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Усреднения в автономной колебательной системе четвертого порядка под действием случайных возмущений в случае кратного действительного и пары сопряженных чисто мнимых корней характеристического уравнения

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Averaging in autonomous fourth order oscillating system under the action of random disturbances in the case of multiple real root and two conjugate pure imaginary roots of characteristic equation

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У роботі вивчається асимптотична поведінка автономної коливної системи, яка описується диференціальним рівнянням четвертого порядку з малими нелінійними зовнішніми збуреннями типу "білого" та центрованого Пуассонівського шумів. Розглядається випадок кратного дійсного кореня і пари спряжених чисто уявних коренів характеристичного рівняння.

Ключові слова: асимптотична поведінка, автономна коливна система, стохастичне диференціальне рівняння.

The asymptotic behavior of autonomous oscillating system describing by differential equation of fourth order with small non-linear external perturbations of "white noise" and centered "Poisson noise" types is studied. Every term of external perturbations has own order of small parameter ε . If small parameter is equal to zero, then general solution of obtained non-stochastic fourth order differential equation has an oscillating part. We consider given differential equation with external stochastic perturbations as the system of stochastic differential equations and study the limit behavior of its solution at the time moment t/ε^k , as $\varepsilon \rightarrow 0$. The system of averaging stochastic differential equations is derived and its dependence on the order of small parameter in every term of external perturbations is studied. The case of multiple real root and two conjugate pure imaginary roots of characteristic equation is considered.

Key Words: asymptotic behavior, autonomous oscillating system, stochastic differential equation.

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1 Introduction

In the papers [1] – [3] it is investigated the behavior, as $\varepsilon \rightarrow 0$, of the fourth order autonomous oscillating system driven by stochastic differential equation

$$x^{IV}(t) + b_1 x'''(t) + b_2 x''(t) + b_3 x'(t) + b_4 x(t) = \varepsilon^{k_1} f_1(x(t), x'(t), x''(t), x'''(t)) + f_\varepsilon(x(t), x'(t), x''(t), x'''(t)) \quad (1)$$

with non-random initial conditions $x(0) = x_0^{(1)}$, $x'(0) = x_0^{(2)}$, $x''(0) = x_0^{(3)}$, $x'''(0) = x_0^{(4)}$, where $\varepsilon > 0$ is a small parameter, $f_\varepsilon(x, x', x'', x''')$ is a

random function such that

$$\begin{aligned} & \int_0^t f_\varepsilon(x(s), x'(s), x''(s), x'''(s)) ds = \\ & = \varepsilon^{k_2} \int_0^t f_2(x(s), x'(s), x''(s), x'''(s)) dw(s) + \\ & + \varepsilon^{k_3} \int_0^t \int_R f_3(x(s), x'(s), x''(s), x'''(s), z) \tilde{\nu}(ds, dz), \end{aligned}$$

$k_i > 0, i = 1, 2, 3$; $f_i, i = 1, 2, 3$ are non-random functions; $w(t)$ is the standard Wiener process; $\tilde{\nu}(dt, dy) = \nu(dt, dy) - \Pi(dy)dt$, $E\nu(dt, dy) = \Pi(dy)dt$, $\nu(dt, dy)$ is the Poisson

measure independent on $w(t)$; $\Pi(A)$ is a finite measure on Borel sets in R .

We will study the asymptotic behavior of given oscillating system, as $\varepsilon \rightarrow 0$, in the case when there exists stable harmonic oscillations at the system under condition $\varepsilon = 0$. Under this condition corresponding characteristic equation has a form

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0.$$

The following cases were considered previously:

1) ([1]) $b_1 > 0, b_3 > 0, b_1b_2 > b_3, b_1^2 > 4(b_2 - \frac{b_3}{b_1}), b_1^2b_4 = b_3(b_1b_2 - b_3)$. In this case the characteristic equation has a roots

$$\lambda_1 = -\eta_1, \lambda_2 = -\eta_2, \lambda_{3,4} = \pm i\omega, \text{ where}$$

$$\eta_{1,2} = \frac{1}{2} \left(b_1 \pm \sqrt{b_1^2 - 4 \left(b_2 - \frac{b_3}{b_1} \right)} \right), \omega^2 = \frac{b_3}{b_1}.$$

