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Журавльов В.М.[1], доцент
Сурков А.М.[2], аспірант

Допустимі сагайдаки та черепичні порядки скінченної глобальної розмірності в $M_n(\mathcal{D})$, $n \leq 5$

[1,2]Київський національний університет імені Тараса Шевченка, Україна, 01601, м. Київ, вул. Володимирська, 64
E-mail: [1]vshur@univ.kiev.ua

Zhuravlev V.N.[1], Associate Professor
Surkov A.M.,[2] PhD student

Admissible quivers and tiled orders of finite global dimension in $M_n(\mathcal{D})$, $n \leq 5$

[1,2]Taras Shevchenko National University of Kyiv, 01033, Kyiv, 64 Volodymyrska st.
E-mail: [1]vshur@univ.kiev.ua

У роботі вивчаються сагайдаки черепичних порядків скінченної глобальної розмірності. Показано, що не кожен сагайдак зведеного черепичного порядку є сагайдаком чере пичного порядку скінченної глобальної розмірності. Для сагайдаків, що мають не більше п'яти вершин і які є сагайдаками черепичних порядків, доведено наступне. Якщо для такого сагайдака існує зведений черепичний порядок скінченної глобальної розмірності з даним сагайдаком, то такий порядок єдиний з точністю до ізоморфізму. Більш того, такий порядок має мінімальну суму елементів матриці показників поміж усіх черепичних порядків з даним сагайдаком. Черепичні порядки скінченної глобальної розмірності в $M_n(K)$, де $M_n(K)$ – повне матричне кільце над полем K, описані у праці [5] для $n \leq 5$. У даній роботі започатковано дослідження черепичних порядків скінченної глобальної розмірності та їх сагайдаків.

Ключові слова: черепичний порядок, допустимий сагайдак, глобальна розмірність.

We study quivers of tiled orders of finite global dimension. It is well known that the quiver of tiled order of a finite global dimension has no loops. Tiled orders associated with the partially ordered sets with disconnected diagrams have infinite global dimension. It is shown that not every quiver of reduced tiled order is quiver of tiled order of finite global dimension. For quivers on at most five vertices and which are tiled orders, proved the following. If for such quiver there is reduced tiled order of finite global dimension with the given quiver, then such tiled order is unique up to isomorphism. Moreover, such tiled order has the minimum sum of all entries of exponent matrix among all tiled orders with the given quiver. Tiled orders of finite global dimension in $M_n(K)$, where $M_n(K)$ is the ring of all $n \times n$ matrices over a ring K were listed in the following paper H.Fujita, Tiled orders of finite global dimension, Trans. Amer. Math. Soc., v. 327, No.2 (1991), pp. 919-920 for $n \leq 5$. In this paper research of tiled orders of finite global dimension and their quivers was founded.

Key words: tiled order, admissible quiver, global dimension of tiled orders.
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1 Introduction

Tiled orders appeared in the 1970s. In the first articles about tiled orders (see [7], [3], [4]) can be found attempts to calculate global dimentions of those tiled orders. V.A. Jategaonkar in [3] proved that up to isomorphism only finite number of tiled orders in $M_n(\mathcal{D})$ of finite global dimension exists. R.B. Tarsy assumed that finite global dimension of tiled order does not exceed $n-1$. H. Fujita described in [5] up to isomorphism all tiled orders in $M_n(\mathcal{D})$ where $n = 4, 5$. Later Fujita disproved Tarsy's hypothesis with counterexample. Some algebraists in order to construct tiled orders of large finite global dimension were using quiver of tiled order.

It the article research of the connections between admissible quivers (namely quivers of tiled orders) and tiled orders of finite global dimension was founded. Primary aim of the article is to prove that for every admissible quiver with n vertices where $n \leq 5$ up to isomorphism no more than one tiled order of finite global dimension might exist. In the case when admissible quiver is quiver of nonisomorphic tiled orders, tiled order of finite global dimension only when sum of entries of exponent matrix is minimal.

