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Системи лінійних функціональних рівнянь в задачі про топологічну спряженість відображень

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Linear functional equations systems in the problem of topological conjugation of mappings

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Вивчено розв'язки лінійних функціональних рівнянь, котрі виникають в задачі встановлення топологічної спряженості унімодальних кусково лінійних відображень інтервалу в себе, котрі складаються з двох кусків лінійності та чий образ містить весь інтервал.

Ми вивчаємо функціональне рівняння, яке отримується після підстановки розв'язку одного з двох лінійних функціональних рівнянь (який знаходиться з точністю до довільної функції) в інше з метою знаходження цієї довільної функції. Ми показуємо складність отримання явних формул для розв'язання функціонального рівняння, яке після такої дії отримується та вивчаємо ітераційні наближення довільної функції, що фігурує у розв'язанні одного з двох функціональних рівнянь.

Ключові слова: Одновимірна динаміка, відображення-капелюшок, топологічна спряженість.

The article deals with topological conjugacy problem for piecewise linear unimodal interval into itself mappings. It is considered such pairs of mappings that the graph of each of them consists of two linear parts and graph of one of them is symmetrical in the center of the function domain. The system of two linear functional equations which determines the topological conjugateness of mentioned maps is studied. The techniques of solving linear functional equations is used for each one of this equations and substituting the solution into another one. As these solutions contain the arbitrary function the substitution makes the second equation to be the equation for that arbitrary function. It is shown that obtained functional equation is complicated and properties of its solutions are studied. We show the complicateness of applying the linear functional equations solving methods for finding the explicit formula for the homeomorphism we study and consider the iterational approximations of the arbitrary function from the explicit formula for the solution of one two functional equations.

Key Words: One-dimensional dynamics, hat mapping, topological conjugateness.

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Introduction

We consider methods of solving the functional equations in the problem of finding the homeomorphism which define the topological conjugateness of mapping

$$f(x) = \begin{cases} 2x, & x < 1/2; \\ 2-2x, & x \geqslant 1/2, \end{cases} \tag{1}$$

and mapping

$$f_v(x) = \begin{cases} \frac{x}{v}, & x \leqslant v; \\ \frac{1-x}{1-v}, & x > v, \end{cases} \tag{2}$$

each of them is defined on the interval $[0; 1]$.

Remind that a mapping f and $\widetilde{f} \in C([0; 1])$ are called topological conjugate if there exists a homeomorphism $h \in C([0; 1])$ such that the following diagram

$$\begin{array}{ccc} [0;\,1] & \xrightarrow{f} & [0;\,1] \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ [0;\,1] & \xrightarrow{\widetilde{f}} & [0;\,1], \end{array}$$

is commutative i.e. the equality

$$h(f(x)) = f_v(h(x)). \tag{3}$$

holds for every $x \in [0; 1]$.

Taking into attention the explicit formulas for mappings f and f_v the commutativeness of the diagram from the definition of the topological conjugate is equivalent to the following system of functional equations.

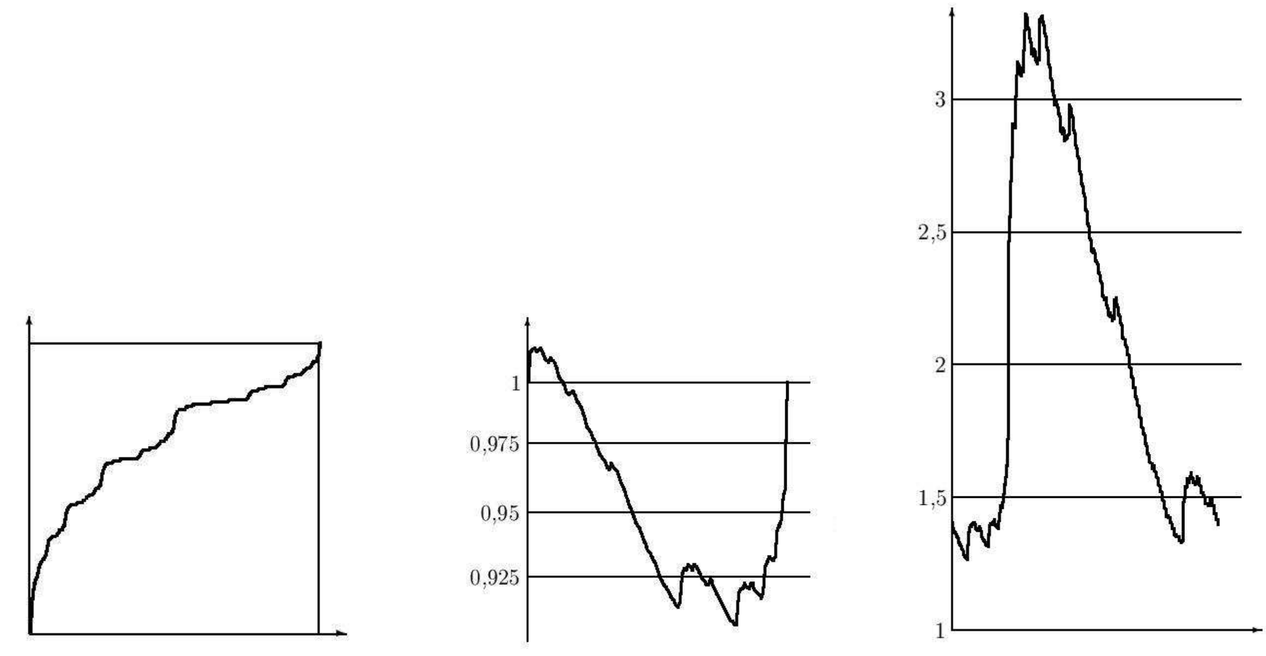
$$\begin{cases} h(2x) = \dfrac{1}{v}\, h(x) & x \leqslant 1/2 \quad (4a) \\ h(2-2x) = \dfrac{1-h(x)}{1-v} & x > 1/2 \quad (4b) \end{cases} \qquad (4)$$

Each of these functional equations is a linear functional equation. Methods of solving linear functional equations are well developed and described for example at [1, 3]. In the same time the mentioned works do not contain any methods of solving systems of linear functional equations.

Note that functional equation (3) is reduced to a system of functional equations (4) only in the assumption that the unknown function h is a homeomorphism. We will prove that functional equations system (4) has the unique solutions which is the homeomorphism which satisfy the functional equation (3). The existence and the uniqueness of the solution of the last equations is proved at [2].

The proving of the uniqueness of the solution of the functional equation (3) will be constructive. That is why our calculations will give us a possibility of using the values table of the mapping h obtained with numerical methods.

Example 1. *With the use of methods which we will show below it is possible to get that for $v = 3/4$ the graph of the mapping h looks as shown at the figure 1a). It is possible to show that this mapping is not differentiable at any open interval.*



a) b) c)

Figure 1:

According to methods of solving functional equations which are described at [1] the solution of (4a) looks as

$$h(x) = x^{-\log_2 v}\omega(\log_2 x), \qquad (5)$$

where $\omega(x)$ is arbitrary periodical period 1 function. The analogical result is also given at [3, p. 408]. Function ω will appear to be one to one defined if we add the demand for the solution of the functional equation (4a) to be also the solution of the equation (4b).