2) ([2]) $b_1 > 0, b_3 > 0, b_4 > b_1b_3/4, b_1^2b_4 = b_3(b_1b_2 - b_3)$. In this case the characteristic equation has a roots

$$\lambda_1 = -\eta + i\nu, \lambda_2 = -\eta - i\nu, \lambda_{3,4} = \pm i\omega, \text{ where}$$

$$\eta = \frac{b_1}{2}, \nu = \frac{1}{2} \sqrt{\frac{b_1(4b_4 - b_1b_3)}{b_3}}, \omega^2 = \frac{b_3}{b_1}.$$

3) ([3]) $b_1 = 0, b_3 = 0, b_2 > 0, b_4 > 0, b_2^2 > 4b_4$. In this case the characteristic equation has a roots

$$\lambda_{1,2} = \pm i\omega_1, \lambda_{3,4} = \pm i\omega_2, \text{ where}$$

$$\omega_1^2 = \frac{1}{2} \left(b_2 + \sqrt{b_2^2 - 4b_4} \right),$$

$$\omega_2^2 = \frac{1}{2} \left(b_2 - \sqrt{b_2^2 - 4b_4} \right).$$

We will consider the equation (1) as the system of stochastic differential equations

$$dy_i(t) = y_{i+1}(t)dt, \quad i = \overline{1,3}$$

$$dy_4(t) = [-(b \cdot y(t)) + \varepsilon^{k_1} f_1(y(t))]dt +$$

$$+ \varepsilon^{k_2} f_2(y(t))dw(t) + \varepsilon^{k_3} \int_{\mathbb{R}} f_3(y(t), z)\tilde{\nu}(dt, dz),$$

$$y(t) = \{y_i(t)\}_{i=\overline{1,4}}, b = (b_4, b_3, b_2, b_1),$$

$$y_i(0) = x_0^{(i)}, \quad i = \overline{1,4},$$

(2)

$(b \cdot y(t))$ – is a scalar product of vectors b and $y(t)$.

In what follows we will use the constant $K > 0$ for the notation of different constants, which are not depend on ε .

2 Main result

This paper deals with the following case:

$b_1 > 0, 4b_2 > b_1^2, b_3 = b_1(b_2 - b_1^2/4), b_4 = b_1^2(b_2 - b_1^2/4)/4$. Characteristic equation has a roots $\lambda_{1,2} = -\eta, \lambda_{3,4} = \pm i\omega$, where $\eta = b_1/2, \omega^2 = 4b_4/b_1^2$.

If $\varepsilon = 0$ then the equation (1) has general solution in the form

$$x(t) = C_1e^{-\eta t} + C_2te^{-\eta t} + A_1 \cos \omega t + A_2 \sin \omega t$$

Let us consider the following representation of the solution $y(t)$ to the system (2):

$$y_1(t) = N_1(t) +$$

$$+ A_1(t) \cos \omega t + A_2(t) \sin \omega t,$$

$$y_2(t) = -\eta N_1(t) + N_2(t) -$$

$$- A_1(t)\omega \sin \omega t + A_2(t)\omega \cos \omega t,$$

$$y_3(t) = \eta^2 N_1(t) - 2\eta N_2(t) -$$

$$- A_1(t)\omega^2 \cos \omega t - A_2(t)\omega^2 \sin \omega t,$$

$$y_4(t) = -\eta^3 N_1(t) + 3\eta^2 N_2(t) +$$

$$+ A_1(t)\omega^3 \sin \omega t - A_2(t)\omega^3 \cos \omega t,$$

(3)

where

$$N_1(t) = (C_1(t) + tC_2(t))e^{-\eta t}, \quad N_2(t) = C_2(t)e^{-\eta t}.$$

We can solve the system of linear equations (3) with respect to $(N_1(t), N_2(t), A_1(t), A_2(t))$ and using the Ito formula we derive the system of stochastic differential equations:

$$dN_1(t) = [-\eta N_1(t) + N_2(t)]dt + \frac{2\eta}{(\eta^2 + \omega^2)^2}dH(t),$$

$$dN_2(t) = -\eta N_2(t)dt + \frac{1}{(\eta^2 + \omega^2)}dH(t),$$

$$dA_1(t) = \frac{(\omega^2 - \eta^2) \sin \omega t - 2\eta\omega \cos \omega t}{\omega(\eta^2 + \omega^2)^2}dH(t),$$

$$dA_2(t) = -\frac{(\omega^2 - \eta^2) \cos \omega t + 2\eta\omega \sin \omega t}{\omega(\eta^2 + \omega^2)^2}dH(t),$$

