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Типи траєкторій періодичних точок трикутних відображень, напівгрупа ітерацій яких скінченна

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Types of periodical points trajectories of triangular maps, whose semigroup of iterations is finite

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Вивчаються неперервні трикутні відображення квадрату $[0, 1]^2$ *в себе, ітерації яких утворюють скінченну групу. Доведено, що якщо напівгрупа ітерацій неперервного трикутного відображення квадрату в себе є скінченною циклічною групою, то воно або тривіальна, або складається з двох, або з чотирьох елементів. Введено поняття канонічного зображення циклічної групи неперервними відображеннями. Доведено, що групи, які зображаються відображеннями інтервалу, мають канонічне зображення, але група* C_4 *не має канонічного зображення трикутними відображеннями. Введено поняття типу траєкторії періодичної точки трикутного відображення квадрату та доведено, що обмеження та співіснування типів періодичних точок періоду 4 для трикутних відображень, що зображають групу* C_4*, відсутні.*

Ключові слова: Трикутне відображення, тип періодичної траєкторії, співіснування типів періодичних траєкторій.

So called triangular continuous square maps, whose iterations semigroup is finite are considered in the article. These are maps of square into itself, such that the first coordinate of the image is independent on the second coordinate of preimage. It is proved that this semigroup is a finite cyclic group only in the case if it is either trivial or has order 2 or 4*. The notion of canonical representation of a cyclic group with triangular maps is introduced as such representations, where group identity corresponds to the identity map as a function, which does not move any point. It is proved that those groups, which can be exactly represented by interval maps have the canonical representation by triangular maps, but the group* C_4 *does not. The notion of a type of a cyclic trajectory of a point is introduced for triangular maps. All possible types of periodical trajectories for continuous interval maps are described. The coexistence of all types of periodical points for triangular map, which represents a group* C_4*, is proved.*

Key Words: Triangular map, periodical trajectory type, coexistence of periodical trajectories types.

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Introduction

Triangular square I^2 into itself maps whose iterations semigroup is finite are considered in the article. Here we denote with I the interval $[0, 1]$.

A map $F : I^2 \to I^2$ are called continuous triangular, if it is of the form

$$F(x, y) = (f(x), g(x, y)) \quad (1)$$

for some continuous functions $f : I \to I$ and $g : I^2 \to I$. This means, that the first coordinate of the image of F is independent on the second coordinate of the argument.

Triangular maps form a semigroup with respect to compositions. Remind, that compositions of a map F with itself is called iterations and the n-th iteration of F is denoted by F^n for $n \geqslant 1$.

Denote by F_{id} a triangular map, which does not move any elements of I^2. This map is the identity element of semigroup of all triangular maps. We will differ representations of monoids (precisely, groups), where identity element is represented by F_{id}, or not.

For any point x_0 of the period n under the action of a map f of the interval I into itself, a permutation $\pi \in S_n$ corresponds to x_0 in the following rule (see. [5, p. 53], or a classical work [4]). Let $x_1 < \ldots < x_n$ be an ordered orbit of x_0. Then permutation π of the set $\{1, \ldots, n\}$ is defined by the rule $f(x_i) = x_{\pi(i)}$ for $1 \leqslant i \leqslant n$.

Definition 1. *The permutation π which is constructed above is called the **type** of periodical point x_0 in the action of the map f.*

The work [4] by O. Sharkovskii contains the world known theorem about the linear ordering of the sets of interval maps and about impossibility to specify this ordering. It is proved that sets B_n of those continuous interval I maps, which have the periodical point of a period n are linearly ordered by inclusion as follows.

$$B_1 \supset B_2 \supset B_{2^2} \supset B_{2^3} \supset \ldots$$
$$\ldots \supset B_{2^2 \cdot 5} \supset B_{2^2 \cdot 3} \supset \ldots \supset B_{2 \cdot 5} \supset B_{2 \cdot 3} \supset \ldots$$
$$\ldots B_9 \supset B_7 \supset B_5 \supset B_3,$$

and all the inclusions are strict i.e. for all different $n_1, n_2 \in \mathbb{N}$ the inequality $B_{n_1} \neq B_{n_2}$ holds.

