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Мартингальні оцінки для варіації і аналог критерію Макенхаупта для зваженої нерівності Пуанкаре на $\mathbb{R}$

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A martingale bound for the variation and an analogue of the Muckenhoupt criterion for a weighted Poincaré inequality on $\mathbb{R}$

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Ця стаття продовжує цикл праць, об'єднаних ідеєю доведення функціональних нерівностей через таку геометричну характеристику мір, як нарізки множин. За допомогою цієї ідеї вже були отримані узагальнення прямої та оберненої логарифмічної нерівностей Соболева. Це стаття компаньйон до роботи Кулика О.М. і Тимошкевича Т.Д. (2014), де запропонований мартингальний метод для доведення зваженої логарифмічної нерівності Соболева. У цій статті ми оцінюємо варіацію, так само за допомогою мартингалів з сигма-алгеброю побудованою на нарізці множин міри. Також наведені два конкретних приклади застосування загальної оцінки, а саме критерій Макенхаупта для класичної нерівності Пуанкаре в $\mathbb{R}$ та зважена нерівність Пуанкаре з хорошим інтегральним ядром.

Ключові слова: Мартингали, нарізка множин, нерівність Пуанкаре.

This article continues a series of works, united by the idea of proof of functional inequalities through such geometrical characteristic of measure as trimmed regions. With this idea we already has received generalization of direct and inverse logarithmic Sobolev inequalities. This paper is a companion to Kulik A.M. and Tymoshkevych T.D. (2014), which proposed martingale method for proving weighted logarithmic Sobolev inequalities. In this paper we estimate the variation, the same way as in the paper of Kulik A.M. and Tymoshkevych T.D. using martingales with sigma-algebra of sets built on trimmed regions of measure. Presented two examples of general assessment, namely the criterion Muckenhoupt for classical Poincare inequality in $\mathbb{R}$ and the weighted Poincare inequality in $\mathbb{R}$ with good integral kernel.

Key Words: Martingale, trimmed regions, Poincaré inequality.
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Introduction

It is said that the *Poincaré inequality* holds true for a probability measure μ on $\mathbb{R}^d$, if for every smooth compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$, the following inequality holds

$$Var_\mu f \le c \int_{\mathbb{R}^d} \|\nabla f\|^2 \, d\mu$$

with some constant c, where

$$Var_\mu f = \int_{\mathbb{R}^d} f^2 \mu - \left(\int_{\mathbb{R}^d} f d\mu \right)^2$$

denotes the *variation* of function f.

The least possible constant c, such that the above inequality holds true for every compactly supported smooth f, is called the *Poincaré constant* for the measure μ.

One possible extension for the Poincaré inequality is given by a *weighted Poincaré inequality* of the form

$$Var_\mu f \le \int_{\mathbb{R}^d} \|K \nabla f\|^2 \, d\mu,$$

where the function K, taking values in $\mathbb{R}^{d\times d}$, has the meaning of a *weight*.

This article is a companion one to [1], where a martingale method is proposed for proving a weighted version of another classical functional inequality, namely, the log-Sobolev inequality. In Theorem 1 below we obtain the martingale representation for variation of function set on $\mathbb{R}^d$. As a corollary of this representation, in Corollary 1 we give another proof of the Muckenhoupt criterion for Poincaré inequality on $\mathbb{R}$, and in Corollary 2 we give sufficient condition for a weighted Poincaré inequality on $\mathbb{R}$ to hold true.

1 A martingale bound for the variation

For probability measure μ in $\mathbb{R}^d$ we fix family of sets

$$\{D_t, t \in [0,1]\},$$

and their complements

$$\{Q_t = \mathbb{R} \setminus D_t, t \in [0,1]\};$$

and put

$$\mathcal{F}_t = \{A \in \mathcal{B}(\mathbb{R})\}, A \cap Q_t \in \{\emptyset, Q_t\}.$$

Properties of family of sets $D_t \in \mathcal{F}_t$:
1) $D_t \nearrow, D_0$ – set of one point, $D_1 = \mathbb{R}^d$;
2) $\mathcal{F}_t \nearrow \mathcal{F}, \mathcal{F}_0 = \{\emptyset, \mathbb{R}^d\}$.
We put

$$\tau(x) = \sup\{t \le 1 : x \in Q_t\}$$

also we put

$$G_t = \frac{1}{\mu(Q_t)} \int_{Q_t} g(y)\mu(dy),$$

then

$$g_t(x) = E(g|\mathcal{F}_t)(x) = g(x)\mathbb{I}_{D_t}(x) + G_t \mathbb{I}_{Q_t}(x).$$