(4)

$$dH(t) = \varepsilon^{k_1} \tilde{f}_1(\omega t, N_1(t), N_2(t), A_1(t), A_2(t))dt +$$

$$+ \varepsilon^{k_2} \tilde{f}_2(\omega t, N_1(t), N_2(t), A_1(t), A_2(t))dw(t) +$$

$$+ \varepsilon^{k_3} \int_{\mathbb{R}} \tilde{f}_3(\omega t, N_1(t), N_2(t), A_1(t), A_2(t), z)\tilde{\nu}(dt, dz),$$

where $\tilde{f}_i(\omega t, N_1, N_2, A_1, A_2), i = 1, 2$ are obtained from $f_i(y(t)), i = 1, 2$ and $\tilde{f}_3(\omega t, N_1, N_2, A_1, A_2, z)$ is obtained from $f_3(y(t), z)$ using (3).

Theorem 2.1. Let $\Pi(\mathbb{R}) < \infty$, $t \in [0, t_0]$, $k = \min(k_1, 2k_2, 2k_3)$. Let us suppose, that functions $f_j, j = \overline{1, 3}$ bounded and satisfy Lipschitz condition on $y_i, i = \overline{1, 4}$. If given below matrix $\bar{\sigma}^2(A_1, A_2)$ is non-negative definite, then

1. If $k_1 = 2k_2 = 2k_3$, then the stochastic process $\xi_\varepsilon(t) = (N_1(t/\varepsilon^k), N_2(t/\varepsilon^k), A_1(t/\varepsilon^k), A_2(t/\varepsilon^k))$ weakly converges, as $\varepsilon \rightarrow 0$, to the stochastic process $\bar{\xi}(t) = (0, 0, \bar{A}_1(t), \bar{A}_2(t))$, where $\bar{A}(t) = (\bar{A}_1(t), \bar{A}_2(t))$ is the solution to the system of stochastic differential equations

$$\begin{aligned} d\bar{A}(t) &= \bar{\alpha}(\bar{A}(t))dt + \bar{\sigma}(\bar{A}(t))d\bar{w}(t), \\ \bar{A}(0) &= (A_1(0), A_2(0)), \end{aligned} \quad (5)$$

where

$$\bar{\alpha}(A_1, A_2) = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}_1(\phi, A_1, A_2) \Psi(\phi) d\phi,$$

$$\begin{aligned} \bar{\sigma}(A_1, A_2) &= \{ \bar{B}(A_1, A_2) \}^{\frac{1}{2}} = \\ &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(\phi, A_1, A_2) \Psi(\phi) \Psi^T(\phi) d\phi \right\}^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \Psi(\phi) &= \frac{1}{\omega(\eta^2 + \omega^2)^2} \times \\ &\times \begin{pmatrix} (\omega^2 - \eta^2) \sin \phi - 2\eta\omega \cos \phi \\ (\eta^2 - \omega^2) \cos \phi - 2\eta\omega \sin \phi \end{pmatrix} \end{aligned}$$

$$\hat{f}_i(\phi, A_1, A_2) = \tilde{f}_i(\phi, 0, 0, A_1, A_2), \quad i = 1, 2$$

$$\hat{f}_3(\phi, A_1, A_2, z) = \tilde{f}_3(\phi, 0, 0, A_1, A_2, z),$$

$$\begin{aligned} \hat{f}(\phi, A_1, A_2) &= \hat{f}_2^2(\phi, A_1, A_2) + \\ &+ \int_R \hat{f}_3^2(\phi, A_1, A_2, z) \Pi(dz), \end{aligned}$$

$\Psi^T(\phi)$ is the vector transpose to vector $\Psi(\phi)$, $\bar{w}(t) = (\bar{w}_i(t), i = 1, 2)$, $\bar{w}_i(t), i = 1, 2$ are independent one-dimensional Wiener processes.

