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Напівгрупа перетворень $\mathcal{IO}_{\mathbb{Z}}$

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Semigroup of transformations $\mathcal{IO}_{\mathbb{Z}}$

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У даній роботі вивчається інверсна напівгрупа монотонних перетворень $\mathcal{IO}_{\mathbb{Z}}$. Описані структура напівгрупи, деякі найпростіші властивості її елементів, ідеали та відношення Гріна напівгрупи. Показано, що існує така система твірних, що довільний елемент напівгрупи $\mathcal{IO}_{\mathbb{Z}}$ можна подати у вигляді добутку не більш ніж чотирьох елементів з цієї системи твірних. А також доведено, що група автоморфізмів напівгрупи $\mathcal{IO}_{\mathbb{Z}}$ ізоморфна нескінченній діедральній групі $\mathbb{D}_\infty$.

Ключові слова: інверсна напівгрупа, напівгрупа перетворень, ідеал, відношення Гріна, автоморфізм.

The segroup theory is a relatively young but important direction in mathematics. It has been studied by many mathematicians during the last decades. Naturally there are issues related to the semigroups of transformations of some ordered set. We consider the so-called order-preserving transformations (a transformation $a : M \to M$ of poset M is called order-preserving, if for all $x < y$ from the domain of a we have $a(x) < a(y)$).

The present paper is dedicated to the inverse semigroup of order-preserving transformations $\mathcal{IO}_{\mathbb{Z}}$. In the first part of this paper we describe a structure of the semigroup $\mathcal{IO}_{\mathbb{Z}}$, properties of multiplication and we show, that there is such generating system, that any element of semigroup $\mathcal{IO}_{\mathbb{Z}}$ can be written as a product of at most four elements from that system. In the second part of this paper we describe ideals and Green's relations of the semigroup $\mathcal{IO}_{\mathbb{Z}}$. And in the end we prove that the group of automorphisms of $\mathcal{IO}_{\mathbb{Z}}$ is isomorphic to the infinite dihedral group $\mathbb{D}_\infty$.

Key Words: inverse semigroup, semigroup of transformations, ideal, Green's relation, automorphism.

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1 Introduction

The important direction in the theory of semigroups of transformations of some set M is the study of semigroups of transformations that somehow consistent with some additional structure on the set M. Usually, this structure is a certain partial order on M. Consistency with the order can be understood in different ways. We consider the so-called order-preserving transformations. A transformation $a : M \to M$ of poset M is called *order-preserving*, if for all $x < y$ from the domain of a we have $a(x) < a(y)$. Many papers are dedicated to the study of semigroups of order-preserving transformations of the linearly ordered sets ([1], [2], [3], [4]). The semigroups of order-preserving transformations of the linearly ordered infinite sets have been studied in papers ([5], [6]). However, at the moment there are many open questions for the case of infinite sets.

In this paper we consider the inverse semigroup $\mathcal{IO}_{\mathbb{Z}}$ of order-preserving transformations. In particular, we describe the ideals of this semigroup, its automorphisms, and show that there exists a system of generators such that every element of the semigroup $\mathcal{IO}_{\mathbb{Z}}$ can be written as a product of at most four elements from this system.

We keep the notations from [3].

2 Main definitions and properties

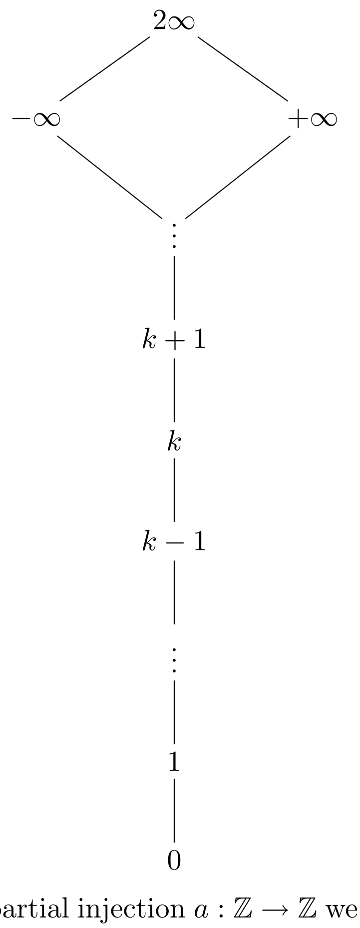
We denote by $\mathbb{Z}$ the set of all integers with the natural linear order.