Theorem 1.

$$Var_\mu(g) = \int_{\mathbb{R}} (g(x) - G_{\tau(x)})^2 \mu(dx).$$

Proof.
By Lemma 1 [1], for $0 < s < t < 1$ the following holds:

$$E(g_t^2 - g_s^2) = \int_{D_t \setminus D_s} (g(x) - G_{\tau(x)})^2 \mu(dx).$$

Also we can see, that

$$\lim_{s\to 0} E_\mu(g_s^2 - (E_\mu g)^2) =$$

$$\lim_{s\to 0} E_\mu(\mathbb{I}_{D_s} g^2) + \lim_{s\to 0}(\mathbb{I}_{Q_s} G_s^2) - \lim_{s\to 0}(E_\mu g)^2 =$$

$$\lim_{s\to 0} \mu(Q_s) G_s^2 - (E_\mu g)^2 =$$

$$\lim_{s\to 0}(\int_{Q_s} g(x)\mu(dx))^2 - (E_\mu g)^2 =$$

$$= (E_\mu g)^2 - (E_\mu g)^2 = 0,$$

and

$$\lim_{t\to 1} |E_\mu(g^2 - g_t^2)| = \lim_{t\to 1} |E_\mu \mathbb{I}_{Q_t}(g^2 - G_t^2)| \le$$

$$\le \lim_{t\to 1} E_\mu(\mathbb{I}_{Q_t} g^2) + \lim_{t\to 1} \mu(Q_t) G_t^2 =$$

$$= \lim_{t\to 1} \mu(Q_t) G_t^2 =$$

$$= \lim_{t\to 1} \mu(Q_t) \Big(\frac{\int_{Q_t} g(x)\mu(dx)}{\mu(Q_t)} \Big)^2 \le$$

$$\le \lim_{t\to 1} \mu(Q_t) \frac{\int_{Q_t} g(x)^2 \mu(dx)}{\mu(Q_t)} = 0.$$

This means that

$$Var(g) = E_\mu(g^2 - (E_\mu g)^2) =$$

$$= \lim_{s\to 0, t\to 1} E_\mu(g_t^2 - g_s^2) =$$

$$= \lim_{s\to 0, t\to 1} \int_{D_t \setminus D_s} (g(x) - G_\tau(x))^2 \mu(dx) =$$

$$= \int_{\mathbb{R}^d} (g(x) - G_\tau(x))^2 \mu(dx).$$

This completes the proof. □

2 Muckenhoupt criterion

Our applications will be formulated for the one-dimensional case, therefore further $\{D_t, t \in [0,1)\}$, is family of segments, $D_1 = \mathbb{R}$ and we put point D_0 equal to median m of measure μ, it means $F_\mu(m) = \frac{1}{2}$. Then we can call one of the endpoints of $D_{\tau(x)}$

$$\{s(x)\} = \Gamma(D_{\tau(x)}) \setminus \{x\}.$$

For the formulation of the application we need the following objects:

$$F(z) = F_\mu(z) - \mathbb{I}_{(D_0;+\infty)}(z)$$

and

$$q_x(z) = F(z)\mathbb{I}_{Q_{\tau(x)}}(z) + F(s(x))\mathbb{I}_{D_{\tau(x)}}(z).$$

Define $u(z) = |\int_m^z \frac{1}{\rho(y)} \mu(dy)|$. Hereafter $\mu_t = \mu(Q_t), t \in [0;1]$.

Corollary 1. *Let $u|F| \le c$, where c is positive constant, then the Poincaré inequality holds true with constant 50c:*

$$Var_\mu(g) \le 50c \int_{\mathbb{R}} g'(z)^2 \mu(dz).$$

Proof.

$$\int_{\mathbb{R}} (g(x) - G_{\tau(x)})^2 \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{1}{\mu^2_{\tau(x)}} \times$$

$$\times \Big(\int_{Q_{\tau(x)}} \int_y^x g'(z) dz \mu(dy) \Big)^2 \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{1}{\mu^2_{\tau(x)}} \Big(\int_{\mathbb{R}} g'(z) q_x(z) dz \Big)^2 \mu(dx) = (1)$$