If for $v = 3/4$ on calculate the function ω from the formula above with the use of the numerical calculations for values of the solution of (3) the get the graph as at the figure 1b).

The solution of the functional equation (4b) looks as

$$h(x) = \frac{1}{2-v} + \left|x - \frac{2}{3}\right|^{-\log_2(1-v)} \cdot \begin{cases} \omega^+\left(\log_2\left|x-\frac{2}{3}\right|\right) & x > \frac{2}{3}; \\ \omega^-\left(\log_2\left|x-\frac{2}{3}\right|\right) & x < \frac{2}{3}. \end{cases} \tag{6}$$

and functions ω^+ and ω^- satisfy the relation

$$\begin{cases} \omega^-(t+1) = -\omega^+(t) \\ \omega^+(t+1) = -\omega^-(t). \end{cases} \tag{7}$$

In the same way like during solving the functional equation (4a) we can find the graph of the function $\omega(x)$ such that

$$\omega\left(\log_2\left|x-\frac{2}{3}\right|\right) = \begin{cases} \omega^+\left(\log_2\left|x-\frac{2}{3}\right|\right) & x > \frac{2}{3}; \\ \omega^-\left(\log_2\left|x-\frac{2}{3}\right|\right) & x < \frac{2}{3}. \end{cases}$$

with the use of homeomorphism h which was found independently earlier. Taking into attention the (7) it is enough to plot the graph of ω only for $x \in [0;\, 2]$. It is presented at the figure 1c).

It was constructed at [2] the approximation of h with piecewise linear mappings h_n for $n \geqslant 1$ such that each of the is not differentiable only at point of the set A_n which is defines as follows.

For every $n \geqslant 1$ denote with A_n, $n \geqslant 1$ the set of all points of the interval $[0;\, 1]$ such that $f^n(A_n) = 0$ and denote also B_n, $n \geqslant 1$ the set of all points of the interval $[0;\, 1]$ such that $f_v^n(B_n) = 0$. It is proved at [2] that sets A_n and B_n are os the same cardinality and that sets $A = \bigcup\limits_{n=1}^{\infty} A_n$ and $B = \bigcup\limits_{n=1}^{\infty} B_n$ are dense in the interval $[0;\, 1]$.

Denote with h_n the increase piecewise linear mapping such that points of A_n maps to B_n and is differentiable on $[0;\, 1] \setminus A_n$. We may consider h_n as an approximation of the mapping h.

The deal of the work is the study of approximations of functions ω which appear in formulas for solutions of functional equations (4a) and (4b) if we construct these approximations with the use of functions h_n.

1 Constructing and the simplest properties of linear functional equations.

Generally said, The system of functional equations (4) does not yield from the functional equation (3). It yields from it only with the assumption that the mapping h which we try to find satisfy the relations

$$\begin{cases} h([0;\, 1/2]) = [0;\, v], \\ h([1/2;\, 1]) = [v;\, 1]. \end{cases}$$

For example we prove that the linear equations system (4) has the unique solution which is the homeomorphic solution of (3) whose existence is proved at [2].

In the same time, the functional equation (4) has only the unique solution which is a mentioned homeomorphism. For example, the following mappings defined with equalities $h_1(x) = 0$ and $h_2(x) = x^*$ for all $x \in [0;\, 1]$ where x^* is a fixed point of f_v will also be solutions of (4).

1.1 A system of linear functional equations as a corollary of the assumption of that mapping which makes the diagram commutative is a homeomorphism

Let a mapping h is a homeomorphism which maps the interval $[0;\, 1]$ into itself and which makes commutative the diagram from the definition of topological conjugateness.

The mapping h moves each fixed point of the mapping f to a fixed point of the mapping f_v.

As 0 is a fixed point of the mapping f and 1 is not a fixed point of the mapping f_v then the equality $h(0) = 0$ holds and yields that homeomorphism increase.

Substitute the value $x = 1/2$ into the functional equation (3) and get $h(1) = f_v(h(1/2))$. As homeomorphism h increase then the last equality yields that $h(1/2) = v$.

So the functional equation (3) can be rewritten as a pair of commutative diagrams

$$\begin{array}{ccc} [0;\, 1/2] & \xrightarrow{x \mapsto 2x} & [0;\, 1], \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ [0;\, v] & \xrightarrow{x \mapsto \frac{x}{v}} & [0;\, 1]; \end{array} \quad \text{and} \quad \begin{array}{ccc} [1/2;\, 1] & \xrightarrow{x \mapsto 2-2x} & [0;\, 1], \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ [v;\, 1] & \xrightarrow{x \mapsto \frac{1-x}{1-v}} & [0;\, 1]. \end{array}$$

These diagrams together are equivalent to the functional equations system (4).

1.2 The uniqueness of the solution of linear functional equations system

Denote with $x^* = \frac{2}{3}$ the positive fixed point of the mapping f i.e. the solution of the equation $x = -2x + 2$.

Lemma 1. *$h(x^*) = \frac{1}{2-v}$ and $h\left(\frac{x^*}{2}\right) = \frac{v}{2-v}$.*

Proof. Substitute x^* into the equation (4b) and get

$$h(x^*) = \frac{1 - h(x^*)}{1-v},$$

whence

$$h(x^*) = \frac{1}{2-v}.$$

If substitute $x = \frac{x^*}{2}$ into the equation (4b) then get

$$h\left(\frac{x^*}{2}\right) = vh(x^*) = \frac{v}{2-v}.$$

□

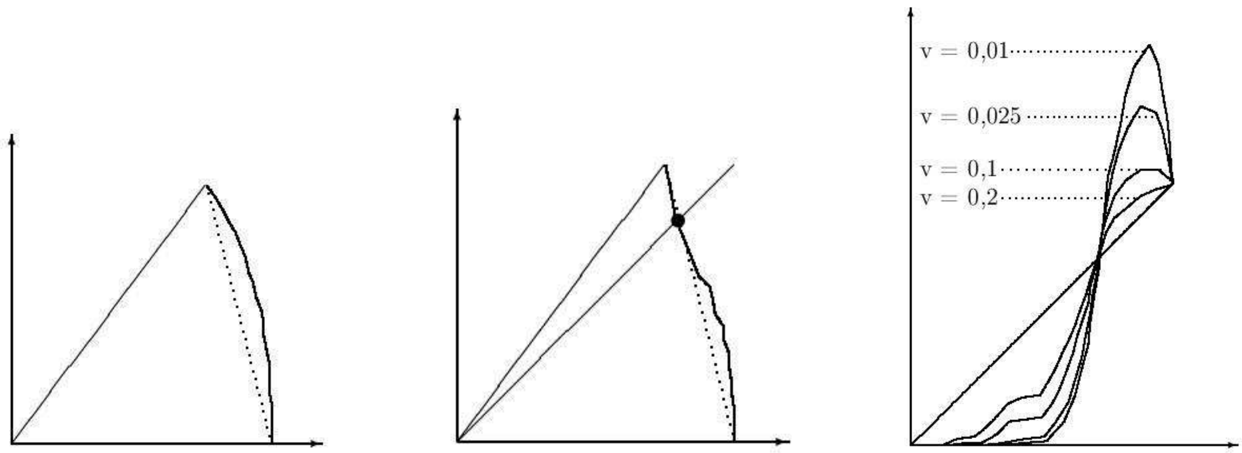
Notation 1. *Call the value of the mapping h at the point x to be* ***uniquely defined*** *if some value $h(x)$ at this point is a corollary of the system of functional equations (4). For example the mapping h is uniquely defined at points 0; x^* and $\frac{x^*}{2}$.*

Lemma 2. *If the mapping h is uniquely defined at point $\widetilde{x}$ then it is uniquely defined at each point of the integer trajectory of this point.*

Proof. Let $\widetilde{x}$ is an arbitrary point such that h is uniquely defined at it.