Definition 2.1. The semigroup of all partial injections $a : \mathbb{Z} \to \mathbb{Z}$, which preserve the natural order on $\mathbb{Z}$ (that is for every $x < y$ from the domain of a we have $a(x) < a(y)$) is called an inverse semigroup, which preserves an order on the set $\mathbb{Z}$. We denote this semigroup by $\mathcal{IO}_{\mathbb{Z}}$.

We define the norm of a subset A on this set by the following rule:

$$||A|| = \begin{cases} k, \text{if } A \text{ is finite and contains } k \text{ elements,} \\ +\infty, \text{if } A \text{ contains an infinite number of elements and is bounded from below,} \\ -\infty, \text{if } A \text{ contains an infinite number of elements and is bounded from above,} \\ 2\infty, \text{if } A \text{ contains an infinite number of elements and is not bounded.} \end{cases}$$

The norms take values in the set $\{0, 1, 2, \ldots\} \cup \{+\infty, -\infty, 2\infty\}$, which is a poset with order determined by following Hasse diagram:



For a partial injection $a : \mathbb{Z} \to \mathbb{Z}$ we denote by $\mathrm{dom}(a)$ and $\mathrm{im}(a)$ the domain and the image of a respectively. If a is monotone, then $||\,\mathrm{dom}(a)|| = ||\,\mathrm{im}(a)||$. In this case the number $\mathrm{nrank}(a) = ||\,\mathrm{dom}(a)|| = ||\,\mathrm{im}(a)||$ is called the normalized rank of a transformation a.

Proposition 1. *For any $a, b \in \mathcal{IO}_{\mathbb{Z}}$*

$$\mathrm{nrank}(ab) \leqslant \min(\mathrm{nrank}(a), \mathrm{nrank}(b)).$$

Proof. For every element $x \in \mathbb{Z}$ we have $(ab)(x) = b(a(x))$. This implies $\mathrm{im}(ab) \subset \mathrm{im}(b)$, so, clearly, we have $||\,\mathrm{im}(ab)|| \leqslant ||\,\mathrm{im}(b)||$. Hence $\mathrm{nrank}(ab) \leqslant \mathrm{nrank}(b)$. Let x be an arbitrary element from $\mathrm{dom}(ab)$. Since $(ab)(x) = b(a(x))$, we have $x \in \mathrm{dom}(a)$. Hence, $\mathrm{dom}(ab) \subset \mathrm{dom}(a)$, which implies inequality $||\,\mathrm{dom}(ab)|| \leqslant ||\,\mathrm{dom}(a)||$. Thus we have $\mathrm{nrank}(ab) \leqslant \mathrm{nrank}(a)$.

For any two subsets A and B of the set $\mathbb{Z}$ with equal norms less then 2∞, there exists a unique element a of the semigroup $\mathcal{IO}_{\mathbb{Z}}$ such that $A = \mathrm{dom}(a)$ and $B = \mathrm{im}(a)$. If $||A|| = ||B|| = 2\infty$, then there are infinitely many elements a of the set $\mathcal{IO}_{\mathbb{Z}}$ such that $A = \mathrm{dom}(a)$ and $B = \mathrm{im}(a)$. In the latter case in order to determine a one needs to specify its value at some point. For convenience, any element $a \in \mathcal{IO}_{\mathbb{Z}}$ with $A = \mathrm{dom}(a)$, $B = \mathrm{im}(a)$, and $m = a(n)$ will be denoted by $\pi_{A,B,n\to m}$, where $n \in A$, $m \in B$. The element $n \in A$ in this notation will be chosen as follows. It is the minimal element of the set A with respect to the order $0 < 1 < -1 < 2 < -2 < \ldots < k < -k < k+1 < -k-1 < \ldots$.

