trialgebras. See [6] for more information about trioids.

In this paper our attention will be aimed to decompositions of trioids. Here we give decompositions of free trioids into tribands and bands of subtrioids. As a consequence, we characterize the least rectangular band congruence, the least left zero congruence and the least right zero congruence on a free trioid.

2 Preliminaries

Recall that a nonempty set T equipped with three binary associative operations $\dashv$, $\vdash$ and $\perp$ satisfying eight axioms: $(x \dashv y) \dashv z = x \dashv (y \vdash z)\,(T1)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)\,(T2)$, $(x \dashv y) \vdash z = x \vdash (y \vdash z)\,(T3)$, $(x \dashv y) \dashv z = x \dashv (y \perp z)\,(T4)$,

$(x \perp y) \dashv z = x \perp (y \dashv z)\,(T5)$, $(x \dashv y) \perp z = x \perp (y \vdash z)\,(T6)$, $(x \vdash y) \perp z = x \vdash (y \perp z)\,(T7)$, $(x \perp y) \vdash z = x \vdash (y \vdash z)\,(T8)$, is called a trioid.

Consider the construction of a free trioid.

Let Y be an arbitrary nonempty set, $\overline{Y} = \{\overline{x} \,|\, x \in Y\}$, $X = Y \cup \overline{Y}$ and $F[X]$ be the free semigroup on X. Let further $P \subset F[X]$ be a subsemigroup which contains words w with the element $\overline{x}$ $(x \in Y)$ occuring in w at least one time.

Let $w \in P$. Denote by $\widetilde{w}$ the word obtained from w by change of all letters $\overline{x}$ $(x \in Y)$ by x. For instance, if $w = x\overline{x}\,\overline{y}x\overline{z}$, then $\widetilde{w} = xxyxz$. Obviously, $\widetilde{w} \in F[X] \backslash P$.

Define operations $\dashv$, $\vdash$ and $\perp$ on P by

$$w \dashv u = w\widetilde{u}, \quad w \vdash u = \widetilde{w}u, \quad w \perp u = wu$$

for all $w, u \in P$. Denote the algebra $(P, \dashv, \vdash, \perp)$ by $Frt(Y)$.

Proposition 1. *$Frt(Y)$ is the free trioid.*

The proof of this statement is the same as the proof of Proposition 1.9 from [4] obtained for the free trioid of rank 1.

Now recall the definition of a dimonoid [7, 8].

A nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1) - (T3)$ is called a dimonoid. If $D = (D, \dashv, \vdash)$ is a dimonoid, then the trioid $(D, \dashv, \vdash, \dashv)$ (respectively, $(D, \dashv, \vdash, \vdash)$) will be denoted by $(D)^{\dashv}$ (respectively, $(D)^{\vdash}$). It is clear that $(D)^{\dashv}$ and $(D)^{\vdash}$ are different as trioids but they coincide as dimonoids.

We need some algebras from [9] which will be used in Section 3.

For an arbitrary nonempty set Y let $Y_{\ell z} = (Y, \dashv)$, $Y_{rz} = (Y, \vdash)$, $Y_{rb} = Y_{\ell z} \times Y_{rz}$ be a left zero semigroup, a right zero semigroup and a rectangular band, respectively. By [9] $Y_{\ell z, rz} = (Y, \dashv, \vdash)$ is the free left zero and right zero dimonoid (or the free left and right diband).

Define operations $\dashv$ and $\vdash$ on Y^2 by

$$(x, y) \dashv (a, b) = (x, b), \quad (x, y) \vdash (a, b) = (a, b)$$

for all $(x, y), (a, b) \in Y^2$. By [9] $(Y^2, \dashv, \vdash)$ is the free (rb, rz)-dimonoid. It is denoted by $Y_{rb,rz}$.

Define operations $\dashv$ and $\vdash$ on Y^2 by

$$(x, y) \dashv (a, b) = (x, y), \quad (x, y) \vdash (a, b) = (x, b)$$

for all $(x, y), (a, b) \in Y^2$. By [9] $(Y^2, \dashv, \vdash)$ is the free $(\ell z, rb)$-dimonoid. It is denoted by $Y_{\ell z, rb}$.

Define operations $\dashv$ and $\vdash$ on Y^3 by

$$(x_1, x_2, x_3) \dashv (y_1, y_2, y_3) = (x_1, x_2, y_3),$$

$$(x_1, x_2, x_3) \vdash (y_1, y_2, y_3) = (x_1, y_2, y_3)$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in Y^3$. The algebra $(Y^3, \dashv, \vdash)$ is denoted by $FRct(Y)$. According to Theorem 1 from [9] $FRct(Y)$ is the free rectangular diband.