All additional facts about tiled orders, exponent matrices and their quivers can be found in [1], [2]. The list of all up to isomorphism tiled orders in $M_n(\mathcal{D})$ where $n = 4, 5$ can be found in [5].

2 Tiled orders over discrete valuation rings and exponent matrices

Recall [6] that a *semimaximal ring* is a semiperfect semiprime right Noetherian ring A such that for each primitive idempotent $e \in A$ the ring eAe is a discrete valuation ring (not necessarily commutative).

Denote by $M_n(\mathcal{D})$ the ring of all $n \times n$ matrices over a ring $\mathcal{D}$.

Theorem 2.1 (see [6]). *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the form*

$$\Lambda = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \ldots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \ldots & \pi^{\alpha_{2n}}\mathcal{O} \\ \ldots & \ldots & \ldots & \ldots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \ldots & \mathcal{O} \end{pmatrix}, \qquad (1)$$

where $n \geq 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and α_{ij} are integers such that

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}, \qquad \alpha_{ii} = 0$$

for all i, j, k.

The ring $\mathcal{O}$ is embedded into its classical division ring of fractions $\mathcal{D}$, and (1) is the set of all matrices $(a_{ij}) \in M_n(\mathcal{D})$ such that

$$a_{ij} \in \pi^{\alpha_{ij}}\mathcal{O} = e_{ii}\Lambda e_{jj},$$

where $e_{11}, \ldots, e_{nn}$ are the matrix units of $M_n(\mathcal{D})$. It is clear that $Q = M_n(\mathcal{D})$ is the classical ring of fractions of Λ.

Obviously, the ring A is right and left Noetherian.

Definition 2.2. *A module M is* distributive *if its lattice of submodules is distributive, i.e.,*

$$K \cap (L + N) = K \cap L + K \cap N$$

for all submodules K, L, and N.

Clearly, any submodule and any factormodule of a distributive module are distributive modules.

A *semidistributive module* is a direct sum of distributive modules. A ring A is *right* (*left*) *semidistributive* if it is semidistributive as the right (left) module over itself. A ring A is *semidistributive* if it is both left and right semidistributive (see [1]).

Theorem 2.3 (see [1]). *The following conditions for a semiperfect semiprime right Noetherian ring A are equivalent:*

- *A is semidistributive;*
- *A is a direct product of a semisimple artinian ring and a semimaximal ring.*

By a *tiled order* over a discrete valuation ring, we mean a Noetherian prime semiperfect semidistributive ring Λ with nonzero Jacobson radical. In this case, $\mathcal{O} = e\Lambda e$ is a discrete valuation ring with a primitive idempotent $e \in \Lambda$.

Definition 2.4. *An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called*

- *an* exponent matrix *if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k;*
- *a* reduced exponent matrix *if $\alpha_{ij} + \alpha_{ji} > 0$ for all i, j, $i \neq j$.*

We use the following notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E}(\Lambda) = (\alpha_{ij})$ is the exponent matrix of the ring Λ, i.e.

$$\Lambda = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O},$$

in which e_{ij} are the matrix units. If a tiled order is *reduced*, i.e. $\Lambda/R(\Lambda)$ is the direct product of division rings, then $\alpha_{ij} + \alpha_{ji} > 0$ if $i \neq j$, i.e., $\mathcal{E}(\Lambda)$ is reduced.

We denote by $\mathcal{M}(A)$ the poset (ordered by inclusion) of all projective right A-modules that are contained in a fixed simple Q-module U. All simple Q-modules are isomorphic, so we can choice one of them. Note that the partially ordered sets $\mathcal{M}_l(A)$ and $\mathcal{M}_r(A)$ corresponding to the left and the right modules are anti-isomorphic.