We sill show that in this case the mapping h is uniquely defined at the point $f(\widetilde{x})$. Really if $\widetilde{x} \leqslant \frac{1}{2}$ then the substitution of the value $x = \widetilde{x}$ into the equation (4a) gives that $h(2\widetilde{x}) = \frac{1}{v}h(\widetilde{x})$ which means that $h(f\widetilde{x})$ is uniquely defined.

Let $\widetilde{x}_*$ be some pre image of the point $\widetilde{x}$. In this case the fact that h is uniquely defined at $\widetilde{x}$ can be proved in the same way or with the substituting $x = 2\widetilde{x}_*$ into the equation (4a) or with substituting $x = \frac{2-\widetilde{x}_*}{2}$ into the equation (4b). □



а) б) в)

Figure 2:

Lemma 3. *The union of all integer trajectories of the point x^* is dense in the set $[0; 1]$ and the mapping h is uniquely defined at each point of this union.*

Let us give some note before the proving the lemma. The pre image of the point $x \in [0; 1]$ under the mapping f looks as follows with given binary code of $x = 0, \alpha_1 \alpha_2, \ldots$.

Remark 1. *If the binary code of the number $x \in [0; 1]$ looks as follows*

$$x = 0, \alpha_1 \alpha_2, \ldots,$$

then the binary code of the pre image x^- under the action of f (i.e. the binary code of such point that the equality $f(x^-) = x$ holds) has one of the *following forms*

$$x^- = \left[\begin{array}{l} 0,0\,\alpha_1\,\alpha_2\ldots\alpha_n\ldots \\ 0,1\,\overline{\alpha}_1\,\overline{\alpha}_2\ldots\overline{\alpha}_n\ldots \end{array} \right.$$

Lemma 3 yields from the more general lemma.

Lemma 4. *For the arbitrary point $x \in [0; 1]$ the union of all its inverse trajectories is dense in* $[0; 1]$.

Proof. This lemma is an immediate corollary from the note 1. It is enough for proving this lemma to prove that for every point $x \in [0; 1]$ and every set $\{\alpha_1\,\alpha_2\,\ldots\,\alpha_n\}$ of binary points there is an integer trajectory of x such that it has a point with the first binary digits be equal to $0,\ \alpha_1 \ldots \alpha_n \ldots$.

This proposition is a trivial corollary of the inductive reasonings for $n \geqslant 1$ which is a number of first chosen digits of the binary code of some pre image. □

Proving of lemma 3. The lemma 1 gives that the value of the mapping h is uniquely defined at point x^*. The lemma 2 gives that the mapping h is uniquely defined at each point of each integer trajectory of the point x^*.

The density of the union of integer trajectories of x^* is proven at the lemma 4. □

The obtain results can be generalized in the following theorem.

Theorem 1. *The system of functional equations (4) has the unique continuous solution which is increase homeomorphism*

2 Solving the functional equations with analytical methods

Each functional equation of the system (4) can be solved and correspond solutions can be represented analytically.

We will introduce the correspond technique below and will do the calculations with each of obtained solutions for showing the complicatedness of the solutions of each functional equation after substituting it into another equation of the system (4).

2.1 Solving of the equation (4a) with further substituting the solution into the equation (4b).

The equation (4a) is a linear functional equation. According to for example [3](p. 408) its solutions looks as

$$h(x) = x^{-\log_2 v}\omega(\log_2 x) \tag{5}$$

where $\omega(x)$ is a arbitrary periodical 1 function.

If substitute the obtained function into the equation (4b) then obtain

$$(2-2x)^{-\log_2 v}\omega(\log_2(2-2x)) = \frac{1 - x^{-\log_2 v}\omega(\log_2 x)}{1-v}.$$

Taking into attention the periodicity of ω this equation can be rewritten as

$$\begin{aligned} &(1-v)(1-x)^{-\log_2 v}\omega(\log_2(1-x)) = \\ &= v(1 - x^{-\log_2 v}\omega(\log_2 x)). \end{aligned} \tag{8}$$

Remark 2. *If consider the equation (8) as a functional equation of the whole real axis then it will appear that the necessary function h is a constant.*

The deal of the remark. If denote $t = 1 - x$ then obtain

$$\begin{aligned} &(1-v)t^{-\log_2 v}\omega(\log_2 t) = \\ &= v(1 - (1-t)^{-\log_2 v}\omega(\log_2(1-t))). \end{aligned}$$

If write x instead of t and take into attention the equation (8) then we may express $(1-x)^{-\log_2 v}\omega(\log_2(1-x))$ from each of the equations and make equal the results after which obtain

$$\frac{v}{1-v}\left(1 - x^{-\log_2 v}\omega(\log_2 x)\right) =$$

$$= 1 - \frac{1-v}{v}\,x^{-\log_2 v}\omega(\log_2 x),$$

whence

$$x^{-\log_2 v}\omega(\log_2 x) = v.$$

The obtained condition means that function h is constant and whence is not invertible. □

The explanation for the remark 2. The equation (8) is obtained with substituting the solution of the functional (4a) into the functional equation (4b).

That is wy the substitution $t = 1 - x$ is in fact the substitution at the equation (4a).

Nevertheless the equation (4a) if obtained from the fact that the diagram

$$\begin{array}{ccc} [0;\ 1/2] & \xrightarrow{f:\,x\mapsto 2x} & [0;\ 1] \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ [0;\ v] & \xrightarrow{f_v:\,x\mapsto x/v} & [0;\ 1] \end{array}$$

commutes. But it is defined only for $x \in [0;\ 1/2]$. As the substitution $t = 1 - x$ for $x \in [0;\ 1/2]$ means that $t \in [1/2;\ 1]$ then after formal rewritten of this substitution one obtain the functional equation for another domain of a function h and formal expressing of

$$(1-x)^{-\log_2 v}\omega(\log_2(1-x))$$

each of two obtained equations is incorrect. □

If use the mapping h which was constructed at the example 1 then numerical methods will let us to get the mapping $\omega(x)$ whose graph is given at the figure 1b) for $x \in [0;\ 1]$.

Remark 3. *Make the remark about the way how the graph of ω was obtained.*

The deal of the remark. As ω is periodic with period 1 then it is enough to find its values at arbitrary interval of the length 1.

If one consider all the values of $x \in [1/2;\ 1]$ with the use of the step which then we will get the table of values of the function $\log_2 x$. Now the equation (5) yields the table of values of the function ω defined at $\log_2 x$ for all x from the former set of values.