2. If $k < k_1$ then in the averaging equation (5) we must put $\hat{f}_1 \equiv 0$; if $k < 2k_2$, then in the averaging equation (5) we must put $\hat{f}_2 \equiv 0$; if $k < 2k_3$ then in the averaging equation (5) we must put $\hat{f}_3 \equiv 0$.

Proof. Let us make a change of variable $t \rightarrow t/\varepsilon^k$ at the system (4) and obtain for the process $\xi_\varepsilon(t) = (N_1^\varepsilon(t), N_2^\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t)) =$

$(N_1(t/\varepsilon^k), N_2(t/\varepsilon^k), A_1(t/\varepsilon^k), A_2(t/\varepsilon^k))$ the system of stochastic differential equations

$$\begin{aligned} dN_1^\varepsilon(t) &= \left[-\frac{\eta}{\varepsilon^k} N_1^\varepsilon(t) + \frac{1}{\varepsilon^k} N_2^\varepsilon(t) \right] dt + \\ &+ \frac{2\eta}{(\eta^2 + \omega^2)^2} dH_\varepsilon(t), \\ dN_2^\varepsilon(t) &= -\frac{\eta}{\varepsilon^k} N_2^\varepsilon(t) dt + \frac{1}{(\eta^2 + \omega^2)} dH_\varepsilon(t), \\ dA_1^\varepsilon(t) &= \Psi_1(\omega t/\varepsilon^k) dH_\varepsilon(t), \\ dA_2^\varepsilon(t) &= \Psi_2(\omega t/\varepsilon^k) dH_\varepsilon(t), \\ dH_\varepsilon(t) &= \varepsilon^{k_1-k} \tilde{f}_1\left(\frac{\omega t}{\varepsilon^k}, \xi_\varepsilon(t)\right) + \\ &+ \varepsilon^{k_2-k/2} \tilde{f}_2\left(\frac{\omega t}{\varepsilon^k}, \xi_\varepsilon(t)\right) dw_\varepsilon(t) + \\ &+ \varepsilon^{k_3} \int_{\mathbb{R}} \tilde{f}_3\left(\frac{\omega t}{\varepsilon^k}, \xi_\varepsilon(t), z\right) \tilde{\nu}_\varepsilon(dt, dz), \end{aligned}$$

where $\Psi_i(\phi), i = 1, 2$ are a components of vector $\Psi(\phi)$, $w_\varepsilon(t) = \varepsilon^{k/2} w(t/\varepsilon^k)$, $\tilde{\nu}_\varepsilon(t, A) = \nu(t/\varepsilon^k, A) - \Pi(A)t/\varepsilon^k$, here A is a Borel set in \mathbb{R} . For each $\varepsilon > 0$ the process $w_\varepsilon(t)$ is standard one-dimensional Wiener process, and $\tilde{\nu}_\varepsilon(t, A)$ is the centered Poisson measure independent on $w_\varepsilon(t)$.

We have $N_2^\varepsilon(t) = \exp\{-\eta t/\varepsilon^k\} C_2(t/\varepsilon^k)$ and process $C_2^\varepsilon(t) = C_2(t/\varepsilon^k)$, satisfy the stochastic differential equations

$$dC_2^\varepsilon(t) = -\frac{e^{\eta t/\varepsilon^k}}{\eta^2 + \omega^2} dH_\varepsilon(t),$$

where $|C_2^\varepsilon(0)| \leq K$.