A trioid $(T, \dashv, \vdash, \perp)$ is called a triband [10], if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are bands.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^3 by

$$(a_1, b_1, c_1) \dashv (a_2, b_2, c_2) = (a_1, b_1, c_1),$$

$$(a_1, b_1, c_1) \vdash (a_2, b_2, c_2) = (a_1, b_2, c_2),$$

$$(a_1, b_1, c_1,) \perp (a_2, b_2, c_2) = (a_1, b_1, c_2)$$

for all $(a_1, b_1, c_1), (a_2, b_2, c_2) \in Y^3$. It is clear that $(Y^3, \perp, \vdash)$ is a rectangular diband [9] and $(Y^3, \dashv)$ is a left zero semigroup. It is immediate to check that $(Y^3, \dashv, \vdash, \perp)$ is a triband. It will be denoted by $Y_{lz,rd}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^3 by

$$(a_1, b_1, c_1) \dashv (a_2, b_2, c_2) = (a_1, b_1, c_2),$$

$$(a_1, b_1, c_1) \vdash (a_2, b_2, c_2) = (a_2, b_2, c_2),$$

$$(a_1, b_1, c_1,) \perp (a_2, b_2, c_2) = (a_1, b_2, c_2)$$

for all $(a_1, b_1, c_1), (a_2, b_2, c_2) \in Y^3$. It is clear that $(Y^3, \dashv, \perp)$ is a rectangular diband [9] and $(Y^3, \vdash)$ is a right zero semigroup. One can check that $(Y^3, \dashv, \vdash, \perp)$ is a triband. It will be denoted by $Y_{rd,rz}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^2 by

$$(a_1, b_1) \dashv (a_2, b_2) = (a_1, b_1),$$

$$(a_1, b_1) \vdash (a_2, b_2) = (a_2, b_2),$$

$$(a_1, b_1) \perp (a_2, b_2) = (a_1, b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in Y^2$. It is clear that $(Y^2, \dashv, \vdash)$ is a left zero and right zero dimonoid [9] and $(Y^2, \perp)$ is a rectangular band. By [10] $(Y^2, \dashv, \vdash, \perp)$ is a triband. It will be denoted by $Y^{rb}_{lz,rz}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on Y^4 by

$$(x_1, x_2, x_3, x_4) \dashv (y_1, y_2, y_3, y_4) = (x_1, x_2, x_3, y_4),$$

$$(x_1, x_2, x_3, x_4) \vdash (y_1, y_2, y_3, y_4) = (x_1, y_2, y_3, y_4),$$

$$(x_1, x_2, x_3, x_4) \perp (y_1, y_2, y_3, y_4) = (x_1, x_2, y_3, y_4)$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in Y^4$. It is clear that $(Y^4, \dashv)$, $(Y^4, \vdash)$ and $(Y^4, \perp)$ are rectangular bands. One routinely verifies that $(Y^4, \dashv, \vdash, \perp)$ is a triband. The triband $(Y^4, \dashv, \vdash, \perp)$ will be denoted by $FRT(Y)$.

A nonempty subset A of a trioid $(T, \dashv, \vdash, \perp)$ is called a subtrioid, if for any $a, b \in T$, a, $b \in A$ implies $a \dashv b$, $a \vdash b, a \perp b \in A$.

If $f : T_1 \to T_2$ is a homomorphism of trioids, then the corresponding congruence on T_1 will be denoted by Δ_f.

3 Decompositions

In this section in terms of tribands of subtrioids we describe the structure of free trioids and characterize the least rectangular band congruence, the least left zero congruence and the least right zero congruence on a free trioid.

In the following, we recall the construction of a triband of subtrioids [10].

Let S be an arbitrary trioid, J be some triband and let $\alpha : S \to J : x \mapsto x\alpha$ be a homomorphism. Then every class of the congruence Δ_α is a subtrioid of the trioid S, and the trioid S itself is a union of such trioids S_ξ, $\xi \in J$, that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \mid (x, t) \in \Delta_\alpha\},$$

$$S_\xi \dashv S_\varepsilon \subseteq S_{\xi \dashv \varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi \vdash \varepsilon},$$

$$S_\xi \perp S_\varepsilon \subseteq S_{\xi \perp \varepsilon}, \quad \xi \neq \varepsilon \Rightarrow S_\xi \cap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a triband of subtrioids (or S is a triband J of subtrioids S_ξ $(\xi \in J)$). If J is an idempotent semigroup (band), then we say that S is a band J of subtrioids S_ξ $(\xi \in J)$. If J is a commutative band, then we say that S is a semilattice J of subtrioids S_ξ $(\xi \in J)$. If J is a left (right) zero semigroup, then we say that S is a left (right) band J of subtrioids S_ξ $(\xi \in J)$.