The set $\mathcal{M}(A)$ is completely determined by the exponent matrix $\mathcal{E}(A) = (\alpha_{ij})$. Namely, if A is reduced, then

$$\mathcal{M}(A) = \{p_i^z \mid i = 1, \ldots n, \text{and } z \in \mathbb{Z}\},$$

where

$$p_i^z \le p_j^{z'} \iff \begin{cases} z - z' \ge \alpha_{ij} & \text{if } \mathcal{M}(A) = \mathcal{M}_l(A) \\ z - z' \ge \alpha_{ji} & \text{if } \mathcal{M}(A) = \mathcal{M}_r(A) \end{cases}$$

Obviously, $\mathcal{M}(A)$ is an infinite periodic set.

Recall that a subset of a poset P is called an *antichain* if all of its elements are pairwise incomparable. The maximal number $w(P)$ of elements in an antichain in P is called the *width* of P. The width of $\mathcal{M}(A)$ is called the *width of a tiled order* A and is denoted by $w(A)$.

Let A and Γ be tiled orders over discrete valuation rings $\mathcal{O}$ and Δ.

3 Quivers of tiled orders

Let I be a two-sided ideal of the tiled order Λ. Obviously,

$$I = \sum_{i,j=1}^{n} e_{ij}\pi^{\beta_{ij}}\mathcal{O},$$

where e_{ij} are matrix units. Denote by $\mathcal{E}(I) = (\beta_{ij})$ the exponent matrix of an ideal I.

Let I and J be two-sided ideals of Λ, $\mathcal{E}(I) = (\beta_{ij})$ and $\mathcal{E}(J) = (\gamma_{ij})$. We have $\mathcal{E}(IJ) = (\delta_{ij})$, where $\delta_{ij} = \min_k(\beta_{ik} + \gamma_{kj})$.

Let R be the Jacobson radical of a reduced tiled order Λ, then $\mathcal{E}(R) = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \ne j$ and $\beta_{ij} = 1$ for $i = 1, \dots, n$.

Let $Q(\Lambda)$ be a quiver of a reduced tiled order Λ ([1]) and let $[Q(\Lambda)]$ be an adjacency matrix of the quiver $Q(\Lambda)$. Obviously, [1, theorema 14.6.2] $[Q(\Lambda)] = \mathcal{E}(R^2) - \mathcal{E}(R)$ and $[Q(\Lambda)]$ is a $(0,1)$-matrix.

Let $\mathcal{E} = (\alpha_{ij})$ be an $n \times n$ reduced exponent matrix. Define the $n \times n$ matrices $\mathcal{E}^{(1)} = (\beta_{ij})$ and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where

$$\beta_{ij} = \begin{cases} \alpha_{ij}, & \text{if } i \ne j, \\ 1, & \text{if } i = j, \end{cases} \qquad \gamma_{ij} = \min_{1 \le k \le n}(\beta_{ik} + \beta_{kj}).$$

Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is a $(0,1)$-matrix. By [1, Theorem 4.1.1 and Corollary 5.3], we have the following assertion.

Theorem 3.1. *The matrix* $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ *is the adjacency matrix of the strongly connected simply laced quiver* $Q = Q(\mathcal{E})$.

Definition 3.2. *The quiver* $Q(\mathcal{E})$ *is called the* quiver of the reduced exponent matrix $\mathcal{E}$. *A strongly connected simply laced quiver is called* admissible *if it is the quiver of a reduced exponent matrix.*

Remark 1. *The quiver* $Q(A)$ *of a reduced tiled order* A *coincides with* $Q(\mathcal{E}(A))$.