Then we can use the Cauchy inequality for (1):

$$\int_{\mathbb{R}} \frac{1}{\mu^2_{\tau(x)}} \Big(\int_{\mathbb{R}} g'(z) q_x(z) dz \Big)^2 \mu(dx) \leq$$

$$\leq \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \Big(\Big(\int_{Q_{\tau(x)}} g'(z) q_x(z) dz \Big)^2 +$$

$$+ \Big(\int_{D_{\tau(x)}} g'(z) q_x(z) \mu(dz) \Big)^2 \Big) \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \Big(\Big(\int_{Q_{\tau(x)}} g'(z) F(z) dz \Big)^2 +$$

$$+ \Big(\int_{D_{\tau(x)}} g'(z) F(s(x)) \mu(dz) \Big)^2 \Big) \mu(dx).$$

We put

$$A = \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \times$$

$$\times \Big(\int_{Q_{\tau(x)}} g'(z) F(z) dz \Big)^2 \mu(dx)$$

$$B = \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \times$$

$$\times \Big(\int_{D_{\tau(x)}} g'(z) F(s(x)) \mu(dz) \Big)^2 \mu(dx).$$

Note that

$$|F(z)|' = \rho(z) sign(m - z), z \neq m$$

$$u(z)' = \frac{sign(z - m)}{\rho(z)}, z \neq m.$$

We use the Cauchy inequality for A:

$$A \leq \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times \int_{Q_{\tau(x)}} \frac{|F(z)|^{2-\beta}}{\rho(z)} dz \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times \int_{Q_{\tau(x)}} |F(z)|^{2-\beta} (u(z) sign(z - m))' dz \mu(dx)$$

$$= \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times (|F(z)|^{2-\beta} u(z) sign(z - m)|_{Q_{\tau(x)}} -$$

$$-(2 - \beta) \int_{Q_{\tau(x)}} |F(z)|^{1-\beta} \times$$

$$\times \rho(z) u(z) sign(z - m) sign(m - z) dz) \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times \Big(- |F(x)|^{2-\beta} u(x) - |F(s(x))|^{2-\beta} u(s(x)) +$$

$$(2 - \beta) \int_{Q_{\tau(x)}} |F(z)|^{1-\beta} \rho(z) u(z) dz) \mu(dx) \Big) \leq$$

$$\leq \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times (2 - \beta) \int_{Q_{\tau(x)}} |F(z)|^{1-\beta} \rho(z) u(z) dz) \mu(dx) \leq$$

$$\leq \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times (2 - \beta) c \int_{Q_{\tau(x)}} |F(z)|^{-\beta} \rho(z) dz) \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times \frac{2 - \beta}{1 - \beta} c \int_{Q_{\tau(x)}} (|F(z)|^{1-\beta} \times$$

$$\times sign(m - z))' dz) \mu(dx) =$$

$$= \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \int_{Q_{\tau(x)}} g'(z)^2 |F(z)|^\beta \mu(dz) \times$$

$$\times\frac{2-\beta}{1-\beta}c(|F(x)|^{1-\beta}+|F(s(x))|^{1-\beta})\mu(dx)=(2)$$

Then, using the Jensen's inequality we obtain

$$\frac{1}{2}(|F(x)|^{1-\beta}+|F(s(x))|^{1-\beta})\leq$$
$$\leq(\frac{1}{2}|F(x)|+\frac{1}{2}|F(s(x))|)^{1-\beta}=$$
$$=\frac{1}{2^{1-\beta}}\mu_{\tau(x)}^{1-\beta}.$$