Let $\omega \in F[X]$ and $w \in Frt(Y)$. Denote the first (respectively, last) letter of ω by $\omega^{(0)}$ (respectively, $\omega^{(1)}$). Suppose that u is the initial (respectively, terminal) subword of w with the minimal length such that $u^{(1)} \in \overline{Y}$ (respectively, $u^{(0)} \in \overline{Y}$). In this case $\widetilde{u^{(1)}}$ (respectively, $\widetilde{u^{(0)}}$) will be denoted by $w^{[0]}$ (respectively, $w^{[1]}$). For every $\omega \in F[X]$ the set of all letters occurring in ω will be denoted by $c(\omega)$ and for every $w \in Frt(Y)$ assume $\widetilde{c}(w) = c(\widetilde{w})$.

Take an arbitrary nonempty finite subset C of Y. Let $B^C(Y)$ be the set of all finite subsets A of Y such that $C \subseteq A$ and let $B_C(Y)$ be a semilattice defined on $B^C(Y)$ by the operation of the set theoretical union.

Let $i, j, k, s \in Y$,

$$L = \{(i, j, k, s), (i, j, k), [i, j, k], [i, j], (i, j), (i, j], [i, j), (i), [i]\}$$

and

$$U_{(i,j,k,s)} = \{w \in Frt(Y) \mid (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) = (i, j, k, s)\},$$

$$U_{(i,j,k)} = \{w \in Frt(Y) \mid (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}) = (i, j, k)\},$$

$$U_{[i,j,k]} = \{w \in Frt(Y) \mid (w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) = (i, j, k)\},$$

$$U_{[i,j]} = \{w \in Frt(Y) \mid (w^{[0]}, w^{[1]}) = (i, j)\},$$

$$U_{(i,j)} = \{w \in Frt(Y) \mid (\widetilde{w}^{(0)}, \widetilde{w}^{(1)}) = (i, j)\},$$

$$U_{(i)} = \{w \in Frt(Y) \mid \widetilde{w}^{(0)} = i\},$$

$$U_{[i]} = \{w \in Frt(Y) \mid \widetilde{w}^{(1)} = i\}.$$

For any $l \in L$ assume l^* be the set containing all components of l. Consider the set

$$U_l^A = \{w \in U_l \mid \widetilde{c}(w) = A\}$$

for $A \in B_{l^*}(Y)$ and $l \in L \backslash \{(i, j], [i, j)\}$.

The following three structure theorems give decompositions of $Frt(Y)$ into tribands of subtrioids.

Theorem 3.1. *Let $Frt(Y)$ be the free trioid.*

(i) *$Frt(Y)$ is a triband $FRT(Y)$ of subtrioids $U_{(i,j,k,s)}$, $(i, j, k, s) \in FRT(Y)$. Every trioid $U_{(i,j,k,s)}$, $(i, j, k, s) \in FRT(Y)$, is a semilattice $B_{(i,j,k,s)^*}(Y)$ of subtrioids $U^A_{(i,j,k,s)}$, $A \in B_{(i,j,k,s)^*}(Y)$.*

(ii) *$Frt(Y)$ is a triband $Y_{lz,rd}$ of subtrioids $U_{(i,j,k)}$, $(i, j, k) \in Y_{lz,rd}$. Every trioid $U_{(i,j,k)}$, $(i, j, k) \in Y_{lz,rd}$, is a semilattice $B_{(i,j,k)^*}(Y)$ of subtrioids $U^A_{(i,j,k)}$, $A \in B_{(i,j,k)^*}(Y)$.*

(iii) *$Frt(Y)$ is a triband $Y_{rd,rz}$ of subtrioids $U_{[i,j,k]}$, $(i, j, k) \in Y_{rd,rz}$. Every trioid $U_{[i,j,k]}$, $(i, j, k) \in Y_{rd,rz}$, is a semilattice $B_{[i,j,k]^*}(Y)$ of subtrioids $U^A_{[i,j,k]}$, $A \in B_{[i,j,k]^*}(Y)$.*

(iv) *$Frt(Y)$ is a triband $Y^{rb}_{lz,rz}$ of subtrioids $U_{[i,j]}$, $(i, j) \in Y^{rb}_{lz,rz}$. Every trioid $U_{[i,j]}$, $(i, j) \in Y^{rb}_{lz,rz}$, is a semilattice $B_{[i,j]^*}(Y)$ of subtrioids $U^A_{[i,j]}$, $A \in B_{[i,j]^*}(Y)$.*

