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Деформації атомів степені 2 на поверхнях з краєм

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Deformations of atoms of complexity 2 on surfaces with boundary

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Розглядаються такі функції Морса на орієнтованих компактних поверхнях з межею, що їх обмеження на межу також є функцією Морса. Знайдено всі можливі деформації функцій в околах критичних рівнів з двома критичними точками до функцій, у яких по одній критичній точці на кожному критичному рівні. Показано, що таку деформацію в кожному з випадків можна здійснити двома способами.

Ключові слова: функція Морса, атом m-функції, топологічна класифікація.

We consider m-functions on a closed oriented two-dimensional manifold (surface) M with boundary ∂M that is smooth functions f such that: 1) f is a Morse function, 2) restriction $f|_{\partial M}$ is a Morse function. We investigate changing of neighborhood of a critical level (f-atom) points under a small deformation of the function. Hear f-atom is considered up to homeomorphism that maps level component of function into level component and preserve the direction of function arising. An atom with one critical point (a simple atom) is stable under smooth deformation of the function. In one a one-parameter family of m-function, an atom with 2 critical points (complexity 2) can appear. Each atom of complexity 2 splits at a small movement in the one-parameter family of m-functions into two simple atoms by two ways. All possible deformations of atoms of complexity 2 of m-functions on surfaces with boundary to simple atoms are considered.

Key words: Morse function, atom of m-function, topological classification.
Статтю представив д. ф.-м. н. проф. Кириченко В.В.

Introduction. Let M be a closed oriented two-dimensional manifold (surface). Let f be a smooth function on M. Consider the Hamiltonian dynamical system given by the equation $\frac{dx}{dt} = \mathrm{sgrad}\, f(x)$, $x \in M$. Then its trajectories lie on the components of the line-level function f. These components are called layers. Homeomorphism of surface reflecting layers on layers is called a layered equivalence. So the layered classification of functions sets the topological classification of Hamiltonian dynamical systems. On the set of all functions one can identify an open everywhere dense subset consisting of simple Morse functions. To study the functions A.Konrod [4] and H.Rib [7] introduced a graph, which is obtained from the surface after collecting each layer to the point. This graph is a complete topological invariant of simple Morse functions.

The analogue of Morse functions for manifolds with boundary are m-functions. The topological properties of m-functions were investigated in [2,3,5,6,8–12]. In [3] all possible atoms of complexity 2 of m-functions on surfaces with boundary are considered. The purpose of the scientific work is to describe all possible deformations of atoms of complexity 2 to simple atoms.

m-function. Simple atoms. Let M is a smooth manifold of dimension n, $f : M \longrightarrow \mathbb{R}$ is a smooth function.

The point $x \in M$ is a *critical* for function f if $df(x) = 0$. Herewith $f(x)$ is a *critical value* for function f.

The critical point is called *non-degenerate* if the second differential $d^2 f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$ is non-degenerate at this point.

Definition. A smooth function on a manifold M is a *Morse function* if it has no degenerate critical points.

Let M is a smooth compact manifold of dimension n with boundary.

Definition. The function $f : M \longrightarrow \mathbb{R}$ is *m-function* if:

a) all its critical points are non-degenerate and do not situated on ∂M;

б) the boundary of manifold can be represented as a union $\partial M = \partial M_{-} \cup \partial M_0 \cup \partial M_{+}$ so that the restriction f_∂ of function f on ∂M_0 is Morse function and when the set $\partial M_{-} \neq \emptyset$ ($\partial M_{+} \neq \emptyset$) then function f takes minimum (maximum) value.

We will call m-functions f and g on the surfaces X^2 and Y^2 *layered equivalent* if there is a diffeomorphism $\lambda : X^2 \longrightarrow Y^2$ which maps connected components line-level of function f into connected components line-level of function g. The pair (X^2, f) is layered equivalent to the pair (Y^2, g).

We will explore the layered equivalence of m-functions in the neighborhoods of their critical values.

Definition. The neighborhood of the P^2 critical layer which is given by the inequality $c-\varepsilon \leqslant f \leqslant c+\varepsilon$ for sufficiently small ε which stratified on the line level of function f and which is considered accurate to the layered equivalence $P^2 = \{x : -\varepsilon \leqslant f(x) - c \leqslant \varepsilon\}$ is called an *atom*.