Remark 1 (see. [5], p. 58). *A cycle of period 3 may be only of the type $\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ or inverse. The cycle of a period 4 may be of some types: for example of $\pi_4^{(1)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ or $\pi_4^{(2)} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$. If the map f has a cycle of the type $\pi_4^{(1)}$ then it is easy to show that it also has a cycle of the period π_3 and whence the Sharkovskii theorem gives that it has points of all periods. In the same time interval map f may have a cyclic point with type $\pi_4^{(2)}$ and have no other point's periods except of those which yield from the Sharkovskii theorem i.e. 2 and 1.*

This remark explains the naturalness of considering of types of points trajectories in the study of properties of periodical points of maps.

It is proved at [7], that Sharkovskii theorem can be generalized to triangular maps.

In our work we will introduce the notion of canonical representation of a cyclic group by triangular map as such representation, where identity element of a group corresponds to F_{id}, mentioned above.

We mention in the Section 1, that only groups of 1, 2, or 4 elements can be exactly represented by triangular maps.

In Section 2 we prove, that there is no canonical representation of the group C_4 by continuous triangular map, whence all its representations are non-canonical.

Is Section 3 we generalize definition 1 for triangular maps and define the type of periodical point of triangular map. We call a type of the periodical trajectory of triangular map representable, if there exists a triangular map with a periodical point of such type. We describe all representable types and prove that any set of different representable types can be the whole set of all types of periodical trajectories of some triangular map.

1 Prelinaries

It is proved at [3] that if the semigroup of iterations of continuous map $f : I \to I$ is a finite group then this group consists of 1, or 2 elements. Also the following lemma is proved.

Lemma 1. *If the map $f \in C^0(I, I)$ has a periodical point with a period $m > 2$ then the semigroup of iterations of f is infinite.*

Simple algebraic reasonings give the following lemma (see. [3]).

Lemma 2. *For an arbitrary map H which maps some set M into its subset the uniformly boundedness of points orbits is equivalent to the fact that iterations semigroup of this map is finite.*

Graphs of maps of I into itself, whose iterations semigroup is a finite group are described at [1]. Also the following theorem is proved.

Theorem 1. *If the map f of the interval I represents the finite group, then there exists an interval $[a, b] \subseteq I$, such that $f(I) = [a, b]$ and one of the following two conditions holds.*

1. For any $x \in [a, b]$ the equality $f(x) = x$ holds and in this case the group of iterations of f is trivial;

2. The graph of f is symmetrical in the line $y = x$ for $x \in [a, b]$ but does not coincide with it. In this case the group of iterations of f is C_2.

All the semigroups which can be represented exactly with triangular maps of n-dimensional cube I^n are described at [2]. It is proved there that if cyclic group has exact representation of triangular maps of I^2 then its order is either 1, or 2, or 4.

Example 1. *Let us give an example of a triangular map F of the square I^2 such that the equality $F = F^n$ holds for its iterations and the condition $F^n = F$ does not hold for $1 < n < 5$.*

The deal of the example. Use the notation of definition 1 for the map F. Let the first coordinate f be defined with the formula $f(x) = 1 - x$ for all $x \in I$. Define the second coordinate g as follows.

Let $x < \frac{1}{2}$. Define $g(x, y) = x$ for $y \in [0, x]$; $g(x, y) = y$ for $y \in (x, 1-x)$ and $g(x, y) = 1 - x$ for $y \in [1 - x, 1]$.

Also for $x \geqslant \frac{1}{2}$ define $g(x, y) = x$ for $y \in [0, 1 - x]$; $g(x, y) = 1 - y$ for $y \in (1 - x, x)$ and finally $g(x, y) = 1 - x$ for $y \in [x, 1]$.

The graphs of $g_{0,4}$ and $g_{0,8}$ are given at figure 1a). Now check the equality $F^5 = F$.