The semigroup $\mathcal{IO}_{\mathbb{Z}}$ contains an identity, which is the identical transformation $id_{\mathbb{Z}}$, and a zero, which is the nowhere defined transformation. The set of idempotents

$$E(\mathcal{IO}_{\mathbb{Z}}) = \{\mathrm{id}_A, A \subset \mathbb{Z}\} := \{\pi_{A,A}, A \subset \mathbb{Z}\}$$

forms a semigroup, which is isomorphic to the semigroup $(2^{\mathbb{Z}}, \cap)$. The idempotent e is primitive element if and only if $e = \mathrm{id}_{\{n\}}$ for some $n \in \mathbb{Z}$. The semigroup $\mathcal{IO}_{\mathbb{Z}}$ is an inverse semigroup since it is a regular semigroup in which idempotents commute. Inverse element of an element $\pi_{A,B,n\to m}$ is $\pi_{B,A,m\to n}$.

Proposition 2. *Each element of the semigroup $\mathcal{IO}_{\mathbb{Z}}$ can be written as a product of at most four elements of the set*

$$G := \{a \in \mathcal{IO}_{\mathbb{Z}} |\, \mathrm{dom}(a) = \mathbb{Z} \text{ or } \mathrm{im}(a) = \mathbb{Z}\}.$$

Proof. Every element of a normalized rank less then 2∞ can be written as a product of two elements of a normalized rank 2∞: $\pi_{A,B,n\to m} = \mathrm{id}_C \cdot b$, where $C \cap \mathrm{dom}(b) = A$ and $b(A) = B$. Every element of a normalized rank 2∞ can be written as a product of two elements from the set G. Indeed, let $a \in \mathcal{IO}_{\mathbb{Z}}$ and $\mathrm{nrank}(a) = 2\infty$, moreover $a = \pi_{A,B,n\to m}$, then $a = \pi_{A,\mathbb{Z},n\to k}\pi_{\mathbb{Z},B,k\to m}$.

3 Ideals and Green's relations

Theorem 3.1. *Let $a \in \mathcal{IO}_{\mathbb{Z}}$. Then*

1) *a principal left ideal generated by an element a has the following form*

$$\mathcal{IO}_{\mathbb{Z}} \cdot a = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{im}(b) \subseteq \operatorname{im}(a)\}$$

2) *a principal right ideal generated by an element a has the following form*

$$a \cdot \mathcal{IO}_{\mathbb{Z}} = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{dom}(b) \subseteq \operatorname{dom}(a)\}$$

3) *a principal two-sided ideal generated by an element a has the following form*

$$\mathcal{IO}_{\mathbb{Z}} \cdot a \cdot \mathcal{IO}_{\mathbb{Z}} = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{nrank}(b) \leqslant \operatorname{nrank}(a)\}$$

Proof.

1) Let $M_a = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{im}(b) \subseteq \operatorname{im}(a)\}$. If $c \in \mathcal{IO}_{\mathbb{Z}}$ then $ca(x) = a(c(x))$ by definition. Thus $\operatorname{im}(ca) \subseteq \operatorname{im}(a)$ and $\mathcal{IO}_{\mathbb{Z}} \cdot a \subseteq M_a$. Now let $\operatorname{im}(b) \subseteq \operatorname{im}(a)$. Denote $A = \operatorname{dom}(b)$, $B = \operatorname{im}(b)$. Then $||A|| = ||B||$. Denote $C = a^{-1}(B)$. Since a^{-1} is partially defined injection, $||C|| = ||B||$. So $||A|| = ||C||$ and there exists a transformation $\alpha : A \to C$. Then $b = \alpha \cdot a$. Hence $M_a \subseteq \mathcal{IO}_{\mathbb{Z}} \cdot a$.

2) Analogous to the proof of statement 1.