So for every $x \in [0;\ 1]$ with some fixed step for example $\frac{1}{n}$ for n is huge enough we will fix $\log_2 x$ and $\omega(\log_2 x) = h(x) \cdot x^{log_2 v}$ whence obtain the dense set of points of the graph of the function ω on the interval $[-1;\ 0]$. □

The function h which is the solution of the system of functional equations (4) is "complicated" yield from the fact that h is not differentiable on any subinterval of $[0;\ 1]$. The form of the equation (5) means that complicatedness of h should come from the complicatedness of ω because the multiplier $x^{-\log_2 v}$ is differentiable at any point.

In the same time we will formulate the list os properties of h of the form (5).

Lemma 5. *If the invertible interval* $[0;\ 1]$ *mapping h is of the form* (5) *then the following hold.*

1. *The mapping h increase;*
2. *For any $n \in \mathbb{N}$ the equality $h\left(\frac{1}{2^n}\right) = v^n$ hold.*

More then this, for all integer t the equality $\omega(t) = 1$ holds.

Proof. Prove at first that function ω is bounded. It is so because of its periodicity it is wholly determined with values of $\omega(\log_2 x)$ for $x \in [1/2;\ 1]$. Nevertheless for such values of x the function h is bounded and the function $x^{-\log_2 v}$ strictly increase. This means that $\omega(\log_2 x)$ is bounded for all $x \in [0;\ 1]$.

If substitute $x = 0$ into the equality (5) then obtain the product of zero times a value of bounded function which means that $h(0) = 0$. This corollary together with the fact of being invertible means that h increase from 0 to 1 for all $x \in [0;\ 1]$.

The condition $h(1) = 1$ yields that after substitution $x = 1$ into the equality (5) obtain $1 = \omega(\log_2 x)$. The periodicity of ω with period 1 means that for every $t \in \mathbb{Z}$ the equality $\omega(t) = 1$ holds.

Plugging $x = \frac{1}{2^n}$ into the equality (5) obtain

$$h\left(\frac{1}{2^n}\right) = v^n\,\omega(\log_2 2^{-n}) = v^n.$$

□

Consider the examples of "simple" mappings ω but such that function h which is determined with (5) is invertible and consider the mapping $\widetilde{f}_v$ which is determined with the commutative diagram

$$\begin{array}{ccc} [0;\ 1] & \xrightarrow{f} & [0;\ 1] \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ [0;\ 1] & \xrightarrow{\widetilde{f}_v} & [0;\ 1]. \end{array} \tag{9}$$

Lemma 6. *If for invertible mapping h of the form* (5) *the diagram* (9) *is commutative then for $x \in [0;\ v]$ the equality*

$$\widetilde{f}_v(x) = \frac{x}{v}$$

holds.

Proof. The condition of that h is invertible yields that $\widetilde{f}_v$ is determined with h. More exactly for every $x \in [0;\ 1]$ the equality $\widetilde{f}_v = h(f(h^{-1}(x)))$ holds. But by lemma 5 for $x \in [0;\ v]$ the inclusion $h^{-1}(x) \in [0;\ 1/2]$ means that $\widetilde{f}_v(x) = h(2h^{-1}(x))$.

The uniqueness of $\widetilde{f}_v$ and the fact that $\widetilde{f}_v(x) = \frac{x}{v}$ satisfies the functional equation above obtain the proposition of lemma. □

The simplest condition is that when ω is periodical is that when it is constant.

The equality $h(1) = 1$ yields that if ω is constant then $\omega(x) = 1$.

Example 2. *Consider the graph of the mapping $\widetilde{f}_v$ which is determined with the commutative diagram (9) for the mapping h of the form (5) if ω is a constant function.*

The deal of the example. If $\omega(x) = 1$ then for every $x \in [0; 1]$ whence $h(x) = x^{-\log_2 x}$. Then by lemma 6 (also it can be shown with the direct calculations) for $x \in [0; v]$ the equality $\widetilde{f}_v(x) = \frac{x}{v}$ holds.

For such function ω the equality $h^{-1}(x) = x^{-\log_v 2}$ holds whence for $x \in [v; 1]$ obtain

$$\widetilde{f}_v(x) = \left(2 - 2x^{-\log_2 v}\right)^{-\log_v 2}.$$

The graph of the mapping $\widetilde{f}_v$ for $v = 3/4$ is given at figure 2a). □

Example 3. *Find the graph of the mapping $\widetilde{f}_v$ which is defined with the commutative diagram (9) for the mapping h of the form (5) if ω is continuous function whose graph consists of two linear branches on the interval* $[1/2; 1]$.

The deal of the example. Plug $x = 3/4$ into the commutative diagram (9) and get

$$\begin{array}{ccc} 3/4 & \xrightarrow{f} & 1/2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ h(3/4) & \xrightarrow{\widetilde{f}_v} & v = h(1/2). \end{array}$$

Define $h(3/4)$ such that this number be the biggest pre image of v under the acting of f_v. In this case the values of f_v and $\widetilde{f}_v$ will coincide on this bigger pre image of v under the acting of f_v.

In another words

$$h\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^{-\log_2 v} \omega\left(\log_2\left(\frac{3}{4}\right)\right) = v^2 - v + 1.$$

For $v = \frac{3}{4}$ obtain $\omega(0,584) \approx 0,915$.

Define ω on the interval $[0; 1]$ as follows. $\omega(x)$ should be piecewise linear whose graph consists of two lines and has the fracture point with coordinates

$$\left(\log_2 \frac{3}{4};\ (v^2 - v + 1) \cdot \left(\frac{3}{4}\right)^{\log_2 v}\right).$$

The graph of mapping $\widetilde{f}_v$ for such ω for $v = 3/4$ is given at 2b).

On this figure we also mark the point of intersection of f with the line $y = x$ because the construction gives that in this case mappings $\widetilde{f}_v$ and$f$ coincide. □

Examples 2 and 3 can be generalized as follows.

Consider the iteration approximations ω_n for the function ω and use them for iteration approximations $\widetilde{h}_n$ of the function h.

For the arbitrary approximation of h on the interval $x \in \left[\frac{1}{2}; 1\right]$ we can obtain the approximation of ω on the interval $[-1; 0]$ with taking into attention the formula (5). As ω is periodical with period 1 then we obtain the approximation of h on the whole $[0; 1]$.

With using of constructed h_n find the values of ω on the set $A_n \cap \left[\frac{1}{2}; 1\right]$.

Denote with $\omega_n(x)$ the mapping whose values are defined with the values of h_n at points of the set $\{\log_2 x,\ x \in A_n \cap \left[\frac{1}{2}; 1\right]\}$ such that $\omega_n(x)$ is linear at all points except the set $A_n \cap \left[\frac{1}{2}; 1\right]$ and is periodical with period 1.

So, the mapping $\widetilde{h}_n$ looks as

$$\widetilde{h}_n(x) = x^{-\log_2 v}\omega_n(\log_2 x), \tag{10}$$

and should be considered as iteration approximation of h.