If the critical value c is local minimum or local maximum then an atom is called an *atom A*. If the critical value c is saddle then the corresponding atom is called *a saddle* .

Atom is called *simple* if m-function in the (P^2, f) is simple. Remaining atoms are called *complex*.

Let c is the critical value of function f on X^2 and c' is the critical value of function g on Y^2. We consider their specific layers: $f^{-1}(c)$ and $g^{-1}(c')$ and suppose that these layers are coherent.

m-functions f and g are called *layered equipped equivalent* in the neighborhood of their particular layers $f^{-1}(c)$ and $g^{-1}(c')$ if there are two positive numbers ε i ε' and diffeomorphism $\lambda : f^{-1}(c-\varepsilon, c+\varepsilon) \longrightarrow g^{-1}(c'-\varepsilon', c'+\varepsilon')$, which converts line-level of function f in line-level of function g and preserves the direction of functions growth that λ maps the range $(f > c)$ into the range $(g > c')$.

We consider the pair (P^2, f) where P^2 is a connected compact surface with non-empty edge ∂P^2 and f is m-function on it which has exactly one critical value C moreover $f^{-1}(c-\varepsilon) \cup f^{-1}(c+\varepsilon) = \partial P^2$. The class of equipped layered equivalence of this pair (P^2, f) is called *f-atom* or equipped atom.

Remark. Each atom corresponds to two f-atoms. Sometimes these two atoms can coincide, be equivalent.

Definition. *f-atom* is an atom with the previous definition for which the splitting of rings into positive and negative is observed .

The simplest examples of atoms is a neighborhood of maxima, minima and saddle critical points on surfaces. When passing through the critical level of m-function of the common situation levels adjustment may occur [2] (fig.1).

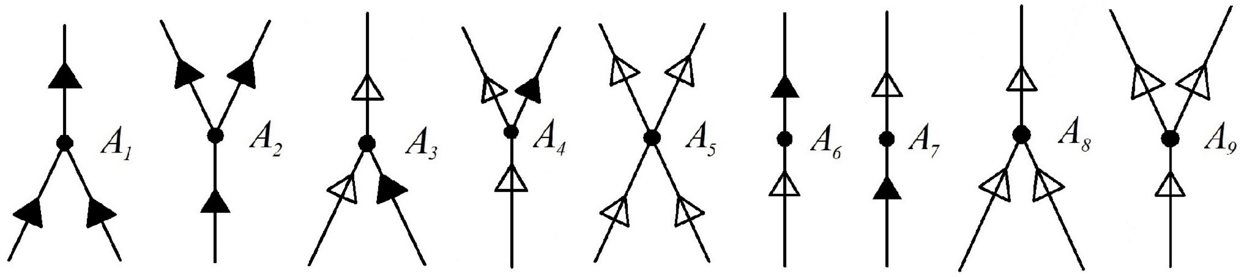

Fig. 1.

Atoms of complexity 2. All possible atoms of complexity 2 of m-functions on surfaces with boundary have been considered in the work [3].

Each atom of complexity 2 splits at a small movement in the one-parameter family of m-functions into two simple atoms by two ways (fig.2-11):

We stretch the points to different critical levels. Fig.2 shows that atom B_1 splits into two simple atoms. Firstly we consider the case when the first critical point is above the second critical point. Having made a small movement we obtain two simple atoms: A_1 and A_6. If we have a second critical point at a higher critical level than the first one we get other simple atoms: A_6 and A_3. And similarly for B_2, whereas B_1 and B_2 are the same as atoms although different as f-atoms.