3) Let $M_a = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{nrank}(b) \leqslant \operatorname{nrank}(a)\}$. For arbitrary $c, d \in \mathcal{IO}_{\mathbb{Z}}$ we have the following inequalities:

$$\operatorname{nrank}(ca) \leqslant \min\{\operatorname{nrank}(c), \operatorname{nrank}(a)\}$$

and

$$\operatorname{nrank}(ad) \leqslant \min\{\operatorname{nrank}(a), \operatorname{nrank}(d)\}.$$

Let $b = c \cdot a \cdot d \in \mathcal{IO}_{\mathbb{Z}}$. Then we have

$$\operatorname{nrank}(b) \leqslant$$
$$\leqslant \min\{\operatorname{nrank}(c), \operatorname{nrank}(a), \operatorname{nrank}(d)\} \leqslant$$
$$\leqslant \operatorname{nrank}(a).$$

Thus $\mathcal{IO}_{\mathbb{Z}} \cdot a \cdot \mathcal{IO}_{\mathbb{Z}} \subseteq M_a$. Let now $\operatorname{nrank}(b) \leqslant \operatorname{nrank}(a)$. Denote $A = \operatorname{dom}(b)$, $B = \operatorname{im}(b)$. Let $C \subseteq \operatorname{dom}(a)$ be an arbitrary subset, such as $||C|| = \operatorname{nrank}(b) = ||A||$, $||D|| = ||a(C)|| = ||C|| = \operatorname{nrank}(b) = ||B||$. There exist some transformations $\alpha : C \to A$ and $\beta : D \to B$. Thus $b = \alpha^{-1} \cdot a\beta$. Hence $M_a \subseteq \mathcal{IO}_{\mathbb{Z}} \cdot a \cdot \mathcal{IO}_{\mathbb{Z}}$.

For every $k \in \mathbb{N}_0$ we denote by

$$I_k = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{nrank}(b) \leqslant k\},$$

$$I_F = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{nrank}(b) < \infty\},$$

$$I_{+\infty} = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{nrank}(b) \leqslant +\infty\},$$

$$I_{-\infty} = \{b \in \mathcal{IO}_{\mathbb{Z}} : \operatorname{nrank}(b) \leqslant -\infty\}.$$

Set $F < \pm\infty$, $F > k$ for all $k \in \mathbb{N}_0$.

Proposition 3. *The sets I_F and I_k, $k \in \mathbb{N}_0$, are two-sided ideals.*

Proof. For the sets I_k, $k \in \mathbb{N}_0$, this statement follows from the definition of an ideal and proposition 1. For arbitrary $a \in I_F$ and $b \in \mathcal{IO}(\mathbb{N}, -\mathbb{N})$ we have $\operatorname{nrank}(bab) \leqslant \operatorname{nrank}(a) < \infty$. So $bab \in I_F$. Hence I_F is a two-sided ideal.

Proposition 4. *$I_{r_1} \subseteq I_{r_2}$ if and only if $r_1 \leqslant r_2$.*

Proof. Obviously, $r_1 \leqslant r_2$ implies $I_{r_1} \subseteq I_{r_2}$. We prove the converse statement by contradiction. Assume that $I_{+\infty} \subset I_{-\infty}$. In particular, it means that some element a with a normalized rank $+\infty$ belongs to the ideal $I_{-\infty}$. By definition of the ideal $I_{-\infty}$ we have that domain of every element of this ideal is bounded from above. However $\operatorname{dom}(a)$ contains an infinite number of positives. We arrive to contradiction. Analogously one can show that $I_{-\infty} \not\subset I_{+\infty}$. This completes the proof.

Proposition 5. *Let $\mathcal{I}$ be the set of principal two-sided ideals. Then every two-sided ideal of the semigroup $\mathcal{IO}_{\mathbb{Z}}$ is a union of at most two sets from the set $\mathcal{I} \cup I_F$.*

Proof. Let $I \subset \mathcal{IO}_{\mathbb{Z}}$ be two-sided ideal. Every two-sided ideal is a union of principal two-sided ideals. Let the set of normalized ranks of elements of the ideal I contain 2∞. Then decomposition of the ideal I into the union of principal ideals contains the principal ideal $\mathcal{IO}_{\mathbb{Z}}$. Therefore, $I = \mathcal{IO}_{\mathbb{Z}}$.

