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Скінченна породженість вінцевих добутків груп за деревовидно впорядкованими множинами

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On finite generation of wreath products of groups indexed by tree ordered sets

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Встановлено достатні умови скінченно породженості у топологічному сенсі вінцевого добутку транзитивних, скінченних, нетривіальних груп підстановок за деревовидно впорядкованими множинами.

Ключові слова: вінцевий добуток, проскінченна група, скінченна породженість.

Let Λ to be a partially ordered set without infinitely increasing chains and let (G_λ, X_λ) be a collection of finite nontrivial permutation groups for $\lambda \in \Lambda$. If wreath product of G_λ by Λ is topologically finitely generated then direct product of abelianizations of G_λ indexed by the set Λ and direct product of groups G_λ indexed by the set of maximal elements of Λ are topologically finitely generated and there exists such constant d that number of generators of G_λ is not greater then a product of d and all $|X_\mu|$ for $\mu > \lambda$ for all $\lambda \in \Lambda$. If all groups G_λ are transitive and direct product of G_λ by the set Λ is topologically finitely generated then wreath product of groups G_λ by Λ is also topologically finitely generated. In this article we provide sufficient condition when the wreath product of transitive, finite, nontrivial groups indexed by tree ordered set is topologically finitely generated.

Key Words: wreath product, profinite group, finite generation.
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1 Introduction

Wreath product is one the standard operations on groups, which plays a fundamental role in many algebraic constructions. There are several ways to generalize wreath product. Hall [6], Silcock [10], Dixon and Fournelle [4] studied restricted generalized wreath products. Different approaches to unrestricted wreath product indexed by poset were proposed by Holland [7], Wells [11] and Feinberg [5]. Later, these definitions were summarized by Behrendt [1].

Lavrenyuk and Oliynyk [8] studied generalized wreath products of groups indexed by tree ordered sets. They obtained an isomorphism criterion for such wreath products for certain ordered sets and permutation groups. In this paper, based on some preliminary results from [9], we provide certain sufficient conditions for finite generation of wreath products indexed by tree ordered sets. These conditions are similar to the conditions given by Bondarenko [2] for infinitely iterated wreath products (Bondarenko's result was further generalized by Detomi and Luchchini [3]).

2 Basic definitions

Let us fix a partially ordered set $(\Lambda, \leqslant)$ and a permutation group (G_λ, X_λ) for each $\lambda \in \Lambda$. Put

$$X = \prod_{\lambda\in\Lambda} X_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} \mid x_\lambda \in X_\lambda\}.$$

By symbol $\emptyset$ we denote the empty set. Then

$$\overline{X_\lambda} = \begin{cases} \{\emptyset\}, & \text{if } \lambda \text{ is maximal element;} \\ \prod\limits_{\mu>\lambda} X_\mu, & \text{otherwise.} \end{cases}$$

The wreath product $W = wr_{\lambda\in\Lambda} G_\lambda$ of groups $(G_\lambda, X_\lambda)_{\lambda\in\Lambda}$ indexed by partially ordered set Λ is the permutation group on the set X that consists of elements

$$g = (g_{(y,\lambda)})_{\lambda\in\Lambda,\ y\in\overline{X_\lambda}}, \quad \text{where } g_{(y,\lambda)} \in G_\lambda.$$

The action of an element g on a point $x = (x_\mu)_{\mu\in\Lambda} \in X$ is given by the following rule:

$$g(x) = (g_{(y_\lambda,\lambda)}(x_\lambda))_{\lambda\in\Lambda} \in X, \quad \text{where } y_\lambda = (x_\mu)_{\mu>\lambda}.$$

The product of two elements $g = (g_{(y,\lambda)} \in G_\lambda)_{\lambda\in\Lambda,\ y\in\overline{X_\lambda}} \in W$ and $h = (h_{(y,\lambda)} \in G_\lambda)_{\lambda\in\Lambda,\ y\in\overline{X_\lambda}} \in W$ is defined by the following rule:

$$g \cdot h = (g_{(y,\lambda)} h_{(g(y),\lambda)})_{\lambda\in\Lambda,\ y\in\overline{X_\lambda}}.$$

The set of all elements of Λ that larger or equal to λ we denote by λ^∇. The symbol $\mathfrak{B}$ stands for the collection of all sets of the form $\cup_{\lambda\in I}\lambda^\nabla$, where I is a finite subset in Λ. The set $\mathfrak{B}$ is a directed set ordered by inclusion, i.e., the set $\Lambda_1 \in \mathfrak{B}$ is less than the set $\Lambda_2 \in \mathfrak{B}$ if and only if $\Lambda_1 \subseteq \Lambda_2$.