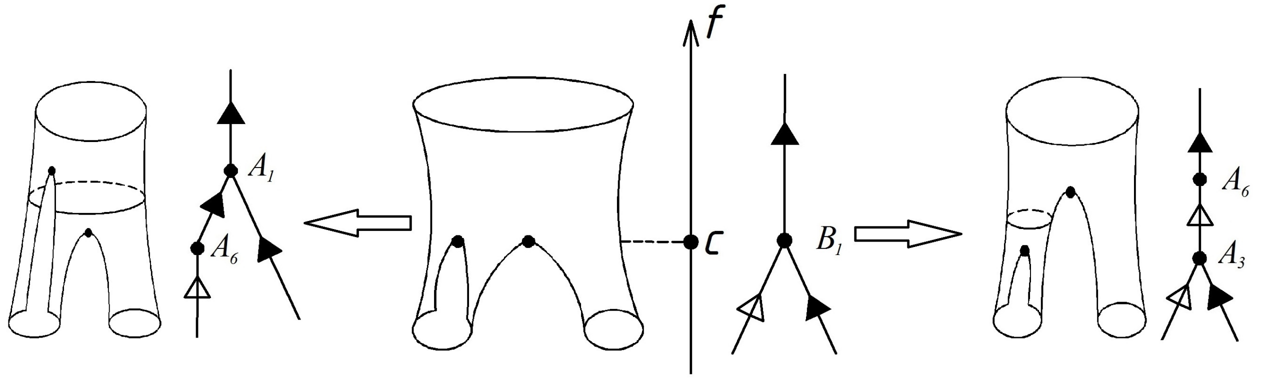

Fig. 2.

Fig.3 shows that atom B_3 splits into two simple atoms by two ways: A_3, A_7 and A_7, A_1. Similarly for B_4, whereas both atoms B_3 and B_4 are the same as atoms although different as f-atoms.

Atom B_5 splits into two simple atoms: A_8, A_5 and A_3, A_6. And similarly for B_6, whereas B_5 and B_6 are the same as atoms although different as f-atoms.

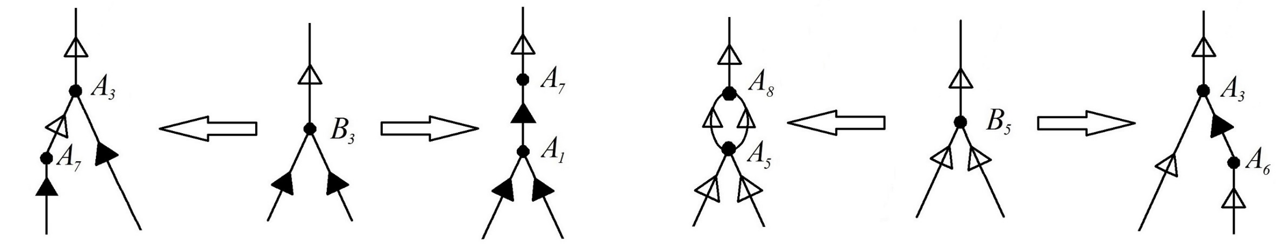

Fig. 3.

Fig.4 shows that atom B_7 splits into two simple atoms by two ways: A_3, A_8 and A_8, A_3. Similarly for B_8.

Atom B_9 splits into two simple atoms: A_5, A_9 and A_9, A_5. And similarly for B_{10}.

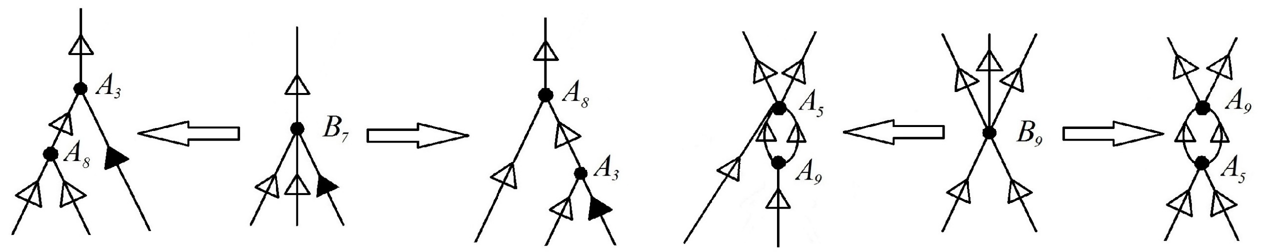

Fig. 4.

Fig.5 shows that atom B_{11} splits into two simple atoms by two ways: A_1, A_1 and A_1, A_1. Similarly for B_{12}.

Atom B_{13} splits into two simple atoms: A_3, A_3 and A_3, A_1. And similarly for B_{14}, whereas both atoms B_{13} and B_{14} are the same as atoms although different as f-atoms.