Proof. (i) Define a map

$$\varphi_{FRT} : Frt(Y) \to FRT(Y) \quad \text{by}$$
$$w \mapsto (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

For arbitrary elements $w,\ u \in Frt(Y)$ obtain

$$(w \dashv u)\varphi_{FRT} = (w\widetilde{u})\varphi_{FRT} =$$
$$= (\widetilde{w\widetilde{u}}^{(0)}, (w\widetilde{u})^{[0]}, (w\widetilde{u})^{[1]}, \widetilde{w\widetilde{u}}^{(1)}) =$$
$$= ((\widetilde{w}\widetilde{u})^{(0)}, w^{[0]}, w^{[1]}, (\widetilde{w}\widetilde{u})^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{u}^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) \dashv (\widetilde{u}^{(0)}, u^{[0]}, u^{[1]}, \widetilde{u}^{(1)}) =$$
$$= w\varphi_{FRT} \dashv u\varphi_{FRT},$$
$$(w \vdash u)\varphi_{FRT} = (\widetilde{w}u)\varphi_{FRT} =$$
$$= (\widetilde{\widetilde{w}u}^{(0)}, (\widetilde{w}u)^{[0]}, (\widetilde{w}u)^{[1]}, \widetilde{\widetilde{w}u}^{(1)}) =$$
$$= ((\widetilde{w}\widetilde{u})^{(0)}, u^{[0]}, u^{[1]}, (\widetilde{w}\widetilde{u})^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, u^{[0]}, u^{[1]}, \widetilde{u}^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) \vdash (\widetilde{u}^{(0)}, u^{[0]}, u^{[1]}, \widetilde{u}^{(1)}) =$$
$$= w\varphi_{FRT} \vdash u\varphi_{FRT},$$
$$(w \perp u)\varphi_{FRT} = (wu)\varphi_{FRT} =$$
$$= (\widetilde{wu}^{(0)}, (wu)^{[0]}, (wu)^{[1]}, \widetilde{wu}^{(1)}) =$$
$$= ((\widetilde{w}\widetilde{u})^{(0)}, w^{[0]}, u^{[1]}, (\widetilde{w}\widetilde{u})^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, u^{[1]}, \widetilde{u}^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}) \perp (\widetilde{u}^{(0)}, u^{[0]}, u^{[1]}, \widetilde{u}^{(1)}) =$$
$$= w\varphi_{FRT} \perp u\varphi_{FRT}.$$

Thus, φ_{FRT} is a surjective homomorphism. It is clear that $U_{(i,j,k,s)}$, $(i,j,k,s) \in FRT(Y)$, is a class of $\Delta_{\varphi_{FRT}}$ which is a subtrioid of $Frt(Y)$. Moreover, for every $(i,j,k,s) \in FRT(Y)$ the map

$$\zeta : U_{(i,j,k,s)} \to\ B_{(i,j,k,s)^*}(Y) : w \mapsto \widetilde{c}(w)$$

is a homomorphism. Indeed,

$$(w \dashv u)\zeta = (w\widetilde{u})\zeta = \widetilde{c}(w\widetilde{u}) = c(\widetilde{w\widetilde{u}}) =$$
$$= c(\widetilde{w}\widetilde{u}) = c(\widetilde{w}) \cup c(\widetilde{u}) = \widetilde{c}(w) \cup \widetilde{c}(u) =$$
$$= w\zeta \cup u\zeta,$$
$$(w \vdash u)\zeta = (\widetilde{w}u)\zeta = \widetilde{c}(\widetilde{w}u) = c(\widetilde{\widetilde{w}u}) =$$
$$= c(\widetilde{w}\widetilde{u}) = c(\widetilde{w}) \cup c(\widetilde{u}) = \widetilde{c}(w) \cup \widetilde{c}(u) =$$
$$= w\zeta \cup u\zeta,$$
$$(w \perp u)\zeta = (wu)\zeta = \widetilde{c}(wu) = c(\widetilde{wu}) =$$
$$= c(\widetilde{w}\widetilde{u}) = c(\widetilde{w}) \cup c(\widetilde{u}) = \widetilde{c}(w) \cup \widetilde{c}(u) =$$
$$= w\zeta \cup u\zeta$$

for all $w, u \in U_{(i,j,k,s)}$. Hence $U_{(i,j,k,s)}$ is a semilattice $B_{(i,j,k,s)^*}(Y)$ of subtrioids $U^A_{(i,j,k,s)}$, $A \in B_{(i,j,k,s)^*}(Y)$.