Let $\Lambda_1, \Lambda_2 \in \mathfrak{B}$ and $\Lambda_1 \subseteq \Lambda_2$. We consider the restriction map $\varphi_{\Lambda_1,\Lambda_2} : W_2 = wr_{\lambda\in\Lambda_2}G_\lambda \to W_1 = wr_{\lambda\in\Lambda_1}G_\lambda$, i.e., for all $g \in W_2$ the element $\varphi_{\Lambda_1,\Lambda_2}(g) \in W_1$ acts as g on the set Λ_1. The map $\varphi_{\Lambda_1,\Lambda_2}$ is well-defined and it is a homomorphism, because for every $\lambda \in \Lambda_2 \setminus \Lambda_1$ and $\mu \in \Lambda_1$ either $\lambda \leqslant \mu$ or elements λ and μ are incomparable. Then the collection of groups $(wr_{\lambda\in\Delta}G_\lambda)_{\Delta\in\mathfrak{B}}$ and homomorphisms $\varphi_{\Lambda_1,\Lambda_2}$ form an inverse system, and we can consider its inverse limit.

Proposition 1. *[9] The wreath product of groups (G_λ, X_λ) indexed by the poset Λ coincides with the inverse limit of groups $(wr_{\lambda\in\Delta}G_\lambda)_{\Delta\in\mathfrak{B}}$.*

We say that a topological group W is *topologically generated by its subset S* if the subgroup generated by S is dense in W. If S is finite, then we say that W is *topologically finitely generated*. By $d(W)$ we denote the minimal number of generators in topological sense.

3 Main theorem

In this section we consider a poset $(\Lambda, \leqslant)$ and a collection of permutation groups $(G_\lambda, X_\lambda)_{\lambda\in\Lambda}$ which satisfy the following restrictions

1) every group G_λ is finite and nontrivial;

2) for every $\lambda \in \Lambda$ the set λ^∇ is finite, i.e., Λ does not contain infinitely increasing chains.

Then the wreath product $W = wr_{\lambda\in\Lambda}G_\lambda$ is a profinite group. This allows us to consider the group W as a topological group.

We introduce some notations that we will use in the article. Let $W = wr_{\lambda\in\Lambda}G_\lambda$. If A is a subset of W then for $\lambda \in \Lambda$ and $x \in \overline{X_\lambda}$ we use notation $A_{(x,\lambda)} = \{g_{(x,\lambda)} \mid g \in A\} \subset G_\lambda$. By $\max\Lambda$ we denote the set of all maximal elements of the poset Λ. Also for $\lambda \in \Lambda$ we define the set $LN(\lambda) = \{\mu \in \Lambda \mid \mu < \lambda$ and there is no $\nu \in \Lambda$ such that $\mu < \nu < \lambda\}$. The commutator subgroup of a group G we denote by G'.

Theorem 3.1. *Let a poset Λ and a collection of transitive permutation groups $(G_\lambda)_{\lambda\in\Lambda}$ satisfy the conditions 1) and 2). Additionally, we suppose that the following conditions hold:*

(i) $LN(\lambda) \bigcap LN(\mu) = \emptyset$ for any different $\mu, \lambda \in \Lambda$;

(ii) there exists a constant C such that $d\left(\prod_{\mu\in LN(\lambda)} G_\mu\right) \leqslant C$ for all $\lambda \in \Lambda$;

(iii) $d\left(\prod_{\mu\in\max\Lambda} G_\mu\right) < \infty$;

(iv) $d\left(\prod_{\lambda\in\Lambda} G_\lambda/{G_\lambda}'\right) < \infty$.

Then the group $W = wr_{\lambda\in\Lambda}G_\lambda$ is topologically finitely generated.