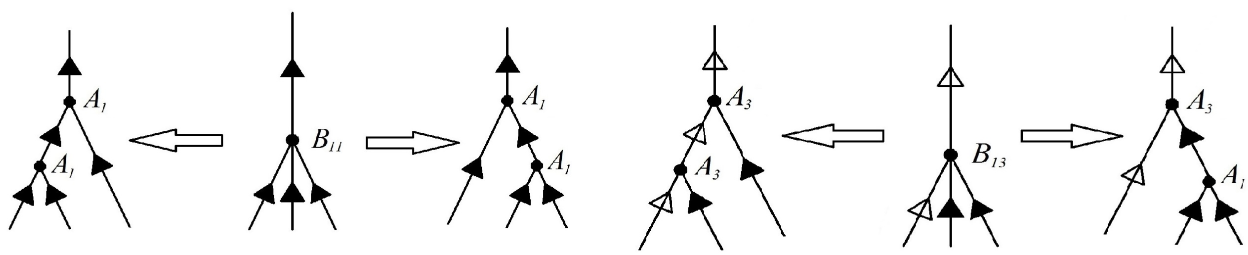

Fig. 5.

Fig.6 shows that atom B_{15} splits into two simple atoms by two ways: A_5, A_3 and A_5, A_4. Similarly for B_{16}.

Atom B_{17} splits into two simple atoms: A_5, A_7 and A_9, A_3. And similarly for B_{18}.

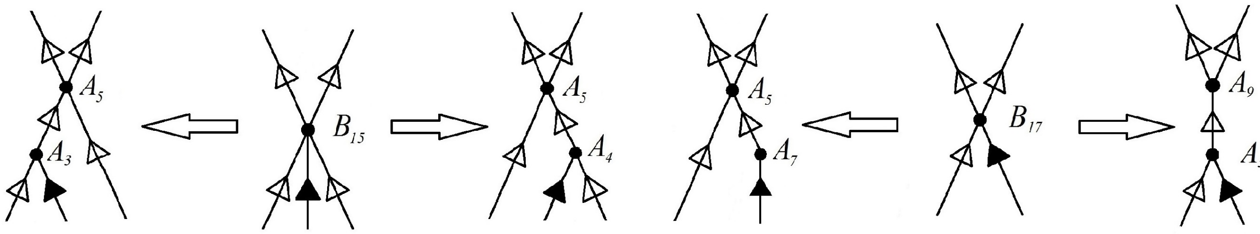

Fig. 6.

Fig.7 shows that atom B_{19} splits into two simple atoms by two ways: A_3, A_9 and A_9, A_3. Similarly for B_{20}.

Atom B_{21} splits into such simple atoms: A_5, A_5 and A_5, A_5.

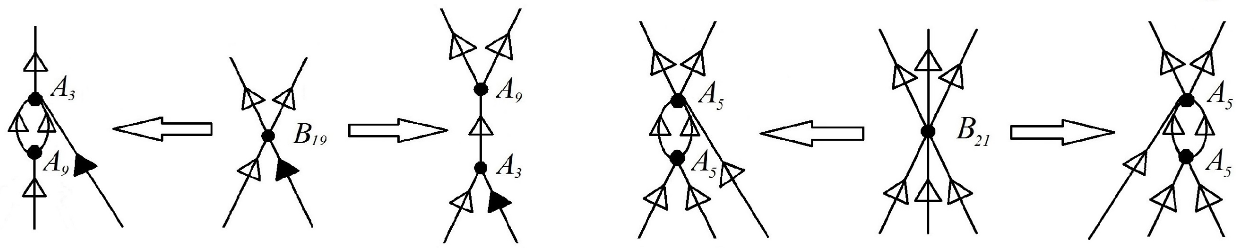

Fig. 7.

Fig.8 shows that atom B_{22} splits into two simple atoms by two ways: A_1, A_2 and A_2, A_1.

And atom B_{23} splits into such simple atoms: A_4, A_4 and A_4, A_4.