From the above it follows that

$$(2)\leq\int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\int_{Q_{\tau(x)}}g'(z)^2|F(z)|^\beta\mu(dz)\times$$
$$\times\frac{2-\beta}{1-\beta}c2^\beta\mu_{\tau(x)}^{1-\beta}\mu(dx)\leq$$
$$\leq\int_{\mathbb{R}}\int_{Q_{\tau(x)}}g'(z)^2|F(z)|^\beta\mu(dz)\times$$
$$\times 2^{\beta+1}\frac{c(2-\beta)}{1-\beta}\mu_{\tau(x)}^{-1-\beta}\mu(dx)=$$
$$=2^{\beta+1}\int_{\mathbb{R}}g'(z)^2|F(z)|^\beta\times$$
$$\times\int_{D_{\tau(z)}}\mu_{\tau(x)}^{-\beta-1}\mu(dx)\mu(dz)=$$
$$=2^{\beta+1}\frac{c(2-\beta)}{1-\beta}\int_{\mathbb{R}}g'(z)^2|F(z)|^\beta\times$$
$$\times\int_{\min(z,s(z))}^{m}\Big(\frac{-\mu_{\tau(x)}^{-\beta}}{\beta}\Big)'d(\mu_{\tau(x)})\mu(dz)=$$
$$=2^{\beta+1}\frac{c(2-\beta)}{1-\beta}\times$$
$$\times\int_{\mathbb{R}}g'(z)^2|F(z)|^\beta\frac{\mu_{\tau(z)}^{-\beta}-1}{\beta}\mu(dz)\leq$$
$$\leq 2^{\beta+1}\frac{c(2-\beta)}{1-\beta}\times$$
$$\times\int_{\mathbb{R}}g'(z)^2\mu_{\tau(z)}^\beta\frac{\mu_{\tau(z)}^{-\beta}}{\beta}\mu(dz)=$$
$$=2^{\beta+1}\frac{c(2-\beta)}{\beta(1-\beta)}\int_{\mathbb{R}}g'(z)^2\mu(dz).$$

To prove inequality we choose family D_t such, that $u(s(x))=u(x)$. Then use the Cauchy inequality for B:

$$B\leq\int_{\mathbb{R}}\frac{2F(s(x))^2}{\mu^2_{\tau(x)}}\times$$
$$\times\int_{D_{\tau(x)}}g'(z)^2u(z)^\alpha\mu(dz)\times$$
$$\times\int_{D_{\tau(x)}}\frac{u(z)^{-\alpha}}{\rho(z)}dz\mu(dx)=$$
$$=\int_{\mathbb{R}}\frac{2F(s(x))^2}{\mu^2_{\tau(x)}}\times$$
$$\times\int_{D_{\tau(x)}}g'(z)^2u(z)^\alpha\mu(dz)\times$$
$$\times\int_{D_{\tau(x)}}\Big(\frac{u(z)^{1-\alpha}sign(m-z)}{1-\alpha}\Big)'dz\mu(dx)=$$
$$=\int_{\mathbb{R}}\frac{2F(s(x))^2}{\mu^2_{\tau(x)}}\times$$
$$\times\int_{D_{\tau(x)}}g'(z)^2u(z)^\alpha\mu(dz)\times$$
$$\times\frac{u(x)^{1-\alpha}+u(s(x))^{1-\alpha}}{1-\alpha}\mu(dx)=$$
$$=\int_{\mathbb{R}}\frac{2F(s(x))^2}{\mu^2_{\tau(x)}}\times$$
$$\times\int_{D_{\tau(x)}}g'(z)^2u(z)^\alpha\mu(dz)\times$$
$$\times\frac{2u(x)^{1-\alpha}}{1-\alpha}\mu(dx)\leq$$
$$\leq\int_{\mathbb{R}}2\int_{D_{\tau(x)}}g'(z)^2u(z)^\alpha\mu(dz)\times$$
$$\times\frac{2u(x)^{1-\alpha}}{1-\alpha}\mu(dx)=$$
$$=\frac{4}{(1-\alpha)}\int_{\mathbb{R}}g'(z)^2u(z)^\alpha\times$$
$$\times\int_{Q_{\tau(z)}}u(x)^{1-\alpha}\mu(dx)\mu(dz)$$
$$=\frac{4}{(1-\alpha)}\int_{\mathbb{R}}g'(z)^2u(z)^\alpha\times$$
$$\times(u^{1-\alpha}F|_{Q_{\tau(z)}}-$$
$$-(1-\alpha)\int_{Q_{\tau(z)}}u(x)^{-\alpha}u'(x)F(x)dx)\mu(dz)$$
$$=\frac{4}{(1-\alpha)}\int_{\mathbb{R}}g'(z)^2u(z)^\alpha\times$$
$$\times(u(z)^{1-\alpha}|F(z)|+u(s(z))^{1-\alpha}|F(s(z))|-$$