Let $\Lambda_1 = \{\lambda \in \Lambda \setminus \max\Lambda \mid G_\lambda \text{ is abelian}\}$. The conditions of the theorem imply that we can choose a finite system of generators for the following groups:

$$\prod_{\mu\in\max\Lambda} G_\mu = \langle p_1, p_2, \ldots, p_\alpha\rangle;$$
$$\prod_{\mu\in LN(\lambda)} G_\mu = \langle c_i^\lambda \mid 1 \leqslant i \leqslant C\rangle,$$
where $c_i^\lambda = (c_{i,\mu}^\lambda)_{\mu\in LN(\lambda)}$;
$$\prod_{\lambda\in\Lambda}(G_\lambda/{G_\lambda}') = \langle s_1', s_2', \ldots, s_\beta'\rangle,\ s_i' = (s_{i,\lambda}')_{\lambda\in\Lambda},$$
where $s_{i,\lambda}' \in G_\lambda/{G_\lambda}'$;
$$\prod_{\lambda\in\Lambda_1} G_\lambda = \langle a_1, a_2, \ldots, a_\gamma\rangle,\ a_i = (a_{i,\lambda})_{\lambda\in\Lambda_1}.$$

For $\lambda \in \Lambda$ we denote by $|\lambda|$ the distance from λ to the maximal element of Λ. Also we denote by $\lambda - i$ an element of Λ at distance i from λ towards the maximal element of Λ. The set λ^∇ is a finite chain for every $\lambda \in \Lambda$. If $\lambda^\nabla = \{\lambda_1 > \lambda_2 > \ldots > \lambda_n\}$ then we will denote each element x in $\overline{X_\lambda}$ by the word $x_{\lambda_1}x_{\lambda_2}\ldots x_{\lambda_n}$.

For each λ we define the following group

$$H_\lambda = \begin{cases} G_\lambda, & \text{if } \lambda \in \Lambda_1 \cup \max\Lambda; \\ {G_\lambda}', & \text{otherwise.} \end{cases}$$

Since the group H_λ is non-trivial, there exists a permutation $\pi_\lambda \in H_\lambda$ such that $\pi_\lambda(1_\lambda) = 2_\lambda$ for two points $1_\lambda \neq 2_\lambda$. Let $\overline{1_\lambda} = 1_{\lambda-|\lambda|} \dots 1_{\lambda-1} 1_\lambda$. Note that $\overline{1_{\lambda-(n)}}$ is the empty word for $n > |\lambda|$.

In order to prove Theorem 3.1 we will use a sufficient condition for finite generation of wreath products established in [9, Lemma 4]. For this it is sufficient to show that for any finite set $\Delta \in \mathfrak{B}$ the following upper bound holds:

$$d(wr_{\lambda\in\Delta} G_\lambda) \leqslant d\left(\prod_{\mu\in\max\Lambda} G_\mu\right) + 3C + 2d\left(\prod_{\lambda\in\Lambda} (G_\lambda/G_\lambda{}')\right).$$

We choose the system of generators for $wr_{\lambda\in\Delta} G_\lambda$ as follows:

(a) We chose elements $\overline{p_i}$ such that $(\overline{p_i})_{(\emptyset,\lambda)} = (p_i)_\lambda$, and $(\overline{p_i})_{(x,\lambda)}$ is trivial for $x \neq \emptyset$.

(b) For all $1 \leqslant i \leqslant C$ we set:

$$(\overline{q_i})_{(x,\lambda)} = \begin{cases} c_{i,\lambda}^{\lambda-1}, & \text{if } |\lambda| = 1 \text{ and } x = 2_{\lambda-1}, \\ e, & \text{otherwise.} \end{cases}$$

(c) We fix a finite collection $\overline{a_1}, \overline{a_2}, \dots, \overline{a_\gamma} \in W$:

$$(\overline{a_i})_{(x,\lambda)} = \begin{cases} a_{i,\lambda}, & \text{if } \lambda \in \Lambda_1 \cap \Delta \\ & \text{and } x = \overline{1_{\lambda-2}} 2_{\lambda-1}, \\ e, & \text{otherwise.} \end{cases}$$

(d) We define two similar collections. Let us define $\overline{b_i}$, $1 \leqslant i \leqslant C$ as follows:

$$(\overline{b_i})_{(x,\lambda)} = \begin{cases} c_{i,\lambda}^{\lambda-1}, & \text{if } |\lambda| \geqslant 2,\ |\lambda| \text{ is even} \\ & \text{and } x = \overline{1_{\lambda-3}} 2_{\lambda-2} 2_{\lambda-1}, \\ e, & \text{otherwise.} \end{cases}$$