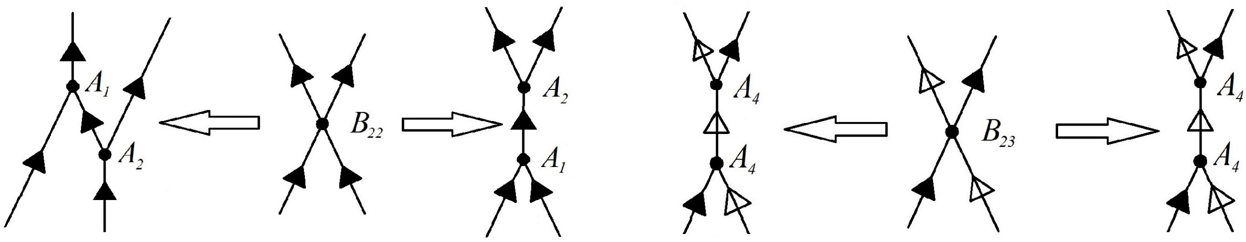

Fig. 8.

Fig.9 shows that atom B_{24} splits into two simple atoms by two ways: A_3, A_2 and A_4, A_3. And in the same way for B_{25}.

Atom B_{26} splits into the following simple atoms: A_3, A_3 and A_5, A_5.

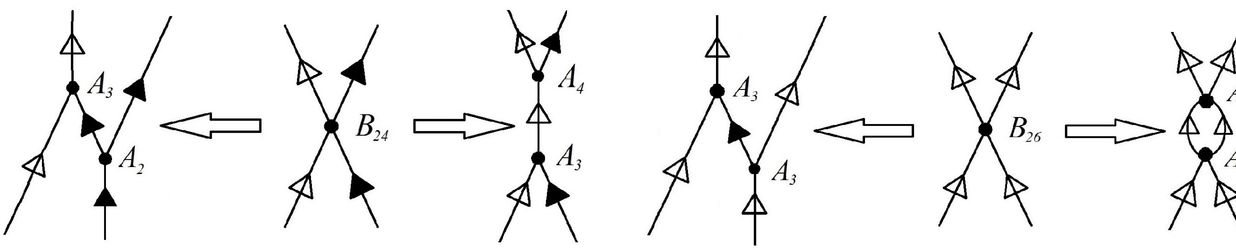

Fig. 9.

Fig.10 shows that atom B_{27} splits into two simple atoms by two ways: A_9, A_7 and A_9, A_7. Similarly for B_{28}.

Atom B_{29} splits into the following simple atoms: A_8, A_9 and A_7, A_6.

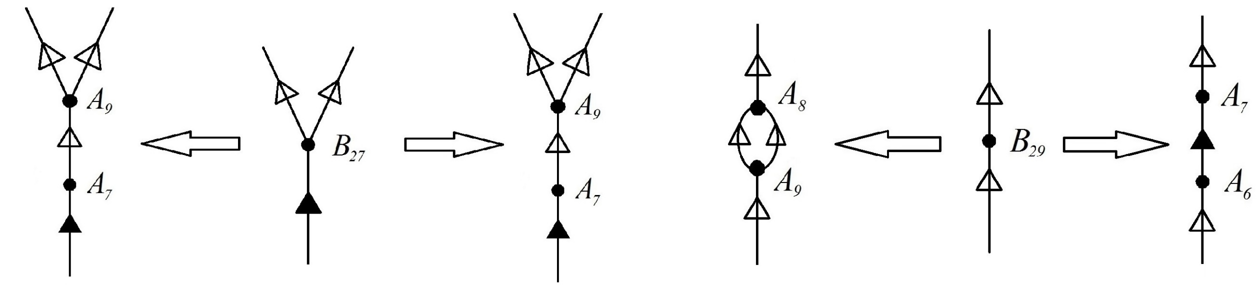

Fig. 10.

Fig.11 shows that atom B_{30} splits into two simple atoms: A_5, A_5 and A_9, A_8.

Atom B_{31} splits into two simple atoms by two ways: A_9, A_9 and A_9, A_9. And similarly for B_{32}.