Definition 3.3. *The quiver* $Q = (VQ, AQ)$ *is called* weighted *if there is a function* $w\colon AQ \to \mathbb{R}$. w *is called weight function. The value of* w *on an arrow of* Q *is called weight of the arrow. The algebraic sum of weights of all arrows of a path is called weight of the path.*

Theorem 3.4. *Simple strongly connected quiver* $Q = (VQ, AQ)$ *is admissible if and only if there is a weight function* $w\colon AQ \to \mathbb{N} \cup \{0\}$ *meeting the following conditions:*
(1) weight of the arrow from the point i *to the point* j *is less than weight of the path from the point* i *to the point* j *of the length* $l \ge 2$,
(2) loop weight in the point i *is less than the weight of any cycle of the length* $l \ge 2$*, passing through* i,
(3) weight of any cycle is always not less than 1,
(4) weight of any loop is 1,
(5) for each point without a loop there is a cycle of the length $l \ge 2$ *of the weight* 1*, passing through this point.*

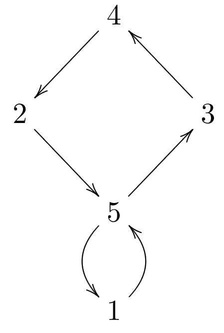
Definition 3.5. *Two reduced tiled orders in* $M_n(D)$ *are isomorphic if and only if their exponent matrices can be obtained from each other by transformations of the following two types :*
(1) subtract an integer from i*-th row and add this number to* i*-th column;*
(2) transpose two rows and corresponding columns.

Quivers of isomorphic tiled orders are isomorphic (see [1])

4 Results

Proposition 4.1. *Not for every admissible quiver* Q *exists tiled order* Λ *of finite global dimension with* $Q(\Lambda) = Q$

Proof. As contrexample can be considered the following quiver.



The quiver is admissible. It is quiver of the tiled order with the following exponent matrix.

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

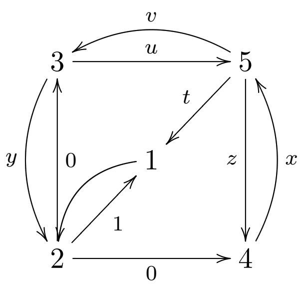
□

Proposition 4.2. *If for admissible quiver Q tiled orders of finite global dimension exist then they are isomorphic.*

Proof. Fujita in the article [5] describes all tiled orders in $M_5(\mathcal{D})$ up to isomorphism of finite global dimension. Fujita provides 40 tiled orders of finite global dimension and their quivers. So as quivers of that tiled orders are not isomorphic then tiled orders are also not isomorphic. □

Proposition 4.3. *Let Λ is tiled order with $\mathcal{E}(\Lambda) = (\alpha_{ij})$ of finite global dimension and L is tiled order with $\mathcal{E}(L) = (\lambda_{ij})$and $Q(L) = Q(\Lambda)$. Then $\sum_{i,j} \alpha_{ij} \leq \sum_{i,j} \lambda_{ij}$.*

Proof. Let's consider the following quiver of tiled order of finite global dimesnion in $M_5(\mathcal{D})$ and calculate minimum sum of exponent matrix of tiled order with the given quiver. Calculations are similar to the rest of tiled orders in [5].



$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$d = \sum_{i,j=1}^{5} \alpha_{ij} = 16.$

Let's find all tiled orders up to isomorphism with the given quiver.

All verteces of the quiver do not have loops. The following corresponding cycles of weight 1 are passing trought points 1 and 4 $1 \to 2 \to 1$ and $4 \to 5 \to 4$. Cycle of weight 1 is passing trought point 3 as well. Consider $3 \to 2 \to 3$ or $3 \to 5 \to 3$. Then $\alpha_{54} + \alpha_{45} = 1, \alpha_{12} + \alpha_{21} = 1$, $\left[\begin{array}{l} \alpha_{35} + \alpha_{53} = 1 \\ \alpha_{32} + \alpha_{23} = 1 \end{array} \right.$.