Also we choose a similar collection of $\overline{h_i}$, $1 \leqslant i \leqslant C$ for the case when $|\lambda|$ is odd:

$$(\overline{h_i})_{(x,\lambda)} = \begin{cases} c_{i,\lambda}^{\lambda-1}, & \text{if } |\lambda| \geqslant 2,\ |\lambda| \text{ is odd} \\ & \text{and } x = \overline{1_{\lambda-3}} 2_{\lambda-2} 2_{\lambda-1}, \\ e, & \text{otherwise.} \end{cases}$$

(e) For each i we choose $s_i = (s_{i,\lambda})_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} G_\lambda$ such that for every $\lambda \in \Delta$ it holds: $s_{i,\lambda} \in s'_{i,\lambda} G_\lambda{}'$ where $s'_{i,\lambda} G_\lambda{}'$ is coset of the group G_λ. Let us fix $\overline{s_i} \in wr_{\lambda\in\Delta} G_\lambda$ as follows

$$(\overline{s_i})_{(x,\lambda)} = \begin{cases} s_{i,\lambda}, & \text{if } |\lambda| > 0 \\ & \text{and } x = \overline{1_{\lambda-2}} 2_{\lambda-1}, \\ e, & \text{otherwise.} \end{cases}$$

Let A be the group generated by all elements from items (a)–(e). Before we prove Theorem 3.1 we need the following auxiliary statement.

Lemma 1. *For any $\lambda \in \Delta$ there are subgroups $F_\lambda, S_\lambda < A$ such that*

$$(F_\lambda)_{(x,\mu)} = \begin{cases} H_\lambda, & \text{if } \mu = \lambda \text{ and } x = \overline{1_{\lambda-1}}; \\ \{e\}, & \text{otherwise.} \end{cases}$$

$$(S_\lambda)_{(x,\mu)} = \begin{cases} H_\lambda, & \text{if } \mu = \lambda \text{ and } x = \overline{1_{\lambda-2}} 2_{\lambda-1}, \\ \{e\}, & \text{otherwise.} \end{cases}$$

Proof. We prove this lemma by induction on $|\nu|$. If $|\nu| = 0$ then ν is a maximal element and elements $\{\overline{p_1}, \overline{p_2}, \dots\}$ from item (a) can generate the required groups. For induction step we assume that for all $|\lambda| < |\nu|$ the induction assumption holds. We suppose that we are able to generate S_ν. Then we can get F_ν by conjugating S_ν. Then we conjugate the group S_ν by the element t given by

$$t_{(x,\lambda)} = \begin{cases} \pi_{\nu-1}, & \text{if } x = \overline{1_{\nu-2}} \text{ and } \lambda = \nu - 1; \\ e, & \text{otherwise.} \end{cases}$$

Element t is in A by the induction assumption. Now we fix $f \in H_\nu$ and $g \in S_\nu$:

$$g_{(x,\lambda)} = \begin{cases} f, & \text{if } x = \overline{1_{\nu-2}} 2_{\nu-1} \text{ and } \lambda = \nu; \\ e, & \text{otherwise.} \end{cases}$$

Since $t(\overline{1_{\nu-1}}) = \overline{1_{\nu-2}} \pi_{\nu-1}(1_{\nu-1}) = \overline{1_{\nu-2}} 2_{\nu-1}$, we get

$$(tgt^{-1})_{(\overline{1_{\nu-1}},\nu)} = t_{(\overline{1_{\nu-1}},\nu)} g_{(t(\overline{1_{\nu-1}}),\nu)} t^{-1}_{(g(t(\overline{1_{\nu-1}})),\nu)} = g_{(\overline{1_{\nu-2}} 2_{\nu-1},\nu)} = f.$$

Other projections of the element $(tgt^{-1})_{(x,\mu)}$ are trivial. In this way we can generate F_ν. So it is enough to generate S_ν.