For $x \in (0, \frac{1}{2})$ and $y \in [0, x)$ we have $F(x, y) = (1 - x, x)$; $F^2(x, y) = F(1 - x, x) = (x, 1 - x)$; $F^3(x, y) = F(x, 1-x) = (1-x, 1-x)$; $F^4(x, y) = F(1 - x, 1 - x) = (x, x)$; $F^5(x, y) = F(x, x) = (1 - x, x)$ whence $F(x, y) = F^5(x, y)$.

For $x \in (0, \frac{1}{2})$ and $y \in [x, 1 - x]$ we have $F(x, y) = (1 - x, y)$; $F^2(x, y) = F(1 - x, y) = (x, 1-y)$; $F^3(x, y) = F(x, 1-y) = (1-x, 1-y)$; $F^4(x, y) = F(1 - x, 1 - y) = (x, y)$ whence $F^5(x, y) = F(x, y)$.

For $x \in (0, \frac{1}{2})$ and $y \in (1 - x, 1]$ we have $F(x, y) = (1 - x, 1 - x)$; $F^2(x, y) = F(1 - x, 1 - x) = (x, x)$; $F^3(x, y) = F(x, x) = (1 - x, x)$; $F^4(x, y) = F(1 - x, x) = (x, 1 - x)$; $F^5(x, y) = F(x, 1 - x) = (1 - x, 1 - x)$ whence $F(x, y) = F^5(x, y)$.

The equality $F = F^5$ for $x \geqslant \frac{1}{2}$ is proved in the same way. □

Definition 2. *A* ***representation*** *of a cyclic group C_n by iterations of the map F is called* ***exact*** *if there is one-to-one correspondence between elements of C_n and iterations of F and this correspondence is concerted with the group operations in C_n.*

Definition 3. *An* ***exact representation*** *of a cyclic group C_n by iterations of the map F is called* ***canonical*** *representation if the identity of a group C_n corresponds to the neutral element of the semigroup of all maps which belongs to the set of iterations of F.*

Remark 2. *Theorem* 1 *yields that canonical representation of either trivial or the group C_2 by the maps of interval I is canonical if and only if f is a bijection i.e. the equality $[a, b] = I$ holds where a and b are from the theorem formulation.*

Remark 3. *It is constructed at the example* 1 *the exact representation of the group C_4 but it is not canonical, because the group identity is represented by a map F^4, which is not identity element of the semigroup of all square I^2 maps.*

2 Canonical representations of cyclic groups with triangular maps

The main theorem of this section is the following.

Theorem 2. *There is no canonical representation of the group C_4 with triangular maps of I^2.*

Remark 4. *If a triangular map F represents the group C_4 canonically, then for every point $(x_1, y_1) \in I^2$ the condition $F^4(x_1, y_1) = (x_1, y_1)$ holds.*

Lemma 3. *If the semigroup of iterations of a triangular map F is finite, then trajectory of any periodical point of the period* 4 *is of the form*

$$\begin{gathered}(x_1, y_1) \to (x_2, y_2) \to (x_1, y_3) \to \\ \to (x_2, y_4) \to (x_1, y_1),\end{gathered} \tag{2}$$

where $y_1 \neq y_3$.

Proof. Let the map F and point (x_1, y_1) be as in the condition of the lemma. Then x_1 is a periodical point of the map f.

If x_1 is a periodical point of the period 4 for the map f then Lemma 1 yields that semigroup of iterations of f is infinite whence semigroup of iterations of F is also infinite.

If x_1 is a fixed point of f then the point y_1 is a periodical point of the map $g_{x_1}(y) = g(x_1, y_1)$ and its period is 4 and trajectory of (x_1, y_1) under the action of F is looks as $F^n(x_1, y_1) = (x_1, g^n_{x_1}(y_1))$. Lemma 1 yields that semigroup of iterations of g_{x_1} is infinite which means that semigroup of iterations of F is also infinite.

So, the period of x_1 as a periodical point of f equals 2 which finishes the proof. □

Lemma 4. *If iterations of the triangular map F represent the group C_4 canonically then for every $x \in I$ the map $g_x(y) = g(x, y)$ is a homeomorphism of I with itself.*

Proof. We will prove this lemma by contradiction.