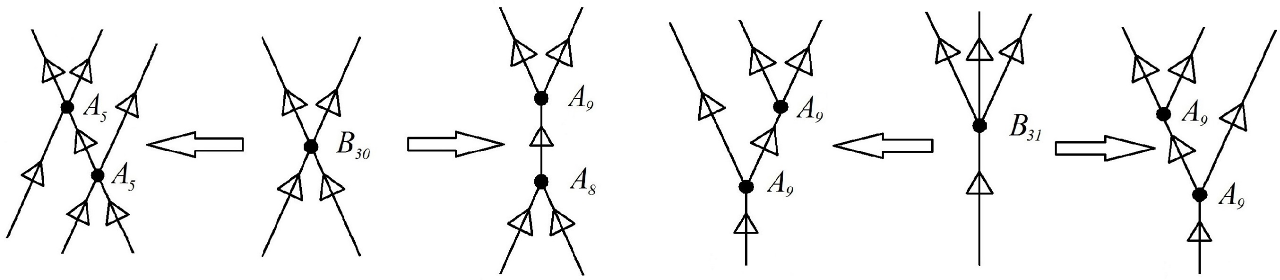

Fig. 11.

Summarizing all of the above the following theorem is just.

Theorem. All possible deformations of atoms of complexity 2 to simple atoms are set in the table:

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
A_1	A_7	A_3	A_6	A_8	A_5	A_3	A_9
A_6	A_2	A_7	A_4	A_5	A_9	A_8	A_4
A_6	A_4	A_7	A_2	A_3	A_7	A_8	A_4
A_3	A_7	A_1	A_6	A_6	A_4	A_3	A_9

B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}
A_5	A_8	A_1	A_2	A_3	A_4	A_5	A_4
A_9	A_5	A_1	A_2	A_3	A_4	A_3	A_5
A_9	A_5	A_1	A_2	A_3	A_2	A_5	A_3
A_5	A_8	A_1	A_2	A_1	A_4	A_4	A_5

B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}
A_5	A_6	A_3	A_8	A_5	A_1	A_4	A_3
A_7	A_5	A_9	A_4	A_5	A_2	A_4	A_2
A_9	A_4	A_9	A_4	A_5	A_2	A_4	A_4
A_3	A_8	A_3	A_8	A_5	A_1	A_4	A_3

B_{25}	B_{26}	B_{27}	B_{28}	B_{29}	B_{30}	B_{31}	B_{32}
A_1	A_3	A_9	A_6	A_8	A_5	A_9	A_8
A_3	A_3	A_7	A_8	A_9	A_5	A_9	A_8
A_3	A_5	A_9	A_6	A_7	A_9	A_9	A_8
A_4	A_5	A_7	A_8	A_6	A_8	A_9	A_8

Conclusions

We have considered all possible deformations of atoms of complexity 2 of m-functions on surfaces with boundary to simple atoms.

The results by O. V. Bolsinov and A. T. Fomenko [1] and O. O. Prishlyak [6] have been generalized in the work.

The obtained results can be used for the global classification of m-functions of complexity 2, as well as an initial step for the study of atoms of greater complexity.

All these results are applied in the study of vector fields of oblique gradient on the manifolds with boundary.