We consider the following cases:

(1). We suppose that the group G_ν is abelian. For an arbitrary $f \in G_\nu$ we can choose a sequence $i_1, i_2, \dots, i_k$ such that $a_{i_1} a_{i_2} \dots a_{i_k} = (g_\mu)_{\mu\in\Lambda_1\cap\Delta}$, where $g_\nu = f$ and g_μ is trivial for $\mu \in \Delta \setminus \{\nu\}$. Now we consider the product $\overline{a_{i_1} a_{i_2}} \dots \overline{a_{i_k}}$. Let $x = (x_\delta)_{\delta>\mu} = \overline{1_{\mu-2}} 2_{\mu-1}$ for some $\mu \in \Lambda_1 \cap \Delta$. The definition of $\overline{a_i}$ implies that $\overline{a_i}(x) = ((\overline{a_i})_{(\overline{1_{\delta-1}},\delta)}(x_\delta))_{\delta>\mu} = (x_\delta)_{\delta>\mu} = x$. So $(\overline{a_i a_j})_{(x,\mu)} = (\overline{a_i})_{(x,\mu)} (\overline{a_j})_{(x,\mu)} = a_{i,\mu} a_{j,\mu}$. We have a similar situation for a more number of multipliers. Therefore $(\overline{a_{i_1} a_{i_2}} \dots \overline{a_{i_k}})_{(x,\mu)} = f$ for $\mu = \nu$ and $x = \overline{1_{\nu-2}} 2_{\nu-1}$; projections are trivial for all other cases. Thus we can generate S_ν.

(2). Let $|\nu| = 1$ and G_ν be non-abelian. We choose t as follows

$$t_{(x,\lambda)} = \begin{cases} \pi_\lambda, & \text{if } x = \emptyset \text{ and } \lambda \neq \nu - 1; \\ e, & \text{otherwise.} \end{cases}$$

Let $q \in \langle \overline{q_i}, 1 \leqslant i \leqslant C \rangle$. We have $(tqt^{-1})_{(x,\lambda)} = t_{(x,\lambda)} q_{(t(x),\lambda)} t^{-1}_{(q(t(x)),\lambda)}$. If $|\lambda| = 0$ and $\lambda \neq \nu - 1$ then $t(1_\lambda) = t_{(\emptyset,\lambda)}(1_\lambda) = \pi_\lambda(1_\lambda) = 2_\lambda$. Then

$$(tqt^{-1})_{(x,\lambda)} = \begin{cases} q_{(2_{\lambda-1},\lambda)}, & \text{if } |\lambda| = 1,\ x = 1_{\lambda-1} \text{ and } \lambda - 1 \neq \nu - 1; \\ q_{(2_{\nu-1},\lambda)}, & \text{if } |\lambda| = 1,\ x = 2_{\nu-1} \text{ and } \lambda - 1 = \nu - 1; \\ e, & \text{otherwise.} \end{cases}$$

We choose an arbitrary $p \in \langle \overline{q_i}, 1 \leqslant i \leqslant C \rangle$. Then

$$[(tqt^{-1}), p]_{(x,\lambda)} = \begin{cases} [q_{(2_{\nu-1},\lambda)}, p_{(2_{\nu-1},\lambda)}], & \text{if} |\lambda| = 1,\ x = 2_{\nu-1} \text{ and } \lambda - 1 = \nu - 1; \\ e, & \text{otherwise.} \end{cases}$$

Thus for an arbitrary $r = (r_\lambda)_{\lambda \in LN(\nu-1)} \in \prod_{\lambda \in LN(\nu-1)} G_\lambda{}'$ there exists $g \in A$ such that $g_{(x,\lambda)}$ is trivial for $x \neq 2_{\nu-1}$ and $g_{(2_{\nu-1},\lambda)} = r_\lambda$. Since $\prod_{\lambda \in LN(\nu-1)} G_\lambda{}' = (\prod_{\lambda \in LN(\nu-1)} G_\lambda)'$ then for an arbitrary $f \in H_\nu$ there exists g such that $d_{(x,\lambda)}$ is trivial for $x \neq 2_{\nu-1}$, $\lambda \neq \nu$ and $d_{(2_{\nu-1},\nu)} = f$ which gives us S_ν.

(3). Let $|\nu| > 1$ and G_ν be non-abelian. We consider the case when $|\nu|$ is even (the proof for the case when of odd $|\nu|$ is similar; we can just replace elements $\overline{b_i}$ by $\overline{h_i}$).