If the constructed $\widetilde{h}_n$ appear to be invertible then there exists the unique mapping $\widetilde{f}_n$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle \widetilde{h}_n}\downarrow & & \downarrow{\scriptstyle \widetilde{h}_n} \\ B & \xrightarrow{\widetilde{f}_n} & B \end{array} \tag{11}$$

commutes. This mapping $\widetilde{f}_n$ can be defined with the formula

$$\widetilde{f}_n = \widetilde{h}_n(f(\widetilde{h}_n^{-1})).$$

For example with the use of notations introduced above the mapping $\widetilde{f}_v$ which was constructed at the example 3 is the mapping $\widetilde{f}_2$ and correspond approximation of h is the mapping $\widetilde{h}_2$.

Nevertheless it may appear that the mapping $\widetilde{h}_n$ will not be monotone and so it will not exist a mapping $\widetilde{f}_n$ which make the diagram (11) commutative.

Example 4. *Consider the case when the mapping $\widetilde{h}_3(x)$ which in fact is dependent on v is non monotone for some v.*

The deal of the example. Give the graphs of mappings $\widetilde{h}_3(x)$ for $v = 0,01$, $v = 0,025$, $v = 0,1$ and $v = 0,2$ on the figure 2c).

As a comment for the given graphs note that each of them satisfy the functional equation (4a)

$$h(2x) = \frac{1}{v}\, h(x),$$

i.e. its form on the each of intervals $\left[\frac{1}{2^{k+1}};\, \frac{1}{2^k}\right]$ is its form on the interval $\left[\frac{1}{2};\, 1\right]$ but being squeezed v^k times.

The mapping $\widetilde{h}_3(x)$ which is calculated for $v = \frac{1}{2}$ is given with the formula $\widetilde{h}_3(x) = x$.

Whence we see that with decreasing of v from $v = \frac{1}{2}$ there exist some "critical value" at which the graph of $\widetilde{h}_3(x)$ becomes to be monotone non monotone. □

Notation 2. *Denote with $\widehat{h}_n(t)$ the function such that the equality*

$$\widehat{h}_n(\log_2 x) = \widetilde{h}_n(x)$$

holds.

Use numbers $\alpha_k = \alpha_{k;\,n}$ to construct the numbers $\widetilde{\alpha}_k = \log_2 \alpha_k$.

Denote with $t_k = t_{k;\,n}$ the extremum of the mapping $\widehat{h}_n$ on the interval $(\widetilde{\alpha}_k;\, \widetilde{\alpha}_{k+1})$. The construction lets to prove that this extremum if unique and the proving is quiet simple.

The condition for $\widetilde{h}_n$ to be monotone is equivalent to that for every k the inclusion

$$t_k \in \mathbb{R}\backslash[\widetilde{\alpha}_k;\, \widetilde{\alpha}_{k+1}] = \mathbb{R}\backslash[\log_2 \alpha_k;\, \log_2 \alpha_{k+1}] \quad (12)$$

holds.

Taking into attention the formula (10) rewrite $\widehat{h}_n(t)$ as

$$\widehat{h}_n(t) = 2^{-t\log_2 v}\omega_n(t).$$

Taking into attention the function $y = \log_2 x$ obtain that being monotone of $\widetilde{h}_n$ is equivalent of being monotone of $\widehat{h}_n$.

The equality $h_n(\alpha_k) = \beta_k$ yields the following equality for the function ω_n

$$\omega_n(\widetilde{\alpha}_k) = \beta_k \cdot 2^{\widetilde{\alpha}_k \log_2 v}.$$

Denote with $\widetilde{\beta}_k = \beta_k \cdot 2^{\widetilde{\alpha}_k \log_2 v} = \beta_k v^{\widetilde{\alpha}_k}$.

Let ω_n be of the form $\omega_n(t) = a_k \cdot t + b_k$ on the interval $(\widetilde{\alpha}_k;\, \widetilde{\alpha}_{k+1})$. Then

$$a_k = \frac{\widetilde{\beta}_{k+1} - \widetilde{\beta}_k}{\widetilde{\alpha}_{k+1} - \widetilde{\alpha}_k}, \quad b_k = \frac{\widetilde{\beta}_k\widetilde{\alpha}_{k+1} - \widetilde{\beta}_{k+1}\widetilde{\alpha}_k}{\widetilde{\alpha}_{k+1} - \widetilde{\alpha}_k}.$$

Find the extremum of the curve which is defined with the equation which determines the mapping $\widehat{h}_n(t)$ on the interval $(\widetilde{\alpha}_k;\, \widetilde{\alpha}_{k+1})$. So, we get

$$\widehat{h}'_n(t) = 2^{-t\log_2 v}(a_k - \log_2 v \ln 2 \cdot (a_k t + b_k))$$

whence extremum of the mapping $\widehat{h}_n$ can be found with the formula

$$t_k = \frac{a_k - b_k \log_2 v \ln 2}{a_k \log_2 v \ln 2} = \frac{1}{\ln v} - \frac{b_k}{a_k}. \qquad (13)$$

The previous calculations give that

$$\frac{b_k}{a_k} = \frac{\widetilde{\beta}_k\widetilde{\alpha}_{k+1} - \widetilde{\beta}_{k+1}\widetilde{\alpha}_k}{\widetilde{\beta}_{k+1} - \widetilde{\beta}_k}.$$

Coming back to previous calculations obtain that

$$\frac{b_k}{a_k} = \frac{\beta_k v^{\log_2 \alpha_k} \log_2 \alpha_{k+1} - \beta_{k+1} v^{\log_2 \alpha_{k+1}} \log_2 \alpha_k}{\beta_{k+1} v^{\log_2 \alpha_{k+1}} - \beta_k v^{\log_2 \alpha_k}}.$$

As $\alpha(k;\, n) = \frac{k}{2^n}$ and $A_n = \{\alpha(k;\, n-1)\}$ obtain that $\alpha_k = \frac{k}{2^{n-1}}$ whence

$$\log_2 \alpha_k = \log_2 k - n + 1.$$

So,

$$\frac{b_k}{a_k} = n - 1 + \frac{\beta_k v^{\log_2 k} \log_2(k+1)}{\beta_{k+1} v^{\log_2(k+1)} - \beta_k v^{\log_2 k}} - $$

$$- \frac{\beta_{k+1} v^{\log_2(k+1)} \log_2 k}{\beta_{k+1} v^{\log_2(k+1)} - \beta_k v^{\log_2 k}}.$$

The example 4 let to come to a natural assumption that for $v \to 0$ the mapping h_n will be non invertible also for huge values of n. With the use of formula (13) we can prove the following lemma.