Let the set of normalized ranks of elements of the ideal I is bounded from above by an element l ($\pm\infty$). Then we take the ideal I_l ($I_{\pm\infty}$) in decomposition of the ideal I. If the set of normalized ranks of elements of the ideal I is not bounded from above, then we take the ideal I_F in decomposition of the ideal I.

Consider the case when the set of normalized ranks of elements of the ideal I is not bounded from above by elements $-\infty$ and $+\infty$. There exist elements $a, b \in I$ such that $\operatorname{nrank}(a) = -\infty$ and $\operatorname{nrank}(b) = +\infty$. So $I_{-\infty} \cup I_{+\infty} = I(a) \cup I(b) \subset I$. Suppose $I \not\subset I_{-\infty} \cup I_{+\infty}$. Then there exists an element $a \in I$ such that $a \in \mathcal{IO}_{\mathbb{Z}} \setminus (I_{-\infty} \cup I_{+\infty})$, that is normalized rank of this element is greater than $-\infty$ or $+\infty$. Therefore $\operatorname{nrank}(a) = 2\infty$. We arrive to contradiction. So $I \subset I_{-\infty} \cup I_{+\infty}$. Hence $I = I_{-\infty} \cup I_{+\infty}$. All other cases are similar.

The next theorem provides a description of Green's relations in $\mathcal{IO}_{\mathbb{Z}}$.

Theorem 3.2. *Let $a, b \in \mathcal{IO}_{\mathbb{Z}}$. Then*

1) $a\mathcal{L}b$ if and only if $\operatorname{im}(a) = \operatorname{im}(b)$;

2) $a\mathcal{R}b$ if and only if $\operatorname{dom}(a) = \operatorname{dom}(b)$;

3) $a\mathcal{H}b$ if and only if $\operatorname{im}(a) = \operatorname{im}(b)$ and $\operatorname{dom}(a) = \operatorname{dom}(b)$;

4) $a\mathcal{J}b$ if and only if $\operatorname{nrank}(a) = \operatorname{nrank}(b)$;

5) $\mathcal{D} = \mathcal{J}$.

Proof.

1) Let $a\mathcal{L}b$. Then $a \in L_b$, $b \in L_a$ and $\operatorname{im}(a) \subseteq \operatorname{im}(b)$, $\operatorname{im}(b) \subseteq \operatorname{im}(a)$. Therefore $\operatorname{im}(a) = \operatorname{im}(b)$. On the other hand, if $\operatorname{im}(a) = \operatorname{im}(b)$, then $b \in \mathcal{IO}_{\mathbb{Z}}a$ and $a \in \mathcal{IO}_{\mathbb{Z}}b$. This implies $\mathcal{IO}_{\mathbb{Z}}b \subseteq \mathcal{IO}_{\mathbb{Z}}a$ and $\mathcal{IO}_{\mathbb{Z}}a \subseteq \mathcal{IO}_{\mathbb{Z}}b$. Thus we have $\mathcal{IO}_{\mathbb{Z}}a = \mathcal{IO}_{\mathbb{Z}}b$, so $a\mathcal{L}b$.

2) Analogous to the proof of statement 1.

3) Let $a\mathcal{H}b$. Then $a\mathcal{L}b$ and $a\mathcal{R}b$ by definition, that is $\operatorname{im}(a) = \operatorname{im}(b)$ and $\operatorname{dom}(a) = \operatorname{dom}(b)$.