We denote

$$Y = \{\overline{1_{\lambda-2}} 2_{\lambda-1} 2_\lambda \mid \lambda \in \Delta, |\lambda| \geqslant 1,\ |\lambda| - |\nu| \text{ is odd}\}.$$

We choose two arbitrary elements d, g from a group $\langle b_i | 1 \leqslant i \leqslant C \rangle$ (from a group $\langle h_i | 1 \leqslant i \leqslant C \rangle$ if $|\nu|$ is odd). We note that if $d_{(x,\lambda)}$ is not trivial then $x \in Y$ (this is true for both cases of even and odd $|\nu|$).

We need to choose an element t such that

$$[tdt^{-1}, g]_{(x,\lambda)} = \begin{cases} [d_{(\overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1},\lambda)}, & g_{(\overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1},\lambda)}], \\ & \text{if } \lambda \in LN(\nu-1); \\ e, & \text{otherwise.} \end{cases} \qquad (1)$$

We choose t using the induction assumption. The equality $(tdt^{-1})_{(x,\lambda)} = t_{(x,\lambda)} d_{(t(x),\lambda)} t^{-1}_{(d(t(x)),\lambda)}$ implies

$$t_{(x,\lambda)} = \begin{cases} \pi_\lambda, & \text{if } x \text{ is the empty word, } |\nu| \text{ is odd and } \lambda \neq \nu - |\nu|; \\ \pi_\lambda, & \text{if } x = \overline{1_{\nu-2}}; \\ \pi_\lambda, & \text{if } x = \overline{1_{\nu-n}} 1_{\lambda-1},\ n \text{ is odd, } 3 \leqslant n \leqslant |\nu| + 1 \text{ and } \lambda \neq \nu - n + 2; \\ \pi_\lambda^{-1}, & \text{if } x = \overline{1_{\nu-n}} 2_{\lambda-1},\ n \text{ is odd, } 3 \leqslant n \leqslant |\nu| + 1 \text{ and } \lambda \neq \nu - 1; \\ e, & \text{otherwise.} \end{cases}$$

In order to prove that t satisfies (1) it is enough to show that an arbitrary word from the set $t(Y) \setminus \{\overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1}\}$ is not a prefix of a word from Y and an arbitrary word from the set $Y \setminus \{\overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1}\}$ is not a prefix of a word from the set $t(Y)$. The word $\overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1}$ must be contained in both sets. If $x = \overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1}$ then $t(x) = x$. The proof goes as follows: fix some word $x \in Y \setminus \{\overline{1_{\nu-3}} 2_{\nu-2} 2_{\nu-1}\}$; compute $t(x)$; we show that $t(x)$ is not a prefix of a word from Y and arbitrary word from Y is not a prefix of $t(x)$. We choose λ and the smallest n such that x can be written as $x = \overline{1_{\nu-n}} 1_{\lambda-|\lambda|+|\nu|-n+1} \ldots 1_{\lambda-2} 2_{\lambda-1} 2_\lambda$. Then $n \leqslant |\nu| + 1$. We consider all possible cases for n.

Let $n \geqslant 3$, $|\lambda| = |\nu| - n + 2$ (then $x = \overline{1_{\nu-n}} 2_{\lambda-1} 2_\lambda$) , $\lambda \neq \nu - 1$. Since $|\lambda| - |\nu|$ is odd, then n is odd. Then $t(x) = \overline{1_{\nu-n}} 2_{\lambda-1} 1_\lambda$. It is obvious that an element $t(x)$ is not a prefix of a word from Y and arbitrary word from Y is not a prefix of $t(x)$.

Let $n \geqslant 3$, $|\lambda| > |\nu| - n + 2$ (then $x = \overline{1_{\nu-n}} 1_{\ldots} \ldots 2_{\lambda-1} 2_\lambda$), n is even. If $\overline{1_{\nu-n}}$ is the empty word then $n = |\nu| + 1$ and the number $|\nu|$ is odd in this case. We have $t(x) = \overline{1_{\nu-n}} 2_{\ldots} \ldots 2_{\lambda-1} 2_\lambda$. It is obvious that an element $t(x)$ is not a prefix of a word from Y. Now we choose some word $y \in Y$ and assume that it is a prefix of $t(x)$. Then $y = \overline{1_{\nu-n}} 2_{\mu-1} 2_\mu$ for some μ. We have $|\mu| = |\nu| - n + 2$, where $|\mu| - |\nu| = -n + 2$ is even. Therefore $y \notin Y$.