Список використаних джерел

1. *Болсинов А.* Интегрируемые гамильтоновые системы. Геометрия, топология, классификация. / А. Болсинов, А. Фоменко. – Киев: Изд-во Дом «Удмуртский университет». – 1999. – С. 66–99.
2. *Пришляк О. О.* Класифікація простих m-функцій на орієнтованих поверхнях / О. О. Пришляк, К. О. Пришляк, К. І. Міщенко, Н. В. Лукова // Журнал обчисл. та прикл. матем. – 2011. – №1(104). – С. 1–12.
3. *Іванюк О. М.* Атоми степені 2 на поверхнях з краєм / О. М. Іванюк, О. О. Пришляк // Proc. Intern. Geom. Center. – 2013. – Vol. 6. – №3. – С. 40–53.
4. *Кронрод А. С.* О функциях двух переменных / А. С. Кронрод // Успехи мат. наук. – 1950. – №5(35). – С. 24–134.
5. *Максименко С. И.* Классификация m-функций на поверхностях / С. И. Максименко // Укр. мат. журн. – 1999. – №8. – С. 1129-1135.
6. *Prishlyak A. O.* Morse functions with finite number of singularities on a plane / A. O. Prishlyak // Methods of func. and topology. – 2002. – Vol. 8. – №1. – P. 75–78.
7. *Reeb G.* Sur les points singuliers de une forme de pfaff completement integrable ou de une function numerique / G. Reeb // Comptes Rendus Hebdomadaires des Seaces de Academie des Sciences. – 1954. – Vol. 222. – P. 847–849.
8. *Лукова Н. В.* Функції загального положення на тривимірних многовидах з межею / Н. В. Лукова, О. О. Пришляк // Вісн. КНУ, Мат. Мех. – 2009. – №21. – С. 32-35.
9. *Пришляк О. О.* Топологічні властивості функцій на тривимірних тілах / О. О. Пришляк, К. О. Пришляк, О. Н. Вятчанінова // Журнал обчисл. та прикл. матем. – 2010. – №2(101). – С. 113-119.
10. *Пришляк О. О.* Класифікація простих m-функцій на орієнтованих поверхнях / О. О. Пришляк, К. О. Пришляк, К. І. Міщенко, Н. В. Лукова // Журнал обчисл. та прикл. матем. – 2011. – №1(104). – С. 116-127.
11. *Пришляк О. О.* М-функції на неорієнтованих поверхнях / О. О. Пришляк, К. О. Пришляк, Н. В. Лукова-Чуйко // Журнал обчисл. та прикл. матем. – 2012. – №2(108). – С. 176-185.
12. *Іванюк О. М.* Молекули m-функцій степені 2 на поверхнях з краєм / О. М. Іванюк, О. О. Пришляк // Proc. Intern. Geom. Center. – 2014. – Vol. 7. – №3. – С. 27–37.

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References

1. BOLSINOV, A. V. and FOMENKO, A. T. (1999), "Integrable Hamiltonian systems", *Geometry, topology, classification*, pp. 66–99.
2. PRISHLYAK, A. O., PRISHLYAK, K. O. , MISCHENKO K. I. and LUKOVA N. V. (2011), "Classification of simple m-functions on oriented surfaces", *Journal of calc. and appl. math*, No. 1 (104), pp. 1–12.
3. IVANYUK, O. M. and PRISHLYAK, A. O. (2013), "Atoms of complexity 2 on surfaces with boundary", *Proc. Intern. Geom. Center*, **v.6**, No. 3, pp. 40–53.
4. KRONROD, A. S. (1950), "About functions of two variables", *Russian Math. Surveys*,No. 5(35), pp. 24–134.
5. MAKSIMENKO, S. I. (1999), "Classification of m-functions on surfaces", *Ukr. Math. Journal*, T. 51, No. 8, pp. 1129–1135.
6. PRISHLYAK, A. O. (2002),"Morse functions with finite number of singularities on a plane", *Methods of func. and topology*, **v. 8**, No. 1, pp. 75–78.
7. REEB, G. (1954), "Sur les points singuliers de une forme de pfaff completement integrable ou de une function numerique", *Comptes Rendus Hebdomadaires des Seaces de Academie des Sciences*, **v. 222**, pp. 847–849.
8. LUKOVA, N. V. and PRISHLYAK, A. O. (2009), "Function in general position on 3-manifolds with boundary", *Bulletin of Taras Shevchenko National Univ. of Kyiv, Math. Mech.*, **v. 21**, pp. 32–35.
9. PRISHLYAK, A. O., PRISHLYAK, K. A. and VYATCHANINOVA, O. N.(2010)," Topological properties of functions on 3-dimensional bodies", *J. Comput. and Applied Math*, No. 2 (101), pp. 113–119.
10. PRISHLYAK, A. O., PRISHLYAK, K. A., MISCHENKO, K. I. and LUKOVA, N. V.(2011), "Classification of simple m-function on oriented surfaces", *J. Comput. and Applied Math*, No. 1 (104), pp. 116–127.
11. PRISHLYAK, A. O., PRISHLYAK, K. A. and LUKOVA-CHUIKO, N. V. (2012)," M-functions on non-oriented surfaces ", *J. Comput. and Applied Math*, No. 2 (108), pp. 176–185.
12. IVANYUK, O. M. and PRISHLYAK, A. O. (2014), "Moleculas of m-functions degree 2 on surfaces with boundary", *Proc. Intern. Geom. Center*, **v. 7**, No. 3, pp. 27—37.