So, from boundedness of functions $f_i, i = \overline{1, 3}$ and condition $\Pi(\mathbb{R}) < \infty$ we have the estimate

$$\begin{aligned} \mathbb{E}|N_2^\varepsilon(t)|^2 &\leq K[e^{-2\eta t/\varepsilon^k} + \varepsilon^k(1 - e^{-2\eta t/\varepsilon^k}) \times \\ &\times (t\varepsilon^{2(k_1-k)} + \varepsilon^{2k_2-k} + \varepsilon^{2k_3-k})]. \end{aligned} \quad (6)$$

For the process $C_\varepsilon(t) = N_1^\varepsilon(t)e^{\eta t/\varepsilon^k} = C_1(t/\varepsilon^k) + C_2(t/\varepsilon^k)t/\varepsilon^k$, using the Ito formula, we obtain the stochastic differential equation

$$\begin{aligned} C_\varepsilon(t) &= C_1(0) + \frac{1}{\varepsilon^k} \int_0^t e^{\eta s/\varepsilon^k} N_2^\varepsilon(s) ds + \\ &+ \frac{2\eta}{(\eta^2 + \omega^2)^2} \int_0^t e^{\eta s/\varepsilon^k} dH_\varepsilon(s), \end{aligned}$$

then for the stochastic process $N_1^\varepsilon(t)$ we have

$$\begin{aligned} N_1^\varepsilon(t) &= e^{-\frac{\eta t}{\varepsilon^k}} C_1(0) + \frac{e^{-\frac{\eta t}{\varepsilon^k}}}{\varepsilon^k} \int_0^t e^{\frac{\eta s}{\varepsilon^k}} N_2^\varepsilon(s) ds + \\ &+ \frac{2\eta e^{-\frac{\eta t}{\varepsilon^k}}}{(\eta^2 + \omega^2)^2} \int_0^t e^{\frac{\eta s}{\varepsilon^k}} dH_\varepsilon(s). \end{aligned}$$

From (6) we derive

$$E \left| \frac{e^{-\frac{\eta t}{\varepsilon^k}}}{\varepsilon^k} \int_0^t e^{\frac{\eta s}{\varepsilon^k}} N_2^\varepsilon(s) ds \right| \leq K \left(\frac{te^{-\frac{\eta t}{\varepsilon^k}}}{\varepsilon^k} + \frac{\varepsilon^{k/2}}{\eta} \left(1 + t^{1/2}\right) \left(1 - e^{-\frac{\eta t}{\varepsilon^k}}\right) \right), \quad (7)$$

and also we have the estimate

$$E \left| \frac{2\eta e^{-\frac{\eta t}{\varepsilon^k}}}{(\eta^2 + \omega^2)^2} \int_0^t e^{\frac{\eta s}{\varepsilon^k}} dH_\varepsilon(s) \right| \leq K \left(\varepsilon^{k_1} \left(1 - e^{-\frac{\eta t}{\varepsilon^k}}\right) + \left(\varepsilon^{k_2} + \varepsilon^{k_3}\right) \left(1 - e^{-\frac{\eta t}{\varepsilon^k}}\right)^{1/2} \right). \quad (8)$$

Because $|C_1(0)| \leq K$, from (6), (7) and (8) we have $\lim_{\varepsilon \rightarrow 0} E|N_2^\varepsilon(t)|^2 = 0$, $\lim_{\varepsilon \rightarrow 0} E|N_1^\varepsilon(t)| = 0$ and it is sufficient to study the behavior, as $\varepsilon \rightarrow 0$, of solution to the system of stochastic differential equations

$$dA_i^\varepsilon(t) = \Psi_i(\omega t/\varepsilon^k) d\hat{H}_\varepsilon(t), \quad i = 1, 2 \quad (9)$$

with initial conditions $A_1^\varepsilon(0) = A_1(0)$, $A_2^\varepsilon(0) = A_2(0)$, where

$$d\hat{H}_\varepsilon(t) = \varepsilon^{k_1-k} \hat{f}_1\left(\frac{\omega t}{\varepsilon^k}, A_1^\varepsilon(t), A_2^\varepsilon(t)\right) + \varepsilon^{k_2-k/2} \hat{f}_2\left(\frac{\omega t}{\varepsilon^k}, A_1^\varepsilon(t), A_2^\varepsilon(t)\right) dw_\varepsilon(t) + \varepsilon^{k_3} \int_{\mathbb{R}} \hat{f}_3\left(\frac{\omega t}{\varepsilon^k}, A_1^\varepsilon(t), A_2^\varepsilon(t), z\right) \tilde{\nu}_\varepsilon(dt, dz),$$