Consider an arbitrary point $(x_1, y_1) \in I^2$ and consider the following two cases.

1. $f(x_1) = x_1$;

2. The point x_1 has a period 2 under the action of the map f.

According to Lemma 3 there are no other possibilities for periods of x_1.

If $f(x_1) = x_1$ then trajectory of (x_1, y_1) is $F^n(x_1, y_1) = (x_1, g^n_{x_1}(y_1)$ and the fact that g_{x_1} is a bijection yields from Remark 2 and Lemma 2.

Let now the period of the point x_1 be equal 2 under the action of the map f. If the lemma does not hold then one of the following two cases holds.

1. For some $x_1 \in I$ and for two distinct $y_1^*, y_2^{**} \in I$ the equality $g(x_1, y_1^*) = g(x_1, y_2^{**})$ holds.

2. For some $x_1 \in I$ the map $g_{x_1}(y) = g(x_1, y)$ maps the interval I into the interval J which does not coincide with I.

Consider each of these cases separately.

If for some $x_1 \in I$ and for points $y_1^*, y_2^{**} \in I$ the equality $g(x_1, y_1^*) = g(x_1, y_2^{**})$ holds, then it means that $F(x_1, y_1^*) = F(x_1, y_2^{**})$. But this equality contradicts to equalities

$$\begin{cases} F^4(x_1, y_1^*) = (x_1, y_1^*); \\ F^4(x_1, y_1^{**}) = (x_1, y_1^{**}), \end{cases}$$

which yield from the fact that representation of C_4 with iterations of F is canonical and from that points y_1^* and y_2^{**} are different.

If for some $x_1 \in I$ the map $g_{x_1}(y) = g(x_1, y)$ maps the interval I into J which does not coincide with I then consider an arbitrary point $y_2 \in I$ such the equation $g(x_1, y) = y_2$ has no solutions for y. Denote $x_2 = f(x_1)$ and consider the trajectory of (x_2, y_2) under the action of F. According to the Remark 4 this trajectory is cyclic i.e. there is some $k \in \mathbb{N}$ such that the equality $F^k(x_2, y_2) = (x_2, y_2)$ holds. It means that $f^{k-1}(x_2) = x_1$, i.e. $F^{k-1}(x_2, y_2) = (x_1, y)$ where $y \in I$ is some point if I. But whence $g_{x_1}(y) = y_2$ which contradicts the way of constructing of y_2.

The obtained contradiction finishes the proof of the lemma. □

There is the following corollary from this lemma and the continuity of F.

Corollary 1. *It iterations of F represent the cyclic group C_4 canonically then for any $x \in I$ the bijective map $g_x(y)$ is either increase for all $y \in I$, or decrease for all $y \in I$.*

Lemma 5. *If for the triangular map F the condition holds then for any $x \in I$ the map $g_x(y) = g(x, y)$ is a bijection then either $F^2 = F$ or the semigroup of iterations of F is infinite.*

Proof. It is enough for proving this lemma to consider the map

$$F^2(x, y) = (f^2(x), g_{f(x)}(g_x(y))).$$

Denote it as $F^2(x, y) = (\widetilde{f}(x), \widetilde{g}(x, y))$.

Lemma 3 and Theorem 1 yield that if the semigroup of iterations of F is finite then for every $x \in I$ the equality $\widetilde{f}(x) = x$ holds.

As for any $x \in I$ the map g_x is a bijection then for any $x \in I$ the map $\widetilde{g}_x$ increase. In this case Theorem 1 yields that the semigroup of iterations of F^2 is finite if and only if for all $x \in I$ and for every $y \in I$ the equality $\widetilde{g}(x, y) = y$ holds. But this equality together with the obtained property of $\widetilde{f}$ means that $F^2 = F$ which finishes the proof. □

Now Theorem 2 is a corollary of Lemma 5.