(ii) Define a map

$$\varphi_{lz,rd} : Frt(Y) \to Y_{lz,rd} \quad \text{by}$$
$$w \mapsto (\widetilde{w}^{(0)}, w^{[0]}, w^{[1]}), w \in Frt(Y).$$

One can check that $\varphi_{lz,rd}$ is a surjective homomorphism and $U_{(i,j,k)}$, $(i,j,k) \in Y_{lz,rd}$, is a class of $\Delta_{\varphi_{lz,rd}}$ which is a subtrioid of $Frt(Y)$. Similarly to (i), the second statement of (ii) can be proved.

(iii) Define a map

$$\varphi_{rd,rz} : Frt(Y) \to Y_{rd,rz} \quad \text{by}$$
$$w \mapsto (w^{[0]}, w^{[1]}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

It can be shown that $\varphi_{rd,rz}$ is a surjective homomorphism and $U_{[i,j,k]}$, $(i,j,k) \in Y_{rd,rz}$, is a class of $\Delta_{\varphi_{rd,rz}}$ which is a subtrioid of $Frt(Y)$. As before, the last statement of (iii) can be proved.

(iv) Define a map

$$\varphi^{rb}_{lz,rz} : Frt(Y) \to Y^{rb}_{lz,rz} \quad \text{by}$$
$$w \mapsto (w^{[0]}, w^{[1]}), w \in Frt(Y).$$

By direct verification we can claim that $\varphi^{rb}_{lz,rz}$ is a surjective homomorphism and $U_{[i,j]}$, $(i,j) \in Y^{rb}_{lz,rz}$, is a class of $\Delta_{\varphi^{rb}_{lz,rz}}$ which is a subtrioid of $Frt(Y)$. As above, the second statement of (iv) can be proved. $\square$

For all $i,j,k \in Y$ let

$$R_{(i,j,k)} = \{w \in Frt(Y) \mid$$
$$(\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}) = (i,j,k)\},$$
$$R_{[i,j,k]} = \{w \in Frt(Y) \mid$$
$$(\widetilde{w}^{(0)}, w^{[1]}, \widetilde{w}^{(1)}) = (i,j,k)\},$$
$$R_{(i)} = \{w \in Frt(Y) \mid w^{[0]} = i\},$$
$$R_{[i]} = \{w \in Frt(Y) \mid w^{[1]} = i\},$$
$$R_{(i,j)} = \{w \in Frt(Y) \mid (\widetilde{w}^{(0)}, w^{[0]}) = (i,j)\},$$
$$R_{[i,j]} = \{w \in Frt(Y) \mid (\widetilde{w}^{(0)}, w^{[1]}) = (i,j)\},$$
$$R_{(i,j]} = \{w \in Frt(Y) \mid (w^{[0]}, \widetilde{w}^{(1)}) = (i,j)\},$$
$$R_{[i,j)} = \{w \in Frt(Y) \mid (w^{[1]}, \widetilde{w}^{(1)}) = (i,j)\}.$$

Consider the set

$$R^A_l = \{w \in R_l \mid \widetilde{c}(w) = A\}$$

for $A \in B_{l^*}(Y)$ and $l \in L \backslash \{(i,j,k,s)\}$.

Theorem 3.2. *Let $Frt(Y)$ be the free trioid.*

(i) *$Frt(Y)$ is a triband $(FRct(Y))^{\dashv}$ of subtrioids $R_{(i,j,k)}$, $(i,j,k) \in (FRct(Y))^{\dashv}$. Every trioid $R_{(i,j,k)}$, $(i,j,k) \in (FRct(Y))^{\dashv}$, is a semilattice $B_{(i,j,k)^*}(Y)$ of subtrioids $R^A_{(i,j,k)}$, $A \in B_{(i,j,k)^*}(Y)$.*

(ii) *$Frt(Y)$ is a triband $(FRct(Y))^{\vdash}$ of subtrioids $R_{[i,j,k]}$, $(i,j,k) \in (FRct(Y))^{\vdash}$. Every trioid $R_{[i,j,k]}$, $(i,j,k) \in (FRct(Y))^{\vdash}$, is a semilattice $B_{[i,j,k]^*}(Y)$ of subtrioids $R^A_{[i,j,k]}$, $A \in B_{[i,j,k]^*}(Y)$.*