where

$$\hat{f}_j(t, A_1, A_2) = \tilde{f}_j(t, 0, 0, A_1, A_2), \quad j = 1, 2$$

$$\hat{f}_3(t, A_1, A_2, z) = \tilde{f}_3(t, 0, 0, A_1, A_2, z).$$

Let us denote $A_\varepsilon(t) = (A_1^\varepsilon(t), A_2^\varepsilon(t))$. Using conditions on coefficients of equation (9) and properties of stochastic integrals we obtain estimates

$$E\|A_\varepsilon(t)\|^2 \leq K[1 + t^2 \varepsilon^{2(k_1-k)} + t(\varepsilon^{2k_2-k} + \varepsilon^{2k_3-k})],$$

$$E\|A_\varepsilon(t) - A_\varepsilon(s)\|^2 \leq K[|t - s|^2 \varepsilon^{2(k_1-k)} + |t - s|(\varepsilon^{2k_2-k} + \varepsilon^{2k_3-k})].$$

Similarly for the process $\zeta_\varepsilon(t) = (\zeta_1^\varepsilon(t), \zeta_2^\varepsilon(t))$, where

$$\begin{aligned} \zeta_i^\varepsilon(t) &= \\ &= \varepsilon^{k_2-k/2} \int_0^t \Psi_i\left(\frac{\omega s}{\varepsilon^k}\right) \hat{f}_2\left(\frac{\omega s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s)\right) dw_\varepsilon(s) + \\ &+ \varepsilon^{k_3} \int_0^t \int_{\mathbb{R}} \Psi_i\left(\frac{\omega s}{\varepsilon^k}\right) \hat{f}_3\left(\frac{\omega s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s), z\right) \tilde{\nu}_\varepsilon(ds, dz), \end{aligned}$$

$i = 1, 2$ we derive estimates

$$\begin{aligned} E\|\zeta_\varepsilon(t)\|^2 &\leq Kt(\varepsilon^{2k_2-k} + \varepsilon^{2k_3-k}), \\ E\|\zeta_\varepsilon(t) - \zeta_\varepsilon(s)\|^2 &\leq K|t - s|(\varepsilon^{2k_2-k} + \varepsilon^{2k_3-k}). \end{aligned}$$

Therefore for stochastic process $\eta_\varepsilon(t) = (A_\varepsilon(t), \zeta_\varepsilon(t))$ conditions of weak compactness [4] are fulfilled

$$\lim_{h \downarrow 0} \lim_{\varepsilon \rightarrow 0} \sup_{|t-s| < h} P\{|\eta_\varepsilon(t) - \eta_\varepsilon(s)| > \delta\} = 0$$

for any $\delta > 0$, $t, s \in [0, T]$,

$$\lim_{N \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} P\{|\eta_\varepsilon(t)| > N\} = 0,$$

and for any sequence $\varepsilon_n \rightarrow 0, n = 1, 2, \dots$ there exists a subsequence $\varepsilon_m = \varepsilon_{n(m)} \rightarrow 0, m = 1, 2, \dots$, probability space, stochastic processes $\bar{A}_{\varepsilon_m}(t) = (\bar{A}_1^{\varepsilon_m}(t), \bar{A}_2^{\varepsilon_m}(t))$, $\bar{\zeta}_{\varepsilon_m}(t)$, $\bar{A}(t) = (\bar{A}_1(t), \bar{A}_2(t))$, $\bar{\zeta}(t)$ defined on this space, such that $\bar{A}_{\varepsilon_m}(t) \rightarrow \bar{A}(t)$, $\bar{\zeta}_{\varepsilon_m}(t) \rightarrow \bar{\zeta}(t)$ in probability, as $\varepsilon_m \rightarrow 0$, and finite-dimensional distributions of $\bar{A}_{\varepsilon_m}(t)$, $\bar{\zeta}_{\varepsilon_m}(t)$ are coincide with finite-dimensional distributions of $A_{\varepsilon_m}(t)$, $\zeta_{\varepsilon_m}(t)$. Since we interesting in limit behaviour of distributions, we can consider processes $A_{\varepsilon_m}(t)$, and $\zeta_{\varepsilon_m}(t)$ instead of $\bar{A}_{\varepsilon_m}(t)$, $\bar{\zeta}_{\varepsilon_m}(t)$. From (9) we obtain equation