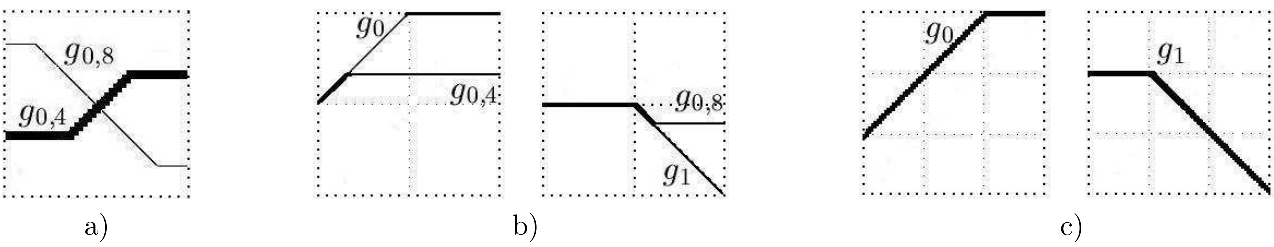

Figure 1:

3 Non canonical representations of cyclic groups

We will consider in this section the question about possible types of periodical points of the period 4 under the action of the triangular map whose semigroup of iterations is finite.

Definition 4. *We will call periodical sequences* $\{(x_k, y_k)\}$ *and* $\{(\widetilde{x}_k, \widetilde{y}_k)\}$ *to have the* ***same type*** *if for some fixed* $t \in \mathbb{N}$ *and all* i, j *inequalities* $y_i \leqslant y_j$ *and* $\widetilde{y}_{i+t} \leqslant \widetilde{y}_{j+t}$ *are equivalent and also inequalities* $x_i \leqslant x_j$ *and* $\widetilde{x}_{i+t} \leqslant \widetilde{x}_{j+t}$ *are equivalent.*

Remark 5. *Note that trajectories of periodical points* (x_1, y_1) *and* $(\widetilde{x}_1, \widetilde{y}_1)$ *under the action of triangular map* F *have the same type then points* x_1 *and* $\widetilde{x}_1$ *as periodical points of the map* f *have the same period and the type of periodical trajectory in the sense of the Definition* 1.

Definition 5. *A* ***type*** *of a periodical sequence* $\mathcal{P} = \{(x_k, y_k)\}$ *is a finite sequence of points* $\mathcal{T} = \{(a_k, b_k)\}$ *of the set* $\mathbb{N}^2$ *which have the following properties:*

1. The set of first values of the first coordinates of points of the set $\mathcal{T}$ *together with the set of the second coordinated* $\mathcal{T}$ *are sets of some first natural numbers.*

2. All elements of $\mathcal{T}$ *are different.*

3. Periodical sequences $\mathcal{P}$ *and* $\widetilde{\mathcal{T}}$ *have the same type where the infinite sequence* $\widetilde{\mathcal{T}}$ *is obtained from the finite* $\mathcal{T}$ *one by its repeating infinite number of times.*

Definition 6. *We will a* ***type of periodical point*** *the type of its trajectory.*

Remark 6. *From Lemma* 3 *yields that a periodical point of the triangular map whose semigroup of iterations is finite may have either type*

$$\mathcal{T}_1 = (1, y_1), (2, y_2), (1, y_3), (2, y_4), \qquad (3)$$

of the type

$$\mathcal{T}_2 = (2, y_1), (1, y_2), (2, y_3), (1, y_4), \qquad (4)$$

for some natural numbers y_1, y_2, y_3 *and* y_4.

Definition 7. *The finite sequence of natural numbers* $\mathcal{T} = \{(a_k, b_k)\}$ *is called an* ***admissible type*** *of periodical point if items 1 and 2 of Definition* 5 *hold.*

Definition 8. *The admissible type* $\mathcal{T}$ *of periodical point is called* ***representable*** *if there exists a continuous triangular map* F *whose semigroup of iterations is a finite group and which has a periodical point of the type* $\mathcal{T}$.

The main result of this section is the following theorem.

Theorem 3. *Each representable type of periodical point* $\mathcal{T}$ *of the period of the form (*3*) and (*4*) is representable.*

Consider some lemmas which let decrease the number of types of periodical points whose representability should be studied.