(iii) *$Frt(Y)$ is a triband $(Y_{lz,rz})^{\dashv}$ of subtrioids $R_{(i)}$, $i \in (Y_{lz,rz})^{\dashv}$. Every trioid $R_{(i)}$, $i \in (Y_{lz,rz})^{\dashv}$, is a semilattice $B_{\{i\}}(Y)$ of subtrioids $R^A_{(i)}$, $A \in B_{\{i\}}(Y)$.*

(iv) *$Frt(Y)$ is a triband $(Y_{lz,rz})^{\vdash}$ of subtrioids $R_{[i]}$, $i \in (Y_{lz,rz})^{\vdash}$. Every trioid $R_{[i]}$, $i \in (Y_{lz,rz})^{\vdash}$, is a semilattice $B_{\{i\}}(Y)$ of subtrioids $R^A_{[i]}$, $A \in B_{\{i\}}(Y)$.*

Proof. (i) Define a map

$$\varphi^{\dashv}_{FRct} : Frt(Y) \to (FRct(Y))^{\dashv} \quad \text{by}$$

$$w \mapsto (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

For any w, $u \in Frt(Y)$ obtain

$$(w \dashv u)\varphi^{\dashv}_{FRct} = (w\widetilde{u})\varphi^{\dashv}_{FRct} =$$
$$= (\widetilde{w\widetilde{u}}^{(0)}, (w\widetilde{u})^{[0]}, \widetilde{w\widetilde{u}}^{(1)}) =$$
$$= ((\widetilde{w}\widetilde{u})^{(0)}, w^{[0]}, (\widetilde{w}\widetilde{u})^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{u}^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}) \dashv (\widetilde{u}^{(0)}, u^{[0]}, \widetilde{u}^{(1)}) =$$
$$= w\varphi^{\dashv}_{FRct} \dashv u\varphi^{\dashv}_{FRct},$$
$$(w \perp u)\varphi^{\dashv}_{FRct} = (wu)\varphi^{\dashv}_{FRct} =$$
$$= (\widetilde{wu}^{(0)}, (wu)^{[0]}, \widetilde{wu}^{(1)}) =$$
$$= ((\widetilde{w}\widetilde{u})^{(0)}, w^{[0]}, (\widetilde{w}\widetilde{u})^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{u}^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}) \dashv (\widetilde{u}^{(0)}, u^{[0]}, \widetilde{u}^{(1)}) =$$
$$= (\widetilde{w}^{(0)}, w^{[0]}, \widetilde{w}^{(1)}) \perp (\widetilde{u}^{(0)}, u^{[0]}, \widetilde{u}^{(1)}) =$$
$$= w\varphi^{\dashv}_{FRct} \perp u\varphi^{\dashv}_{FRct}.$$

Similarly for $\vdash$. So, $\varphi^{\dashv}_{FRct}$ is a surjective homomorphism. It is evident that $R_{(i,j,k)}$, $(i,j,k) \in (FRct(Y))^{\dashv}$, is a class of $\Delta_{\varphi^{\dashv}_{FRct}}$ which is a subtrioid of $Frt(Y)$. Furthermore, it is not hard to prove that for every $(i,j,k) \in (FRct(Y))^{\dashv}$ the map

$$R_{(i,j,k)} \to \ B_{(i,j,k)^*}(Y) : w \mapsto \widetilde{c}(w)$$

is a homomorphism. Hence $R_{(i,j,k)}$ is a semilattice $B_{(i,j,k)^*}(Y)$ of subtrioids $R^A_{(i,j,k)}$, $A \in B_{(i,j,k)^*}(Y)$.

(ii) Define a map

$$\varphi^{\vdash}_{FRct} : Frt(Y) \to (FRct(Y))^{\vdash} \quad \text{by}$$
$$w \mapsto (\widetilde{w}^{(0)}, w^{[1]}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

It is not difficult to show that $\varphi^{\vdash}_{FRct}$ is a surjective homomorphism and $R_{[i,j,k]}$, $(i,j,k) \in (FRct(Y))^{\vdash}$, is a class of $\Delta_{\varphi^{\vdash}_{FRct}}$ which is a subtrioid of $Frt(Y)$. Similarly to the case (i), the second assertion of (ii) can be proved.

(iii) Define a map

$$\varphi^{\dashv}_{lz,rz} : Frt(Y) \to (Y_{lz,rz})^{\dashv} \quad \text{by}$$
$$w \mapsto w^{[0]}, w \in Frt(Y).$$

We can show that $\varphi^{\dashv}_{lz,rz}$ is a surjective homomorphism and $R_{(i)}$, $i \in (Y_{lz,rz})^{\dashv}$, is a class of $\Delta_{\varphi^{\dashv}_{lz,rz}}$ which is a subtrioid of $Frt(Y)$. Similarly to (i), the last statement of (iii) can be proved.