Lemma 7. *For any $n \in \mathbb{N}$ there exists $v_0 \in (0;\, 1)$ such that for every $v \in (0;\, v_0)$ the mapping $\widetilde{h}_n(x)$ is monotone on the interval $\left[\frac{2^{n-1}-1}{2^{n-1}};\, 1\right]$.*

Proof. Let us find evident formulas for $\beta(n;\ 2^{n-1}-1)$. We know about this point that

$$\begin{cases} f_v^n(\beta(n;\ 2^{n-1}-1)) = 0; \\ f_v^{n-1}(\beta(n;\ 2^{n-1}-1)) = 1. \end{cases}$$

Find the tangent of the mapping f_v^{n-1} on the last interval where it is monotone. This tangent equals $\frac{1}{v-1}\cdot\frac{1}{v^{n-2}}$ which means that

$$\beta(n;\ 2^{n-1}-1) = 1 + v^{n-2}\cdot(v-1).$$

With the use of notions which were introduced during finding of t_k defined with (13) we get $k = 2^{n-1}-1$, $\beta_k = 1+v^{n-2}\cdot(v-1)$, $\beta_{k+1} = 1$, $\alpha_k = 1-\frac{1}{2^{n-1}}$, $\alpha_{k+1} = 1$. So, $\log_2\alpha_{k+1} = 0$ whence

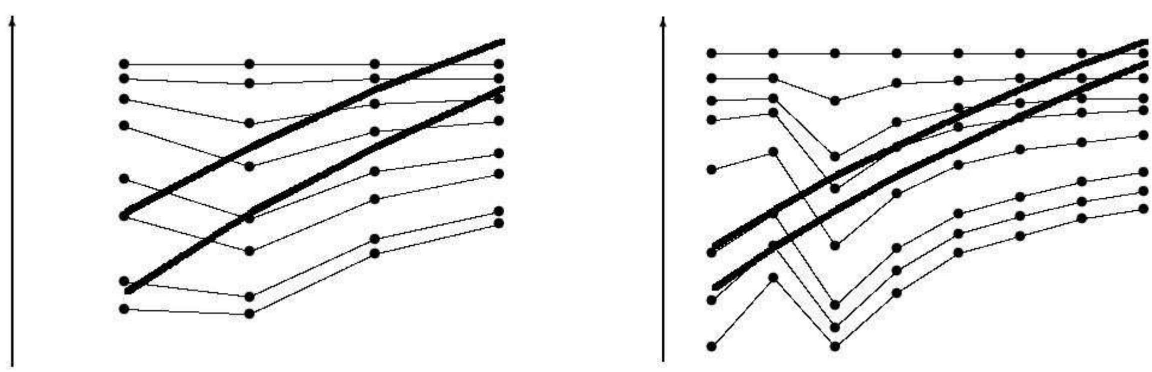
$$t_k = \frac{1}{\ln v} + \frac{\log_2\alpha_k}{1-(1+v^{n-2}\cdot(v-1))v^{\log_2\alpha_k}}.$$

Note that inequality $t<0$ holds because for $v\in(0;1)$ there are inequalities $\ln v<0$ and $\log_2(1-2^{1-n})<0$. In this case the denominator of the fraction is positive.

Also for fixed n and $v\to 0$ we have that $v^{\log_2\alpha_k}\to\infty$ whence $t_k\to 0$.

The obtained value of limit proves lemma. □

With the use of formula (13) we can study the example 4 more attentively if consider $n=3$ and study the behavior of numbers t_2 and t_3 dependently on v.



a) b)

Figure 3:

Example 5. *Consider the function $t_2(v)$ for $n=3$. We will which un succeed attempt of calculation of the limit $\lim\limits_{v\to1} t_2(v)$. We will show that this limit exists and equals 1 in the time when numerical methods look like the limit does not exist.*

The deal of the example. For $n=3$ we have a partition of the interval $\left[\frac{1}{2};\ 1\right]$ to 2 intervals with 3 points $\alpha_2=\frac{1}{2}$, $\alpha_3=\frac{3}{4}$ and $\alpha_4=1$. Correspondingly $\beta_2=v$, $\beta_3=\max f_v^{-1}(v)=1-v(1-v)$ and $\beta_4=1$.

Then

$$t_2 = \frac{1}{\ln v} + \frac{v^2-v^2\log_2 3}{(v^2-v+1)v^{\log_2 3}-v^2}.$$

For the function $\widetilde{h}_2$ being continuous on the interval int is necessary and sufficient the inclusion

$$t_2\in\mathbb{R}\backslash[-1;\ \log_2 3-2]\approx\mathbb{R}\backslash[-1;\ -0,415]$$

hold.

The numerical analysis of the obtained function $t_2(x)$ lets make a conclusion that it decrease for $x\in[0;\ 0,5]$ and has an asymptote for $x=0,5$ i.e. $\lim\limits_{x\to0,5-} t_2=-\infty$ and $\lim\limits_{x\to0} t_2=-1$.

If try plot the graph of $t_2(x)$ for $x\in[0,5;\ 1]$ then numerical methods for $x\in[0,99999;\ 1]$ show that it becomes to be non monotone. So, for $t\in[0,6;\ 0,999999]$ graph of $t_2(x)$ is given at figure 4a).

Nevertheless the graph of $t_2(x)$ for $x\in M=[0,99999999;\ 0,999999995]$ is given at the figure 4b). Strictly said, the given sketch is not exactly the graph of $t_2(x)$ but the set of its points whose x-coordinated are uniformly distributed on

M and are all these points are connected with lines.

Remind that the obvious continuality of the mapping $t_2(v)$ yields that the mapping $t_2(v)$ has on the set M such points that correspond mapping $\widehat{h}_3$ is not continuous on the interval $[\alpha_2;\, \alpha_3]$.

Further experiments show that for each of intervals of the form $\left[1-\frac{1}{10^{m+1}};\, 1-\frac{1}{10^m}\right]$ the mapping $t_2(v)$ has such point that correspond mapping $\widehat{h}_3$ is not continuous on correspond point of $[\alpha_2;\, \alpha_3]$.

In the same time these experiments do not represent the mathematical reality of the function under consideration, because the limit

$$\lim_{v\to 1}\left(\frac{1}{\ln v}+\frac{v^2(1-\log_2 3)}{(v^2-v+1)v^{\log_2 3}-v^2}\right)$$

is a limit of difference of two expressions such that each of them tends to $+\infty$ so trying to calculate this limit directly with calculating of each term may accumulate a huge calculation mistake. Let us prove properly that the limit which was reminded above exists.

As

$$\lim_{v\to\infty} t_2(v)+1=\frac{1}{0}+\frac{0}{0}=\frac{0}{0},$$

make the sum a proper fraction to make possible to differentiate is with the L'Hopital's rule. So,

$$t_2+1=$$
$$=\frac{(v^2-v+1)v^{\log_2 3}-v^2+\ln v(v^2-v^2\log_2 3)}{\ln v((v^2-v+1)v^{\log_2 3}-v^2)}.$$

Denote with

$$s(v)=(v^2-v+1)v^{\log_2 3}-v^2+\ln v(v^2-v^2\log_2 3)$$

and

$$p(v)=\ln v((v^2-v+1)v^{\log_2 3}-v^2).$$

Then$s'(v)=(2v-1)v^{\log_2 3}+\log_2 3(v^2-v+1)v^{\log_2 3-1}-2v+(v-v\log_2 3)+2(v-v\log_2 3)\ln v$, whence $s'(1)=0$. Also $p'(v)=(v-1)v^{\log_2 3}+v^{\log_2 3-1}-v+((2v-1)v^{\log_2 3}+\log_2 3(v^2-v+1)v^{\log_2 3-1}-2v)\ln v$, whence $p'(1)=0$.