Lemma 6. *If for some natural numbers* y_1, y_2, y_3 *and* y_4 *the type of periodical points* $\mathcal{T}_1 = [(1, y_1), (2, y_2)\ (1, y_3), (2, y_4)]$ *is representable then type of periodical point* $\mathcal{T}_2 = [(2, y_1), (1, y_2), (2, y_3), (1, y_4)]$ *is also representable.*

Proof. Let triangular map $F = (f, g)$ have a periodical point of the type $\mathcal{T}_1$. Consider the map $\widetilde{F}$, defined as $\widetilde{F}(x, y) = F(f(x), g(1 - x, y))$.

The periodical point which whose period type is $\mathcal{T}_1$ under the map F will be of the type $\mathcal{T}_2$ under the map $\widetilde{F}$.

Note that all trajectories of $\widetilde{F}$ are symmetrical in the line $x = 0,5$ to the correspond trajectories of F. □

Lemma 6 lets to prove Theorem 3 only for admissible trajectories types $\mathcal{T}$ of period 4 of the form (3) and not to consider types of the form (4).

Lemma 7. *If for some natural numbers* y_1, y_2, y_3 *and* y_4 *the type of trajectory* $\mathcal{T}$ *of the form (*3*) is representable the a type of trajectory* $\widetilde{\mathcal{T}} = [(1, y_1), (2, y_4), (1, y_3), (2, y_2)]$ *is also representable.*

Proof. If the map F has a point of the type $\mathcal{T}$ then this point has a type $\widetilde{\mathcal{T}}$ under the action of the map $\widetilde{F} = F^3$. □

Lemma 8. *If the type* $\mathcal{T}$ *of a periodical point of the form (*3*) is representable then the type*

$$\widetilde{\mathcal{T}} = (1, y_{\pi(1)}), (2, y_{\pi(2)}), (1, y_{\pi(3)}), (2, y_{\pi(4)})$$

of periodical point is also representable for any permutation $\pi \in S_4$.

Proof. Definition 5 of a periodical trajectory yields that type of a periodical point $\mathcal{T}$ of the form (3) will not change if we "permutate cyclicly" points of the sequence $\mathcal{T}$ i.e. obtain

$$\mathcal{T}_1 = (2,\, y_2), (1,\, y_3), (2,\, y_4), (1,\, y_1),$$

$$\mathcal{T}_2 = (1,\, y_3), (2,\, y_4), (1,\, y_1), (2,\, y_2)$$

and

$$\mathcal{T}_3 = (2,\, y_4), (1,\, y_1), (2,\, y_2), (1,\, y_3).$$

Now lemma yields from Lemmas 6 and 7. □

Lemma 9. *Type of trajectory $\mathcal{T}$ of the form (3) where $\{y_1, y_2, y_3, y_4\} = \{1, 2\}$ is representable.*

Proof. Lemma yields from the example 1 and Lemma 8. □

Lemma 10. *Type of periodical point $\mathcal{T}$ of the form (3) where $\{y_1, y_2, y_3, y_4\} = \{1, 2, 3\}$ is representable.*

Proof. Taking into attention Lemma 8 it is enough to prove that there exists a triangular map F whose iterations form an exact representation of the group C_4 and which has a periodical point of the type

$$\mathcal{T} = (1,\, 1), (2,\, 1), (1,\, 2), (2,\, 3).$$

We will construct the map F with the periodical point of the type T in the same way as it was done at the Example 1.

Let the map f be defined with the formula $f(x) = 1 - x$ for all $x \in I$.

For $x \leqslant 1/2$ let g be defined as $g_x(y) = y + 1/2$ for $y \in [0,\, 0,5 - x)$ and let be $g_x(y) = 1 - x$ for $y \geqslant 0,5 - x$.

For $x > 1/2$ take $g_x(y) = 0,5$ for $y \in [0,\, 0,5]$; $g(y) = 1 - y$ for $y \in (0,5,\, x)$ and $g_x(y) = 1 - x$ for $y > x$.

Graphs of $g_0(y)$, $g_{0,4}(y)$, $g_{0,8}(y)$ and $g_1(y)$ are given at the figure 1b).