(iv) Define a map

$$\varphi^{\vdash}_{lz,rz} : Frt(Y) \to (Y_{lz,rz})^{\vdash} \quad \text{by}$$
$$w \mapsto w^{[1]}, w \in Frt(Y).$$

It is immediate to check that $\varphi^{\vdash}_{lz,rz}$ is a surjective homomorphism and $R_{[i]}$, $i \in (Y_{lz,rz})^{\vdash}$, is a class of $\Delta_{\varphi^{\vdash}_{lz,rz}}$ which is a subtrioid of $Frt(Y)$. As above, the second statement of (iv) can be proved. $\square$

Theorem 3.3. *Let $Frt(Y)$ be the free trioid.*

(i) *$Frt(Y)$ is a triband $(Y_{lz,rb})^{\dashv}$ of subtrioids $R_{(i,j)}$, $(i,j) \in (Y_{lz,rb})^{\dashv}$. Every trioid $R_{(i,j)}$, $(i,j) \in (Y_{lz,rb})^{\dashv}$, is a semilattice $B_{(i,j)^*}(Y)$ of subtrioids $R^A_{(i,j)}$, $A \in B_{(i,j)^*}(Y)$.*

(ii) *$Frt(Y)$ is a triband $(Y_{lz,rb})^{\vdash}$ of subtrioids $R_{[i,j]}$, $(i,j) \in (Y_{lz,rb})^{\vdash}$. Every trioid $R_{[i,j]}$, $(i,j) \in (Y_{lz,rb})^{\vdash}$, is a semilattice $B_{[i,j]^*}(Y)$ of subtrioids $R^A_{[i,j]}$, $A \in B_{[i,j]^*}(Y)$.*

(iii) *$Frt(Y)$ is a triband $(Y_{rb,rz})^{\dashv}$ of subtrioids $R_{(i,j]}$, $(i,j) \in (Y_{rb,rz})^{\dashv}$. Every trioid $R_{(i,j]}$, $(i,j) \in (Y_{rb,rz})^{\dashv}$, is a semilattice $B_{(i,j]^*}(Y)$ of subtrioids $R^A_{(i,j]}$, $A \in B_{(i,j]^*}(Y)$.*

(iv) *$Frt(Y)$ is a triband $(Y_{rb,rz})^{\vdash}$ of subtrioids $R_{[i,j)}$, $(i,j) \in (Y_{rb,rz})^{\vdash}$. Every trioid $R_{[i,j)}$, $(i,j) \in (Y_{rb,rz})^{\vdash}$, is a semilattice $B_{[i,j)^*}(Y)$ of subtrioids $R^A_{[i,j)}$, $A \in B_{[i,j)^*}(Y)$.*

Proof. (i) Define a map

$$\varphi^{\dashv}_{lz,rb} : Frt(Y) \to (Y_{lz,rb})^{\dashv} \quad \text{by}$$

$$w \mapsto (\widetilde{w}^{(0)}, w^{[0]}), w \in Frt(Y).$$

We can directly prove that $\varphi^{\dashv}_{lz,rb}$ is a surjective homomorphism and $R_{(i,j)}$, $(i,j) \in (Y_{lz,rb})^{\dashv}$, is a class of $\Delta_{\varphi^{\dashv}_{lz,rb}}$ which is a subtrioid of $Frt(Y)$. Similarly to (i) from Theorem 3.1, the second statement of (i) can be proved.

(ii) Define a map

$$\varphi^{\vdash}_{lz,rb} : Frt(Y) \to (Y_{lz,rb})^{\vdash} \quad \text{by}$$

$$w \mapsto (\widetilde{w}^{(0)}, w^{[1]}), w \in Frt(Y).$$

It can easily be checked that $\varphi^{\vdash}_{lz,rb}$ is a surjective homomorphism and $R_{[i,j]}$, $(i,j) \in (Y_{lz,rb})^{\vdash}$, is a class of $\Delta_{\varphi^{\vdash}_{lz,rb}}$ which is a subtrioid of $Frt(Y)$. In the same way as above, the second assertion of (ii) can be proved.

(iii) Define a map

$$\varphi^{\dashv}_{rb,rz} : Frt(Y) \to (Y_{rb,rz})^{\dashv} \quad \text{by}$$

$$w \mapsto (w^{[0]}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

By immediate verification we can state that $\varphi^{\dashv}_{rb,rz}$ is a surjective homomorphism and $R_{(i,j]}$, $(i,j) \in (Y_{rb,rz})^{\dashv}$, is a class of $\Delta_{\varphi^{\dashv}_{rb,rz}}$ which is a subtrioid of $Frt(Y)$. As before, the last assertion of (iii) can be proved.