Find second derivatives $s''(v)$ and $p''(v)$ to get $s''(1)=3+(\log_2 3)^2-2\log_2 3\approx 2,3422$ and $p''(1)=2\log_2 3-2\approx 1,1699$. That is why

$$t_2(v)\to\frac{s''(1)}{p''(1)}-1\approx 1,0020.$$

□

Example 6. *Consider the function $t_3(v)$ for $n=3$. We will show that function $t_3(v)$ makes the correspond function $\widehat{h}_3(x)$ non monotone for all $v<v_0$ where $v_0=0,18867\pm 0,00001$. This result is consistent with the one of the example 4.*

The deal of the example. Study the question of being monotone of the mapping $\widehat{h}_3(x)$ on the interval $[\alpha_3;\, \alpha_4]$ for $n=3$. In this case

$$t_3=\frac{1}{\ln v}+\frac{v^2\log_2 3-2v^2}{v^2-(v^2-v+1)v^{\log_2 3}}.$$

For the violation of being monotone of $\widetilde{h}_3$ it is necessary the inclusion

$$t_3\in[\log_2 3-2;\, 0]\approx[-0,415;\, 0].$$

The graph of $t_3(v)$ for $v\in[0;\, 0,21]$ is given at the figure 4c).

As it is shown at the example 4 there is a point near $v_0\approx 0,2$ such that $\widetilde{h}_3$ is non monotone near it for all $v\leqslant v_0$. Our calculations show that this value is about

$$v_0\in[0,18868;\, 0,18869],$$

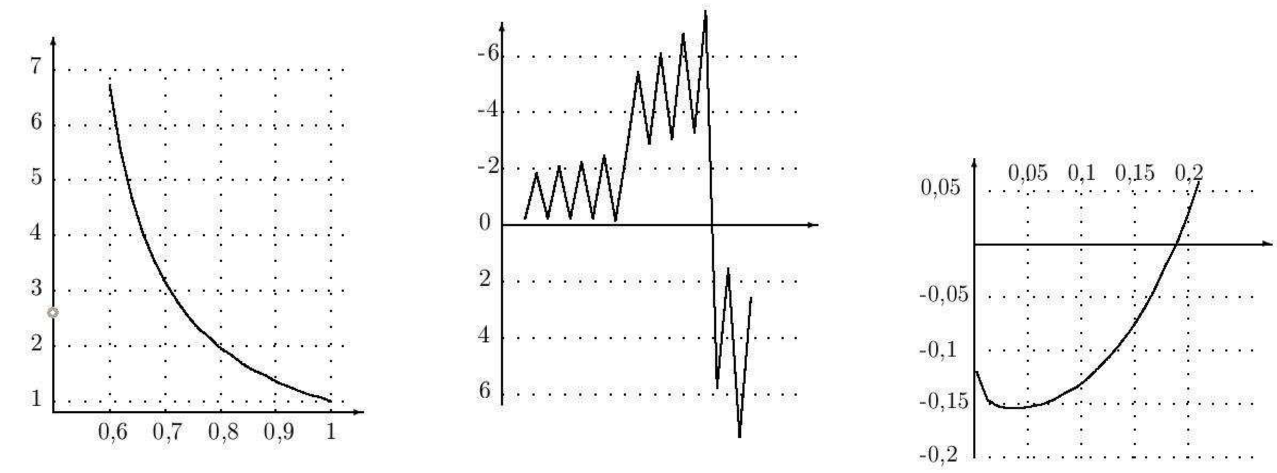
i.e.

$$v_0\approx 0,18867\pm 0,00001.$$

As about $t_3(v)$ on the interval $[0,2;\, 0,5]$, it has an asymptote at the point $0,5$ and increase at the whole interval $[0,2;\, 0,5]$ to positive infinity. □

We will give now the similar investigation for $n=4$. For every v and for every interval $[\frac{1}{2};\, \frac{5}{8}]$, $[\frac{5}{8};\, \frac{3}{4}]$, $[\frac{3}{4};\, \frac{7}{8}]$ and $[\frac{7}{8};\, 1]$ find the value t_k with the use of formula (13).

For different values v and every $k\in[4;\, 7]$ find $t_k(v)$ and this will give us 4 points. Just for convenience concatenate these points to polygons i.e. obtain a polygon for each of v. Also add the following two curves (and put with the bold) to the picture. Let one of them to connects numbers $\{\widetilde{\alpha}_k\}$ and another connects numbers $\{\widetilde{\alpha}_{k+1}\}$ for the same $k\in[4;\, 7]$. The previous explanations give that the mapping $\widehat{h}_{n;\,v}$ will be monotone if and only if each of obtained 4 points will be between these two bold curves, i.e. under then bottom and above then the lower of them.



a) b) c)

Figure 4:

The figure 3a contains these curves for values $v_1 = 0,15$, $v_2 = 0,1$, $v_3 = 0,07$, $v_4 = 0,03$, $v_5 = 0,17$, $v_6 = 0,01$, $v_7 = 0,001$ and $v_8 = 0,00001$. The top curves on the picture correspond to the lower values of v.

As we can see and this corresponds to lemma 7 if the value of v is small enough then curves become close to the horizontal line $y = 1$ and the last point of each of these curves (that one which corresponds to the interval $[\alpha_7; 1]$) tends to 1 for $v \to 0$ and so belongs to the interval $(\widetilde{\alpha}_7; \widetilde{\alpha}_8) = (\widetilde{\alpha}_7; 0)$. This means that $\widetilde{h}_n$ is non monotone on the interval $(\alpha_7; \alpha_8) = (\alpha_7; 1)$.

The figure 3b contains the analogues calculations for $n = 5$ and values $v_1 = 0,125$, $v_2 = 0,1$, $v_3 = 0,15$, $v_4 = 0,05$, $v_5 = 0,02$, $v_6 = 0,01$, $v_7 = 0,001$ and $v_8 = 0,000000001$.

2.2 Solving the equation (4b) with further substitution o the equation (4a).

The solution of the functional equation (4b) looks as

$$h(x) = \frac{1}{2-v} + \left|x - \frac{2}{3}\right|^{-\log_2(1-v)} \times$$
$$\times \begin{cases} \omega^+\left(\log_2\left|x-\frac{2}{3}\right|\right) & x > \frac{2}{3}; \\ \omega^-\left(\log_2\left|x-\frac{2}{3}\right|\right) & x < \frac{2}{3}. \end{cases} \tag{6}$$

where functions ω^+ and ω^- satisfy the equation

$$\begin{cases} \omega^-(t+1) = -\omega^+(t) \\ \omega^+(t+1) = -\omega^-(t). \end{cases} \tag{7}$$

In the same way like in the previous section we can use the found h for getting the function ω.

The equation (7) yields that function ω^+ and ω^- are periodical with the period 2.

Consider the function h for

$$x \in \left[\frac{35}{48}; \frac{11}{12}\right].$$

When the variable x become equal to all the values of the mentioned interval then he values of $\log_2\left(x - \frac{2}{3}\right)$ will become equal to all values of the interval $[-4; -2]$ whose length is 2. This will make possible to find the function ω^+ on the interval of its periodicity.

Lemma 8. *If for the invertible mapping h of the form (6) the diagram (9) is commutative then for $x \in [v; 1]$ the equation*

$$\widetilde{f}_v(x) = \frac{1-x}{1-v}$$

holds.