$$-(1-\alpha)\int_{Q_{\tau(z)}} u(x)^{-\alpha}u'(x)F(x)dx)\mu(dz) \le$$

$$\le \frac{4}{(1-\alpha)}\int_{\mathbb{R}} g'(z)^2 u(z)^{\alpha}\times$$

$$\times(cu(z)^{-\alpha}+cu(s(z))^{-\alpha}+$$

$$+(1-\alpha)\int_{Q_{\tau(z)}} u(x)^{-\alpha}|F(x)||u'(x)|dx)\mu(dz) =$$

$$= \frac{4}{(1-\alpha)}\int_{\mathbb{R}} g'(z)^2 u(z)^{\alpha}\times$$

$$\times(2cu(z)^{-\alpha}+(1-\alpha)\int_{Q_{\tau(z)}} u(x)^{-1-\alpha}\times$$

$$\times|u(x)F(x)||u'(x)|dx)\mu(dz) \le$$

$$\le \frac{4}{(1-\alpha)}\int_{\mathbb{R}} g'(z)^2 u(z)^{\alpha}\times$$

$$\times(2cu(z)^{-\alpha}+c(1-\alpha)\times$$

$$\times\int_{Q_{\tau(z)}}\Big(\frac{u(x)^{-\alpha}}{-\alpha}sign(m-x)\Big)'dx)\mu(dz) =$$

$$= \frac{4}{(1-\alpha)}\int_{\mathbb{R}} g'(z)^2 u(z)^{\alpha}\times$$

$$\times(2cu(z)^{-\alpha}+\frac{2c(1-\alpha)}{\alpha}u(z)^{-\alpha})\mu(dz) =$$

$$= \frac{8c}{(1-\alpha)}\int_{\mathbb{R}} g'(z)^2 u(z)^{\alpha}\times$$

$$\times(u(z)^{-\alpha}+\frac{(1-\alpha)}{\alpha}u(z)^{-\alpha})\mu(dz) =$$

$$= \frac{8c}{(1-\alpha)\alpha}\int_{\mathbb{R}} g'(z)^2\mu(dz).$$

Then, for $\alpha=\beta=\frac{1}{2}$

$$Var_\mu(g)=\int_{\mathbb{R}}(g(x)-G_{\tau(x)})^2\mu(dx) \le A+B \le$$

$$2^{\beta+1}\frac{c(2-\beta)}{\beta(1-\beta)}+\frac{8c}{(1-\alpha)\alpha}\int_{\mathbb{R}} g'(z)^2\mu(dz) \le$$

$$\le 50c\int_{\mathbb{R}} g'(z)^2\mu(dz).$$

3 Weighted Poincaré inequality

Corollary 2.

$$Var_\mu(g) \le 16\int_{\mathbb{R}} g'(z)^2\frac{F(z)^2}{\rho(z)^2}\mu(dz).$$

Proof. To prove inequality we choose family D_t such, that $|F(s(x))|=|F(x)|$.

$$Var_\mu(g)=\int_{\mathbb{R}}(g(x)-G_{\tau(x)})^2\mu(dx) \le$$

$$\int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\Big(\Big(\int_{Q_{\tau(x)}} g'(z)\frac{F(z)}{\rho(z)}\mu(dz)\Big)^2+$$

$$+\Big(\int_{D_{\tau(x)}} g'(z)\frac{F(s(x))}{\rho(z)}\mu(dz)\Big)^2\Big)\mu(dx).$$

We put

$$C=\int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\times$$

$$\times\Big(\int_{Q_{\tau(x)}} g'(z)\frac{F(z)}{\rho(z)}\mu(dz)\Big)^2\mu(dx)$$

$$D=\int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\times$$

$$\times\Big(\int_{D_{\tau(x)}} g'(z)\frac{F(s(x))}{\rho(z)}\mu(dz)\Big)^2\mu(dx).$$

We use the Cauchy inequality for C:

$$C=\int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\Big(\Big(\int_{Q_{\tau(x)}} g'(z)|F(z)|^{\frac{5}{4}}\times$$

$$\times\frac{1}{|F(z)|^{\frac{1}{4}}\rho(z)}\mu(dz)\Big)^2\mu(dx) \le$$

$$\le \int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\int_{Q_{\tau(x)}} g'(z)^2\frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2}\mu(dz)\times$$

$$\times\Big(\int_{Q_{\tau(x)}}\frac{1}{|F(z)|^{\frac{1}{2}}}\mu(dz)\Big)\mu(dx) =$$

$$= \int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\int_{Q_{\tau(x)}} g'(z)^2\frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2}\mu(dz)\times$$