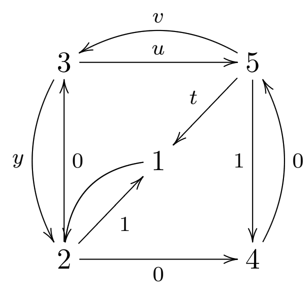
Let's assume that $\alpha_{1j} = 0 \ \forall j$ then $\alpha_{21} = 1$. From the following equalities $\alpha_{13} = \alpha_{12} + \alpha_{23}$, $\alpha_{14} = \alpha_{12} + \alpha_{24}$, $\left[\begin{array}{l} \alpha_{15} = \alpha_{13} + \alpha_{35} \\ \alpha_{15} = \alpha_{14} + \alpha_{45} \end{array} \right.$ we obtain $\alpha_{23} = 0$, $\alpha_{24} = 0$ and $\left[\begin{array}{l} \alpha_{35} = 0 \\ \alpha_{45} = 0 \end{array} \right.$.

Then $\left[\begin{array}{l} \alpha_{45} = x = 0 \\ \alpha_{35} = u = 0 \end{array} \right.$, $\left[\begin{array}{l} u + v = 1 \\ 0 + y = 1 \end{array} \right.$, $x + z = 1$. Moreover, $\begin{cases} u + v \geq 1 \\ 0 + y \geq 1 \end{cases}$.

According to (1) in 3.4 the following inequalities are true $u < 0 + y + 0 + x$, $v < t + 0 + 0$, $y < u + t + 0$, $\alpha_{23} = 0 < 0 + x + v$, $\alpha_{24} = 0 < 0 + u + z$, $z < t + 0 + 0$, $z < v + y + 0$, $t < v + y + 1$, $1 < 0 + x + t$, $1 < 0 + u + t$.

Examine the inequality $0 = \alpha_{24} < \alpha_{23} + \alpha_{35} + \alpha_{54} = 0 + u + z$. If $x = 1$ then $u = 0$ and $z = 0$. That's contradiction with $0 < u + z$. So $x = 0$ and z = 1.

Then quiver Q is

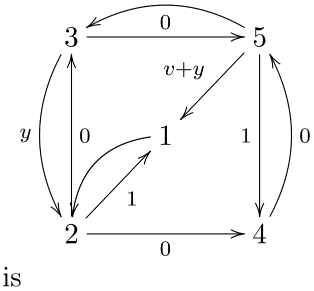


And the following equalities hold $\left[\begin{array}{l} u + v = 1 \\ y = 1 \end{array} \right.$, $\begin{cases} u + v \geq 1 \\ y \geq 1 \end{cases}$, $u < y, v < t, y < u + t, 0 < v$, $0 < u + 1, 1 < t, 1 < v + y$, $t < v + y + 1, 1 < t, 1 < u + t$.

Suppose that $u \geq 1$ then $1 \leq u < y, 1 \leq v$ so $y \geq 2, u + v \geq 2$. But $\left[\begin{array}{l} u + v = 1 \\ y = 1 \end{array} \right.$. We got contradiction. Then $u = 0$. $0 < y < t, 0 < v < t, t < v + y + 1$, $\left[\begin{array}{l} v = 1 \\ y = 1 \end{array} \right.$, $\begin{cases} v \geq 1 \\ y \geq 1 \end{cases}$.

If $v = 1$ then $y < t < y + 2$ so $t = y + 1 = y + v$. If $y = 1$ then $v < t < v + 2$ so $t = v + 1 = v + y$. Then $t = v + y$. $\alpha_{31} = min(y + 1, 0 + t) = y + 1$.

Finally, quiver is



and exponent matrix is

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ y+1 & y & 0 & 1 & 0 \\ v+y & v+y & v & 0 & 0 \\ v+y & v+y & v & 1 & 0 \end{pmatrix}.$$

$d = 4(v+y) + 2y + 2v + 4 = 6(v+y) + 4$. So $d_{min} = 6(1+1) + 4 = 16$. □

5 Conclusion

If for admissible quiver there is tiled order of finite global dimension with the given quiver then the tiled order is unique up to isomorphism. Furthermore, sum of entries of exponent matrix of the tiled order is minimal among all tiled orders with the given quiver.

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