Proof. The prove of this lemma is analogous to the prove of the lemma 6. □

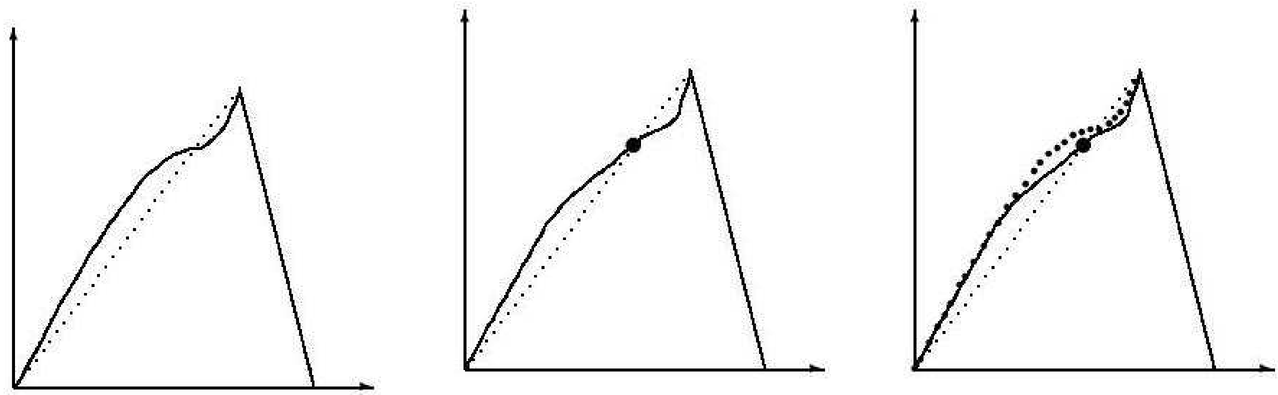
Example 7. *Obtain the graph of the mapping $\widetilde{f}_v$ which is defined with the commutative diagram (9) for the mapping h of the form (6) if ω is the function of the simplest form.*

The deal of the example. The equation (7) makes impossible the case when the function ω is constant and does not equal to 0. So the simplest case for ω is that when $\omega = \omega_0$ for $x < 0$ and $\omega = -\omega_0$ for $x > 0$.

Then the equality $h(0) = 0$ gives that

$$\omega_0 = \frac{-1}{2-v} \cdot \left(\frac{2}{3}\right)^{\log_2(1-v)}.$$

In this case the graph of the mapping $\widetilde{f_v}(x) = h(f(h^{-1}(x)))$ which for $v = \frac{3}{4}$ is given at figure 5a). □



a) b) c)

Figure 5:

Example 8. *Obtain the graph of the mapping $\widetilde{f_v}$ which is defined with the commutative diagram (9) for the mapping h of the form (6) if ω is a function such that ω^+ and ω^- are the simplest but non constant.*

The deal of the example. Consider the functions ω^+ and ω^- to be continuous.

The equality $h(0) = 0$ gives that

$$0 = \frac{1}{2-v} + \left(\frac{2}{3}\right)^{-\log_2(1-v)} \omega^-\left(\log_2 \frac{2}{3}\right),$$

i.e.

$$\omega^-(1-\log_2 3) = \frac{1}{v-2}\left(\frac{2}{3}\right)^{\log_2(1-v)} \quad (14)$$

As the left pre image of the point $\frac{1}{2}$ for the mapping f goes to

$$h\left(\frac{1}{4}\right) = v^2,$$

under the mapping h, i.e.

$$v^2 = \frac{1}{2-v} + \left(\frac{5}{12}\right)^{-\log_2(1-v)} \omega^-\left(\log_2 \frac{5}{12}\right),$$

whence the periodicity of ω^- with period 2 gives that

$$\omega^-\left(\log_2 \frac{5}{3}\right) = \left(v^2 - \frac{1}{2-v}\right)\left(\frac{5}{12}\right)^{\log_2(1-v)} \quad (15)$$

Construct the function ω^- as follows. The values of ω^- at points $\log_2 \frac{5}{3} \approx 0,737$ and $2+\log 2\frac{5}{3} \approx 2,737$ are equal and are defines with the equality (15) (according to (7) function ω^- is periodical with period 2).

The value of ω^- at point $3 - \log_2 3 \approx 1,415$ is defined with (14) (because of the periodicity of ω^- the equality (14) can really be used for finding ω^- at this point).

In this case we make the function ω^- to be linear at each of intervals $[\log_2 \frac{5}{3}; 3 - \log_2 3]$ and $[3-\log_2 3; 2+\log 2\frac{5}{3}]$ and is periodical with period 2.

We will construct the function ω^+ with using the function ω^- and taking into attention the equation (7). The graph of the mapping $\widetilde{f_v}$ which was constructed in such a way for $v = 3/4$ is given at the figure 5b).

If compare the graph from the previous example with that one which was constructed yet then see the plot as at figure 5c) (the dots are used for the first graph). □

Plug the solution (6) of the functional equation (4b) into the functional equation (4a) and obtain

$$\frac{v}{2-v} + v\left|2x-\tfrac{2}{3}\right|^{-\log_2(1-v)} \times$$
$$\times\ v\cdot\begin{cases}\omega^+\left(\log_2\left|2x-\tfrac{1}{3}\right|\right) & x>\tfrac{2}{3};\\ \omega^-\left(\log_2\left|2x-\tfrac{1}{3}\right|\right) & x<\tfrac{2}{3}\end{cases} =$$
$$= \frac{1}{2-v} + \left|x-\tfrac{2}{3}\right|^{-\log_2(1-v)} \times$$
$$\times\ \cdot\begin{cases}\omega^+\left(\log_2\left|x-\tfrac{2}{3}\right|\right) & x>\tfrac{2}{3};\\ \omega^-\left(\log_2\left|x-\tfrac{2}{3}\right|\right) & x<\tfrac{2}{3}.\end{cases}$$

Here the unknown functions ω^+ and ω^- are connected with the equation (7). The complicatedness of this equation in comparison with linear functional equation is obvious. The question on the invertibility of the mappings h which are obtained in such a way is so complicated as in the previous case.

3 The final remarks

We have shown in the work that methods of solving the linear functional equations appear to be practically non useful in the case of the simplest generalization of functional linear equations i.e. for the system of two linear functional equations.

In the same time the problems which were stated in the work may be generalized to the case when f_v is more complicated the that which we consider, for example has home the one point when it is not differentiable.

In spite of the experimental material which is given in the work we were failed to find the reason why the mentioned sequence $\widetilde{h}_n$ sometimes appear to be non monotone. So, we can no formulate the necessary an sufficient conditions for this sequence to be consisted of monotone functions.

Our calculations show that in the case which is considered in more details during the work the mapping $\widehat{h}_n$ becomes close to be constantly equal $\underline{1}$ for $v\to 0$. This means that for all n the mapping $\widetilde{h}_n$ becomes non monotone when $v\to 0$. Nevertheless we were failed to find the exact prove that these $\widetilde{h}_n$ really tend to be constant.

In any way we have found the list of constants like

$$v_0 \approx 0,18867 \pm 0,00001$$

from the example 6 which characterize the topological conjugation of f and each of the mappings f_v. We have calculated these constants only numerically and have failed to catch their deep meaning.

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