$$\begin{aligned} A_{\varepsilon_m}(t) &= A(0) + \int_0^t \alpha_{\varepsilon_m}(s, A_{\varepsilon_m}(s)) ds + \zeta_{\varepsilon_m}(t), \\ A(0) &= (A_1(0), A_2(0)), \end{aligned} \quad (10)$$

where $\alpha_\varepsilon(t, A) = (\alpha_1^\varepsilon(t, A_1, A_2), \alpha_2^\varepsilon(t, A_1, A_2))$,

$$\alpha_i^\varepsilon(t, A_1, A_2) = \varepsilon^{k_1-k} \Psi_i\left(\frac{\omega t}{\varepsilon^k}\right) \hat{f}_1\left(\frac{\omega t}{\varepsilon^k}, A_1, A_2\right),$$

$i = 1, 2$. It should be noted that process $\zeta_\varepsilon(t)$ is the vector-valued square integrable martingale with matrix characteristic

$$\begin{aligned} \langle \zeta_i^\varepsilon, \zeta_j^\varepsilon \rangle(t) &= \\ &= \int_0^t \sigma_i^\varepsilon(s, A_1^\varepsilon(s), A_2^\varepsilon(s)) \sigma_j^\varepsilon(s, A_1^\varepsilon(s), A_2^\varepsilon(s)) ds + \\ &+ \frac{1}{\varepsilon^k} \int_0^t \int_{\mathbb{R}} \gamma_i^\varepsilon(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z) \times \\ &\times \gamma_j^\varepsilon(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z) \Pi(dz) ds, \end{aligned}$$

$i, j = 1, 2$, where

$$\begin{aligned} \sigma_i^\varepsilon(s, A_1^\varepsilon(s), A_2^\varepsilon(s)) &= \\ &= \varepsilon^{k_2-k/2} \Psi_i \left(\frac{\omega s}{\varepsilon^k} \right) \hat{f}_2 \left(\frac{\omega s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s) \right), \end{aligned}$$

$$\begin{aligned} \gamma_i^\varepsilon(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z) &= \\ &= \varepsilon^{k_3} \Psi_i \left(\frac{\omega s}{\varepsilon^k} \right) \hat{f}_3 \left(\frac{\omega s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s), z \right), \end{aligned}$$

$i = 1, 2$.

For processes $A_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ following estimates hold

$$\begin{aligned} \mathbb{E} \|A_\varepsilon(t) - A_\varepsilon(s)\|^4 &\leq K[\varepsilon^{4(k_1-k)}|t-s|^4 + \\ &+ \mathbb{E} \|\zeta_\varepsilon(t) - \zeta_\varepsilon(s)\|^4], \end{aligned} \quad (11)$$

$$\begin{aligned} \mathbb{E} \|\zeta_\varepsilon(t) - \zeta_\varepsilon(s)\|^4 &\leq K[(\varepsilon^{4k_2-2k} + \\ &+ \varepsilon^{4k_3-2k})|t-s|^2 + \varepsilon^{4k_3-3k/2}|t-s|^{3/2} + \\ &+ \varepsilon^{4k_3-k}|t-s|], \end{aligned} \quad (12)$$

$$\begin{aligned} \mathbb{E} \|A_\varepsilon(t) - A_\varepsilon(s)\|^8 &\leq K, \\ \mathbb{E} \|\zeta_\varepsilon(t) - \zeta_\varepsilon(s)\|^8 &\leq K. \end{aligned} \quad (13)$$

Since $A_{\varepsilon_m}(t) \rightarrow \bar{A}(t)$, $\zeta_{\varepsilon_m}(t) \rightarrow \bar{\zeta}(t)$ in probability, as $\varepsilon_m \rightarrow 0$, then, using (13), from (11) and (12) we obtain estimates

$$\begin{aligned} \mathbb{E} \|\bar{A}(t) - \bar{A}(s)\|^4 &\leq K(|t-s|^4 + |t-s|^2), \\ \mathbb{E} \|\bar{\zeta}(t) - \bar{\zeta}(s)\|^4 &\leq C|t-s|^2. \end{aligned}$$

Therefore processes $\bar{A}(t)$ and $\bar{\zeta}(t)$ satisfy the Kolmogorov's continuity condition [5].