4) Let $a\mathcal{J}b$. Then $a \in J_b$, $b \in J_a$ and $\operatorname{nrank}(a) \leqslant \operatorname{nrank}(b)$, $\operatorname{nrank}(b) \leqslant \operatorname{nrank}(a)$. Therefore $\operatorname{nrank}(a) = \operatorname{nrank}(b)$. On the other hand if $\operatorname{nrank}(a) = \operatorname{nrank}(b)$, then $b \in \mathcal{IO}_{\mathbb{Z}}a\mathcal{IO}_{\mathbb{Z}}$ and $a \in \mathcal{IO}_{\mathbb{Z}}b\mathcal{IO}_{\mathbb{Z}}$. This implies $\mathcal{IO}_{\mathbb{Z}}a\mathcal{IO}_{\mathbb{Z}} \subseteq \mathcal{IO}_{\mathbb{Z}}b\mathcal{IO}_{\mathbb{Z}}$ and $\mathcal{IO}_{\mathbb{Z}}b\mathcal{IO}_{\mathbb{Z}} \subset \mathcal{IO}_{\mathbb{Z}}a\mathcal{IO}_{\mathbb{Z}}$. Thus we have $\mathcal{IO}_{\mathbb{Z}}a\mathcal{IO}_{\mathbb{Z}} = \mathcal{IO}_{\mathbb{Z}}b\mathcal{IO}_{\mathbb{Z}}$, so $a\mathcal{J}b$.

5) $\mathcal{D} \subseteq \mathcal{J}$ by definition. Let $a\mathcal{J}b$, then $\operatorname{nrank}(a) = \operatorname{nrank}(b)$ and $||\operatorname{dom}(a)|| = ||\operatorname{im}(b)||$. So there exists $c \in \mathcal{IO}_{\mathbb{Z}}$ such that $\operatorname{dom}(c) = \operatorname{dom}(a)$ and $\operatorname{im}(c) = \operatorname{im}(b)$. This implies $a\mathcal{R}c$ and $c\mathcal{L}b$. Then $a(\mathcal{R} \circ \mathcal{L})b$. Thus $a\mathcal{D}b$. Hence $\mathcal{J} \subseteq \mathcal{D}$.

4 Automorphisms

Recall that the infinite dihedral group $\mathbb{D}_\infty$ is the group of symmetries (both rotations and reflections) of the set of integers $\mathbb{Z}$ with respect to composition.

Theorem 4.1. $\operatorname{Aut}(\mathcal{IO}_{\mathbb{Z}}) \simeq \mathbb{D}_\infty$.

Proof. Let f be an automorphism of $\mathcal{IO}_{\mathbb{Z}}$. An automorphism maps an ideal to an ideal, moreover the relation of inclusion is preserved. One can conclude from the diagram of ideals that $f(I_k) = I_k$ for every $k \in \mathbb{N}_0$, $f(I_F) = I_F$ and $f(I_{\pm\infty}) = I_{\pm\infty}$ or $f(I_{\pm\infty}) = I_{\mp\infty}$. Let $a \in \mathcal{IO}_{\mathbb{Z}}$ be an element of a normalized rank k for some $k \in \mathbb{N}_0$. Then $a \in I_k \setminus I_{k-1}$. This implies $f(a) \in I_k$ and $f(a) \notin I_{k-1}$. Thus $f(a) \in I_k \setminus I_{k-1}$ and $\operatorname{nrank}(f(a)) = k$. Similarly one can prove that $\operatorname{nrank}(f(a)) = \pm\infty$ for a of normalized rank $\pm\infty$ in case $f(I_{\pm\infty}) = I_{\pm\infty}$ and $\operatorname{nrank}(f(a)) = \mp\infty$ for a of normalized rank $\pm\infty$ in case $f(I_{\pm\infty}) = I_{\mp\infty}$. Let a be an element of normalized rank 2∞. Then it does not belong to any ideal I_k, $I_{+\infty}$ and $I_{-\infty}$ for every $k \in \mathbb{N}_0$. Thus $f(a)$ does not belong to any of these ideals. Hence it has normalized rank 2∞.