Let $n \geqslant 3$, $|\lambda| > |\nu| - n + 2$ (then $x = \overline{1_{\nu-n}} 1_{\ldots} \ldots 2_{\lambda-1} 2_\lambda$), n is odd. Since $|\lambda| - |\nu|$ is odd, then $|\lambda| - |\nu| + n$ is even. As $|\lambda| - |\nu| + n > 2$ then $|\lambda| - |\nu| + n \geqslant 4$ (therefore $x = \overline{1_{\nu-n}} 1_{\ldots} 1_{\ldots} \ldots 2_{\lambda-1} 2_\lambda$). Then $t(x) = \overline{1_{\nu-n}} 1_{\ldots} 2_{\ldots} \ldots 2_{\lambda-1} 2_\lambda$. It is obvious that an element $t(x)$ is not a prefix for some word in Y. Now we choose some word $y \in Y$ and assume that it is a prefix of $t(x)$. Then $y = \overline{1_{\nu-n}} 1_{\mu-2} 2_{\mu-1} 2_\mu$ for some μ. We have $|\mu| = |\nu| - n + 3$, where $|\mu| - |\nu| = -n + 3$ is even. Therefore $y \notin Y$.

Let $n < 3$ (then $x = \overline{1_{\nu-2}}\ldots 2_{\lambda-1}2_\lambda$). Therefore $|\lambda| - |\nu - 2| \geqslant 2$. Since $|\lambda| - |\nu|$ is odd, then $|\lambda| - |\nu - 2|$ is odd and $|\lambda| - |\nu - 2| \geqslant 3$ (then $x = \overline{1_{\nu-2}}1_{\ldots}\ldots 2_{\lambda-1}2_\lambda$). Then $t(x) = \overline{1_{\nu-2}}2_{\ldots}\ldots 2_{\lambda-1}2_\lambda$. It is obvious that an element $t(x)$ is not a prefix of a word from Y. Now we choose some word $y \in Y$ and assume that it is a prefix of $t(x)$. Then $y = \overline{1_{\nu-2}}2_{\mu-1}2_\mu$ for some μ. Then $|\mu| = |\nu|$ and therefore $|\mu| - |\nu| = 0$ is even. Hence $y \notin Y$.

So the condition 1 holds. Therefore for an arbitrary $r = (r_\lambda)_{\lambda\in LN(\nu-1)} \in \prod_{\lambda\in LN(\nu-1)} H_\lambda$ there exists $g \in A$ such that $g_{(x,\lambda)}$ is trivial for $x \neq \overline{1_{\nu-3}}2_{\nu-2}2_{\nu-1}$ and $g_{(\overline{1_{\nu-3}}2_{\nu-2}2_{\nu-1},\lambda)} = r_\lambda$. Since $\prod_{\lambda\in LN(\nu-1)} G_\lambda{}' = (\prod_{\lambda\in LN(\nu-1)} G_\lambda)'$ then for an arbitrary $f \in H_\nu$ there exists g such that $d_{(\overline{1_{\nu-3}}2_{\nu-2}2_{\nu-1},\nu)} = f$ and $d_{(x,\lambda)}$ is trivial for other cases. To complete the proof we conjugate received element by the following element

$$(t)_{(x,\lambda)} = \begin{cases} \pi_{\nu-2}, & \text{if } x = \overline{1_{\nu-3}},\ \lambda = \nu - 2; \\ e, & \text{otherwise.} \end{cases}$$

Then

$$(tdt^{-1})_{(x,\lambda)} = \begin{cases} f, & \text{if } x = \overline{1_{\nu-2}}2_{\nu-1},\ \lambda = \nu; \\ e, & \text{otherwise.} \end{cases}$$

We have shown how to generate S_ν what finishes the proof of the lemma. □

Now we prove the main theorem.