(iv) Define a map

$$\varphi^{\vdash}_{rb,rz} : Frt(Y) \to (Y_{rb,rz})^{\vdash} \quad \text{by}$$

$$w \mapsto (w^{[1]}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

One can prove that $\varphi^{\vdash}_{rb,rz}$ is a surjective homomorphism and $R_{[i,j)}$, $(i,j) \in (Y_{rb,rz})^{\vdash}$, is a class of $\Delta_{\varphi^{\vdash}_{rb,rz}}$ which is a subtrioid of $Frt(Y)$. Similarly to (i) from Theorem 3.1, the second statement of (iv) can be proved. $\square$

The following structure theorem gives decompositions of $Frt(Y)$ into bands of subtrioids.

Theorem 3.4. *Let $Frt(Y)$ be the free trioid.*

(i) $Frt(Y)$ is a rectangular band Y_{rb} of subtrioids $U_{(i,j)}$, $(i,j) \in Y_{rb}$. Every trioid $U_{(i,j)}$, $(i,j) \in Y_{rb}$, is a semilattice $B_{(i,j)^*}(Y)$ of subtrioids $U^A_{(i,j)}$, $A \in B_{(i,j)^*}(Y)$.

(ii) $Frt(Y)$ is a left band Y_{lz} of subtrioids $U_{(i)}$, $i \in Y_{lz}$. Every trioid $U_{(i)}$, $i \in Y_{lz}$, is a semilattice $B_{\{i\}}(Y)$ of subtrioids $U^A_{(i)}$, $A \in B_{\{i\}}(Y)$.

(iii) $Frt(Y)$ is a right band Y_{rz} of subtrioids $U_{[i]}$, $i \in Y_{rz}$. Every trioid $U_{[i]}$, $i \in Y_{rz}$, is a semilattice $B_{\{i\}}(Y)$ of subtrioids $U^A_{[i]}$, $A \in B_{\{i\}}(Y)$.

Proof. (i) Define a map

$$\varphi_{rb} : Frt(Y) \to Y_{rb} \quad \text{by}$$

$$w \mapsto (\widetilde{w}^{(0)}, \widetilde{w}^{(1)}), w \in Frt(Y).$$

One can verify that φ_{rb} is a surjective homomorphism and $U_{(i,j)}$, $(i,j) \in Y_{rb}$, is a class of $\Delta_{\varphi_{rb}}$ which is a subtrioid of $Frt(Y)$. Similarly to (i) from Theorem 3.1, the last assertion of (i) can be proved.

(ii) Define a map

$$\varphi_{lz} : Frt(Y) \to Y_{lz} \quad \text{by}$$

$$w \mapsto \widetilde{w}^{(0)}, w \in Frt(Y).$$

It is easily shown that φ_{lz} is a surjective homomorphism and $U_{(i)}$, $i \in Y_{lz}$, is a class of $\Delta_{\varphi_{lz}}$ which is a subtrioid of $Frt(Y)$. As above, the second statement of (ii) can be proved.

(iii) Define a map

$$\varphi_{rz} : Frt(Y) \to Y_{rz} \quad \text{by}$$

$$w \mapsto \widetilde{w}^{(1)}, w \in Frt(Y).$$

It can easily be checked that φ_{rz} is a surjective homomorphism and $U_{[i]}$, $i \in Y_{rz}$, is a class of $\Delta_{\varphi_{rz}}$ which is a subtrioid of $Frt(Y)$. As before, the last assertion of (iii) can be proved. $\square$

If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that operations of $(T, \dashv, \vdash, \perp)/_{\rho}$ coincide and it is a rectangular band (respectively, left zero semigroup, right zero semigroup), then we say that ρ is a rectangular band congruence (respectively, left zero congruence, right zero congruence).

From Theorem 3.4 we obtain

Corollary 1. *Let $Frt(Y)$ be the free trioid.*

(i) $\Delta_{\varphi_{rb}}$ is the least rectangular band congruence on $Frt(Y)$.

(ii) $\Delta_{\varphi_{lz}}$ is the least left zero congruence on $Frt(Y)$.

(iii) $\Delta_{\varphi_{rz}}$ is the least right zero congruence on $Frt(Y)$.

Proof. (i) It is well-known that Y_{rb} is the free rectangular band. By Theorem 3.4 (i) we obtain (i).

The proofs of (ii) and (iii) are similar. $\square$

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