$$\times 2|F|^{\frac{1}{2}}sign(m-z)|_{Q_{\tau(x)}}\mu(dx) =$$

$$= \int_{\mathbb{R}}\frac{2}{\mu^2_{\tau(x)}}\int_{Q_{\tau(x)}} g'(z)^2\frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2}\mu(dz)\times$$

$$\times(-4|F(x)|^{\frac{1}{2}})\mu(dx) =$$

$$= \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2} \int_{D_{\tau(z)}} \frac{2}{\mu^2_{\tau(x)}} \times$$

$$\times(-4|F(x)|^{\frac{1}{2}}\mu(dx))\mu(dz) =$$

$$= \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2} \times$$

$$\times \int_{D_{\tau(z)}} \frac{-2}{|F(x)|^{\frac{3}{2}}} \mu(dx)\mu(dz) \leq$$

$$\leq \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2} 8(F(z)^{-\frac{1}{2}} - 2)\mu(dz) \leq$$

$$\leq \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{\frac{5}{2}}}{\rho(z)^2} 8|F(z)|^{-\frac{1}{2}}\mu(dz) =$$

$$= 8 \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^2}{\rho(z)^2} \mu(dz).$$

We use the Cauchy inequality for D:

$$D = \int_{\mathbb{R}} \frac{2}{\mu^2_{\tau(x)}} \times$$

$$\times\Big(\int_{D_{\tau(x)}} g'(z) \frac{|F(s(x))|}{\rho(z)} \mu(dz)\Big)^2 \mu(dx)$$

$$= \frac{1}{2} \int_{\mathbb{R}} \Big(\int_{D_{\tau(x)}} g'(z) \frac{1}{\rho(z)} \mu(dz)\Big)^2 \mu(dx) \leq$$

$$\leq \frac{1}{2} \int_{\mathbb{R}} \int_{D_{\tau(x)}} g'(z)^2 \frac{|F(z)|^{1+\beta}}{\rho(z)^2} \mu(dz) \times$$

$$\times \int_{D_{\tau(x)}} |F(z)|^{-1-\beta} \mu(dz)\mu(dx) =$$

$$= \frac{1}{2} \int_{\mathbb{R}} \int_{D_{\tau(x)}} g'(z)^2 \frac{|F(z)|^{1+\beta}}{\rho(z)^2} \mu(dz) \times$$

$$\times \frac{2}{\beta}(\frac{1}{2^{-\beta}} - |F(x)|^{-\beta})\mu(dx) =$$

$$= \frac{1}{\beta} \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{1+\beta}}{\rho(z)^2} \times$$

$$\times \int_{Q_{\tau(z)}} (|F(x)|^{-\beta} - \frac{1}{2^{-\beta}})\mu(dx)\mu(dz) \leq$$

$$\leq \frac{1}{\beta} \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{1+\beta}}{\rho(z)^2} \times$$

$$\times \int_{Q_{\tau(z)}} |F(x)|^{-\beta} \mu(dx)\mu(dz) =$$

$$= \frac{2}{\beta(1-\beta)} \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^{1+\beta}}{\rho(z)^2} \times$$

$$\times|F(z)|^{1-\beta}\mu(dz) =$$

$$= \frac{2}{\beta(1-\beta)} \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^2}{\rho(z)^2} \mu(dz).$$

Then we can put $\beta = \frac{1}{2}$, and obtain

$$Var_\mu(g) \leq C + D \leq 16 \int_{\mathbb{R}} g'(z)^2 \frac{|F(z)|^2}{\rho(z)^2} \mu(dz).$$

□

Список використаних джерел

[1] *Kulik A.M.* A martingale bound for the entropy associated with a trimmed filtration on $\mathbb{R}^d$ / A.M. Kulik, T.D. Tymoshkevych. — Modern Stochastics: Theory and Applications.— 2014.— Vol. 1. — No. 2. p. 151—165

[2] *Muckenhoupt B.* Hardy's inequality with weights / B. Muckenhoupt.— Studia Math. — 1972.— p. 31—38

References

1. KULIK, A.M. TYMOSHKEVYCH, T.D. (2014) A martingale bound for the entropy associated with a trimmed filtration on $\mathbb{R}^d$ *Modern Stochastics: Theory and Applications.* Vol. 1. No. 2. p. 151—165.

2. MUCKENHOUPT, B. (1972) Hardy's inequality with weights *Studia Math.* p. 31–38.

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