Let us consider the case $k_1 = 2k_2 = 2k_3$. Under these condition we have for $i, j = 1, 2$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{t} \int_0^t \alpha_i^\varepsilon(s, A_1, A_2) ds &= \bar{\alpha}_i(A_1, A_2), \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{t} \int_0^t \left[\sigma_i^\varepsilon(s, A_1, A_2) \sigma_j^\varepsilon(s, A_1, A_2) + \right. & \\ \left. + \frac{1}{\varepsilon^k} \int_R \gamma_i^\varepsilon(s, A_1, A_2, z) \gamma_j^\varepsilon(s, A_1, A_2, z) \Pi(dz) \right] ds &= \\ &= \bar{B}_{ij}(A_1, A_2), \end{aligned} \quad (14)$$

where functions $\bar{\alpha}^{(i)}(A_1, A_2)$ and $\bar{B}(A_1, A_2) = \{\bar{B}_{ij}(A_1, A_2), i, j = 1, 2\}$ are defined in the condition of theorem. Since processes $\bar{A}(t), \bar{\zeta}(t)$ are continuous, then from (Lemma 1 [6]) and relationships (10), (14) it follows

$$\begin{aligned} \bar{A}(t) &= A(0) + \int_0^t \bar{\alpha}(\bar{A}_1(s), \bar{A}_2(s)) ds + \bar{\zeta}(t), \\ A(0) &= (A_1(0), A_2(0)), \end{aligned} \quad (15)$$

where $\bar{\zeta}(t)$ is continuous vector-valued martingale with matrix characteristic

$$\langle \bar{\zeta}^{(i)}, \bar{\zeta}^{(j)} \rangle(t) = \int_0^t \bar{B}_{ij}(\bar{A}_1(s), \bar{A}_2(s)) ds, \quad i, j = 1, 2.$$

Hence [7] there exists Wiener process $\bar{w}(t) = (\bar{w}_i(t), i = 1, 2)$, such that

$$\begin{aligned} \bar{\zeta}(t) &= \int_0^t \bar{\sigma}(\bar{A}_1(s), \bar{A}_2(s)) d\bar{w}(s), \\ \bar{\sigma}(A_1, A_2) &= \{\bar{B}(A_1, A_2)\}^{1/2}. \end{aligned} \quad (16)$$

Relationships (15), (16) mean that process $\bar{A}(t)$ satisfies equation (5). Under conditions of theorem the equation (5) has unique solution. Therefore process $\bar{A}(t)$ does not depend on choosing of sub-sequence $\varepsilon_m \rightarrow 0$, and finite-dimensional distributions of process $A_{\varepsilon_m}(t)$ converge to finite-dimensional distributions of process $\bar{A}(t)$. Since processes $A_{\varepsilon_m}(t)$ and $\bar{A}(t)$ are Markov processes then using the conditions for weak convergence of Markov processes we finish the proof of statement 1 of theorem.

Let us consider the case $k < k_1$. Then the corresponding terms in the coefficients $\alpha_\varepsilon^{(i)}(t, A_1, A_2)$, $i = 1, 2$ of equation (10) tend to zero, as $\varepsilon \rightarrow 0$.

In the case $k < 2k_2$ in (14) we have $\sigma_i^\varepsilon(t, A_1, A_2) \sigma_j^\varepsilon(t, A_1, A_2) = O(\varepsilon^{2k_2-k})$, $i, j = 1, 2$. In the case $k < 2k_3$ in (14) we have

$$\begin{aligned} \frac{1}{\varepsilon^k} \int_R \gamma_i^\varepsilon(s, A_1, A_2, z) \gamma_j^\varepsilon(s, A_1, A_2, z) \Pi(dz) &= \\ &= O(\varepsilon^{2k_3-k}) \end{aligned}$$

Repeating with obvious modifications the proof of statement 1) of theorem we obtain proof of the statement 2). \square

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