The fact that this map really has that properties as it is necessary can be proved with the direct checking in the same way as it was done at the Example 1. □

Lemma 11. *The type of the sequence $\mathcal{T}$ of the form (3) where $\{y_1, y_2, y_3, y_4\} = \{1, 2, 3, 4\}$ is representable.*

Proof. The idea of the prove of this lemma is also analogical to the proof of the Lemma 9 and the Example 1.

According to Lemma 8 it is enough to prove that there exists a triangular map F whose iterations represent the group C_4 which have periodical point of the type

$$\mathcal{T} = (1,\, 1), (2,\, 2), (1,\, 3), (2,\, 4).$$

Take $f(x) = 1 - x$ for all $x \in I$.

For $x \leqslant 1/2$ take $g_x(y) = 1/3 + x/3$ for $y \leqslant x/3$; $g_x(y) = y + 1/3$ for $y \in (x/3,\, 2/3 - x)$ and $g_x(y) = 1 - x$ for $y \geqslant 2/3 - x$.

For $x > 1/2$ take $g_x(y) = 1/3 + x/3$ for $y \leqslant 2/3 - x/3$; $g_x(y) = 1 - y$ for $y \in (2/3 - x/3,\, x)$ and $g_x(y) = 1 - x$ for $y \geqslant x$.

Graphs of maps $g_0(y)$ and $g_4(y)$ are given at figure 1c). □

Theorem 3 yields from Lemmas 9, 10 and 11.

4 Coexistence of types of periodical points

Remark 4 shows that the problem about coexistence of cycles of periodical points for interval maps was considered at first by A. Sharkovskii at his classical work [4]. In fact it is in this work the notion of type of periodical point of continuous interval map was given at first and the impotentness of some additional characteristic of a cycle except its period was explained.

Lemma 12. *If a triangular map F whose semigroup of iterations if finite has a periodical point of period 4 then it has a periodical point of the type $\mathcal{T} = (1,\, 1), (2,\, 1)$.*

Proof. Let $(x_1,\, y_1)$ be a periodical point of the period 4. According to Lemma 3 its trajectory is of the form (2) i.e. x_1 is a periodical point of f and its period is 2.

Consider the map

$$\widetilde{g}(y) = g_{x_2}(g_{x_1}(y)).$$

As a continuous interval I into I map it has a periodical point i.e. there exists a point y_0 such that trajectory of the point $(x_1,\, y_0)$ looks as

$$(x_1,\, y_0) \to (x_2,\, y_0) \to (x_1,\, y_0).$$

This point has that type which is necessary. □

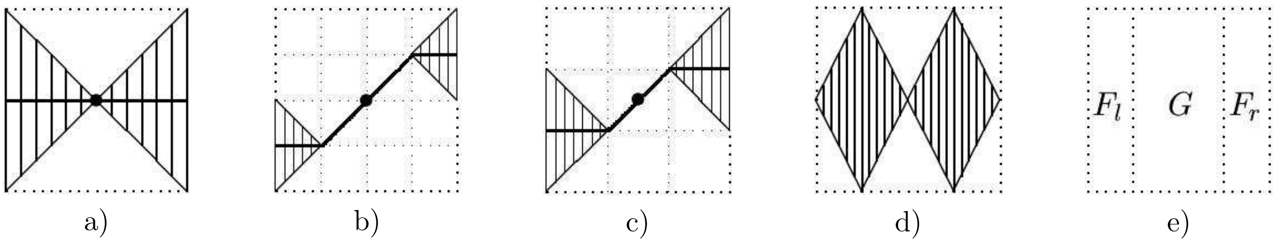

Figure 2:

Remark 7. *Note that all triangular maps which were constructed in the process of proving of Lemmas* 9, 10 *and* 11 *have the following properties.*

1. $g_{0,5}(y) = y$ *for all* $y \in I$ *in particularly for* $x = 0,5$ *the unique periodical point of* F *is its fixed point* $(0,5,\, 0,5)$.

2. The points of period 4 *have the same type for each of three constructed maps.*

Taking into attention the Remark 7 we may obtain the following construction of triangular maps which have points of period 4 of different types.