Proof. For each $\lambda \in \Delta$ we define the group $\widehat{G_\lambda}$ as follows

$$(\widehat{G_\lambda})_{(x,\mu)} = \begin{cases} G_\lambda, & \text{if} x = \overline{1_{\lambda-2}}2_{\lambda-1} \text{and} \mu = \lambda; \\ \{e\}, & \text{otherwise.} \end{cases}$$

By Lemma 5 from [9] it is enough to show that all groups $\widehat{G_\lambda}$ are contained in the group generated by A. We fix $\nu \in \Delta$ and $g \in G_\nu$. If ν is a maximal element then we can generate the group $\widehat{G_\nu}$ using elements from the item (a) . Now we consider the case when ν is not a maximal element. There exists $g' \in G_\nu/G_\nu{}'$ such that $g \in g'G_\nu{}'$. Now we choose a sequence $i_1, \ldots, i_k$ such that $s'_{i_1,\nu} \cdot \ldots \cdot s'_{i_k,\nu} = g'$ and $s'_{i_1,\lambda} \cdot \ldots \cdot s'_{i_k,\lambda} = e$ in $\lambda \in \Delta \setminus \{\nu\}$. Then we consider the product $\overline{s} = \overline{s_{i_1}} \cdot \ldots \cdot \overline{s_{i_k}}$. Let us compute $\overline{s_i}(\overline{1_{\lambda-2}}2_{\lambda-1}) = ((\overline{s_i})_{(\overline{1_{\mu-1}},\mu)}(1_\mu))_{\mu>\lambda-1}(\overline{s_i})_{(\overline{1_{\lambda-2}},\lambda-1)}(2_{\lambda-1}) = \overline{1_{\lambda-2}}2_{\lambda-1}$. Therefore

$$\begin{cases} \overline{s}_{(x,\lambda)} \in g'G_\nu{}', & \text{if } x = \overline{1_{\nu-2}}2_{\nu-1} \text{ and } \lambda = \nu; \\ \overline{s}_{(x,\lambda)} \in G_\lambda{}', & \text{if } x = \overline{1_{\lambda-2}}2_{\lambda-1} \text{ and } \lambda \neq \nu; \\ \overline{s}_{(x,\lambda)} = e, & \text{otherwise.} \end{cases}$$

By Lemma 1 for an arbitrary $\lambda \in \Delta \setminus \{\nu\}$ we can choose elements $w_\lambda \in A$ such that $(w_\lambda)_{(\overline{1_{\lambda-2}}2_{\lambda-1},\lambda)} = (\overline{s_{i_1}} \cdot \ldots \cdot \overline{s_{i_k}})^{-1}_{(\overline{1_{\lambda-2}}2_{\lambda-1},\lambda)} \in H_\lambda$. Then $(w_\lambda)_{(\overline{1_{\lambda-2}}2_{\lambda-1},\lambda)} = (\overline{s})^{-1}_{(\overline{1_{\lambda-2}}2_{\lambda-1},\lambda)}$ and $(w_\lambda)_{(x,\mu)}$ is trivial in the other case. We can also choose $w_\nu \in A$ such that $(w_\nu)_{(\overline{1_{\nu-2}}2_{\nu-1},\nu)} = (\overline{s})^{-1}_{(\overline{1_{\nu-2}}2_{\nu-1},\nu)} g$ and $(w_\nu)_{(x,\mu)}$ is trivial in the other case. Let $\Delta = \{\lambda_1, \ldots, \lambda_{|\Delta|}\}$. Then we consider the product $w = w_{\lambda_1} \cdot \ldots w_{\lambda_{|\Delta|}}$

$$\begin{cases} (w)_{(\overline{1_{\lambda-2}}2_{\lambda-1},\lambda)} &= (\overline{s})^{-1}_{(\overline{1_{\lambda-2}}2_{\lambda-1},\lambda)}, \lambda \in \Delta \setminus \{\nu\}; \\ (w)_{(\overline{1_{\nu-2}}2_{\nu-1},\nu)} &= (\overline{s})^{-1}_{(\overline{1_{\nu-2}}2_{\nu-1},\nu)} g; \\ (w)_{(x,\mu)} &= e, \text{ otherwise.} \end{cases}$$

The product $\overline{s}w$ is a required element:

$$\begin{cases} (\overline{s}w)_{(\overline{1_{\nu-2}}2_{\nu-1},\nu)} &= g; \\ (\overline{s}w)_{(x,\mu)} &= e, \text{ otherwise.} \end{cases}$$

Thus we can generate $\widehat{G_\nu}$ for any $\nu \in \Delta$. □

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