An automorphism f induces a bijection φ on the set of primitive elements. The map φ specifies a permutation τ_φ on the set $\mathbb{Z}$ by the following rule: $\tau_\varphi(x) = y$ if and only if $\varphi(id_{\{x\}}) = id_{\{y\}}$. So $\pi_{\{i\},\{j\}} = id_{\{i\}}\pi_{\{i\},\{j\}}id_{\{j\}}$. This implies $f(\pi_{\{i\},\{j\}}) = f(id_{\{i\}})f(\pi_{\{i\},\{j\}})f(id_{\{j\}})$ or $f(\pi_{\{i\},\{j\}}) = id_{\{\tau(i)\}}f(\pi_{\{i\},\{j\}})id_{\{\tau(j)\}}$. Since f preserves normalized ranks of the type k, where $k \in \mathbb{N}_0$, then $\operatorname{nrank} f(\pi_{\{i\},\{j\}}) = 1$. On the other hand $\operatorname{dom}(id_{\{\tau(i)\}}f(\pi_{\{i\},\{j\}})id_{\{\tau(j)\}})$ does not contain any element $\tau(z)$, if $z \neq i$. Thus $\operatorname{dom} f(\pi_{\{i\},\{j\}}) = \{\tau(i)\}$. Analogously $\operatorname{im} f(\pi_{\{i\},\{j\}}) = \{\tau(j)\}$. Therefore $f(\pi_{\{i\},\{j\}}) = \pi_{\{\tau(i)\},\{\tau(j)\}}$.

Let $a \in \mathcal{IO}_{\mathbb{Z}}$ be an element such that $i \in \operatorname{dom}(a)$ and $a(i) = j$. Then $\operatorname{id}_{\{i\}} \cdot a \cdot \operatorname{id}_{\{j\}} = \pi_{\{i\},\{j\}}$. Hence $\operatorname{id}_{\{\tau(i)\}} f(a) \operatorname{id}_{\{\tau(j)\}} = \pi_{\{\tau(i)\},\{\tau(j)\}}$. So $\tau(i) \in \operatorname{dom}(f(a))$ and $f(a)(\tau(i)) = \tau(j)$. Since $i \in \operatorname{dom}(a)$ is arbitrary, then $\operatorname{dom}(f(a)) \supset \tau(\operatorname{dom}(a))$. For every $i \notin \operatorname{dom}(a)$ we have $\operatorname{id}_{\{i\}} \cdot a = 0$. So $\operatorname{id}_{\{\tau(i)\}} f(a) = 0$ and $\tau(i) \notin \operatorname{dom} f(a)$. Thus $\operatorname{dom} f(a) = \tau(\operatorname{dom}(a))$ and $\varphi(a)(\tau(i)) = \tau(j)$ for every $i \in \operatorname{dom}(a)$. Thus, an automorphism f is completely determined by permutation τ_f, which it induces on the set $\mathbb{Z}$.

If there exist pairs $i < j$, $t < s$ such that $\tau_f(i) < \tau_f(j)$, $\tau_f(s) < \tau_f(t)$, then we have $f(a) = \pi_{\{\tau_f(i),\tau_f(j)\},\{\tau_f(t),\tau_f(s)\}} \notin \mathcal{IO}_{\mathbb{Z}}$ for $a = \pi_{\{i,j\},\{t,s\}} \in \mathcal{IO}_{\mathbb{Z}}$. We arrive to contradiction. So for all i, j such that $i < j$ we have $\tau_f(i) < \tau_f(j)$ or for all i, j such that $i < j$ we have $\tau_f(i) > \tau_f(j)$.

If $\tau_f(i) < \tau_f(j)$ for every i, j such that $i < j$, then $\tau_f(i) = i + t$. In this case $f(a) = \pi_{\mathrm{dom}(a),\mathrm{im}(a)+t,n\to a(n)+t}$, $t \in \mathbb{Z}$. If $\tau_f(i) > \tau_f(j)$ for every i, j such that $i < j$, then $\tau_f(i) = -i - t$. In this case $f(a) = \pi_{\mathrm{dom}(a),\mathrm{im}(a)+t,n\to a(n)+t}$, $t \in \mathbb{Z}$.

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