Consider the transformation Ψ_1 which acts on the set of triangular maps and moves the map $F(x,\, y) = (f(x),\, g(x,\, y))$ to $\Psi_1(F)$ which is defines as $(\Psi_1(F))(x,\, y) = (\widetilde{f}(x),\, g(x,\, y))$ where the function $\widetilde{f}$ is defined as follows. If $x \leqslant 1/4$ then $\widetilde{f}(x) = f(-2x + 1/2)$; if $1/4 < x \leqslant 3/4$ then $\widetilde{f}(x) = f(2x - 1/2)$ and if $x \geqslant 3/4$ then $\widetilde{f}(x) = f(-2x + 3/2)$.

For example the figure 2a) contains a shaded area of the square I^2 which is the periodical points set of the map F from the Example 1. Sets of periodical points of maps which were constructed at Lemmas 10 and 11 are given at figures 2b) and 2c). The bold curve at all three figures denotes the periodical points of period 2 and the point at the center of a rectangle is a fixed point.

The following lemma holds.

Lemma 13. *Let the triangular map* F *is such that its the first coordinate is defined as* $x \mapsto 1 - x$ *and* $F(0,5,\, y) = 0,5$ *for all* $y \in I$ *then the following holds for* $H = \Psi_1(F)$.

1. $H(0,5,\, y) = H(0,\, y) = H(1,\, y) = 0,5$ *for all* $y \in I$*;*

2. Sets of types of periodical points of maps F *and* H *coincide.*

3. Semigroups of iterations of the maps F *and* H *coincide.*

This lemma is proved with the direct checking.

For example the set of periodical points of map $\Psi_1(F)$ from the Example 1 if given at the figure 2d).

Define a transformation Ψ_2 which makes possible to construct the triangular map $\Psi_2(F,\, G)$ with two given triangular maps F and G and Ψ_2 will be such that the set of types of periodical points of the image map will be union of types of periodical points of F and G.

Additionally assume that as the triangular map F, as triangular map G are defined with the formula $x \to 1 - x$ for the first coordinate. Let F_l be the restriction of F to the set $[0,\, 1/2] \times I$ and let F_r be restriction of F to the set $[1/2,\, 1] \times I$. Then the transformation $\Psi_2(F,\, G)$ is schematically shown on the figure 2e) and it can be defined analytically as follows. $(\Psi_2(F,\, G))(x,\, y) = F(2x,\, y)$ for $x \leqslant 1/4$, $(\Psi_2(F,\, G))(x,\, y) = G(2x - 1/2,\, y)$ for $1/4 < x \leqslant 3/4$ and $(\Psi_2(F,\, G))(x,\, y) = F(2x - 1,\, y)$ for $x > 3/4$. The following lemma yields from the construction of Ψ_2.

Lemma 14. *Let the triangular map* F *and* G *are as in Lemma* 13. *Then the following holds for* $H = \Psi_2(\Psi_1(F),\, \Psi_2(G))$.

1. $H(0,5,\, y) = H(0,\, y) = H(1,\, y) = 0,5$ *for all* $y \in I$*;*

2. Sets of types of periodical points of H *is a union of sets of types of periodical points of* F *and types of periodical points of* G*;*

3. If semigroups of iterations of the maps F *and* G *are finite groups the the semigroup of iterations of* H *is bigger of these two groups.*

The reasonings above can be generalized in the following theorem.

Theorem 4. *For arbitrary representable types* $\mathcal{T}_1, \ldots, \mathcal{T}_k$ *of periodical points of period* 4 *there is a triangular map* F *whose iterations exactly represents a group* C_4 *and which has periodical points of each of types* $\mathcal{T}_1, \ldots, \mathcal{T}_k$ *and has no periodical points of another period 4 types.*

Proof. As is was noted in the Remark 7 the points of period 4 have the same type for each maps which were constructed at Lemmas 9, 10 and 11.

The constructions which were presented in the proof of further lemmas till Lemma 13 are just transformations which construct new map all phose periodical period 4 points have the same type.

Now application of transformations Ψ_1 and Ψ_2 introduced above finishes the proof. □

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