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Деформація векторних полів на неорієнтованих поверхнях

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Deformation of vector fields on non-orientable surfaces

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Розглядаються деформації векторних полів на неорієнтованих поверхнях використовуючи поняття атомів та молекул функції за Фоменком. Описуються всі можливі переходи для неорієнтованих атомів, складність яких не перевищує 3, за допомогою операцій ковзання і скорочення ручок. Складено таблицю всіх суміжних графів; доведено, що два поля можна з'єднати шляхом, якщо існує відповідна послідовність суміжних графів, а також, що якщо двом сім'ям відповідають однакові оснащені послідовності, то вони топологічно еквівалентні.

Ключові слова: Атоми та молекули функцій Морса, f-граф, деформація векторного поля, топологічна класифікація.

Deformations of vector fields on non-orientable surfaces are investigated using notions of atoms and molecules of function by Fomenko. In general position vector field is a Morse-Smale field. But in one-parametrized family of vector fields this condition for vector field can false. Handle decomposition of surface can be constructed using neighborhoods of stable manifolds of singular points. The atom of function, which is the neighborhood of critical level, can be described using handle attaching. Deformation of vector fields correspond transition of correspondent atoms. We describe all possible transitions for non-oriented atoms of complexity does not exceed 3 by means of operations of the slide and the reduction of the handles. We made a table of all adjacent graphs and proved that two fields can be connected by path if there exists a correspondent sequence of related graphs connecting corresponding atoms. When there several transition from one atom to another, we fix one of them using equipment. If two families correspond to the identical equipped sequences, then they are topologically equivalent.

Key words: Atoms and molecules of Morse functions, f-graph, deformation of vector field, topological classification.

Статтю представив д. ф.-м. н. проф. Кириченко В.В.

Introduction. Bifurcation theory of dynamical systems is a theory that studies the changes of the qualitative picture of the partition of the phase space, depending on the variation of the parameter (or several parameters).

Bifurcation is obtaining a new quality in the movements of a dynamic system at a small change in its parameters. The central concept of bifurcation theory is the concept of stable system. They take any dynamic system and consider a manyparametric family of dynamical systems that the original system is obtained as a special case - at any one value of the parameter(s). If the value of the parameters that are sufficiently close to this, there remains a qualitative picture of the partition of the phase space in the path, then the system is called rough. Otherwise, if a neighborhood does not exist, the system is called unstable.

Bifurcation theory considers the small parameter changes, and we consider major option changes when passing several bifurcations. In bifurcation theory invariants are applied, and we will use atoms and molecules introduced into the [1] for Morse functions. In the two-dimensional case Morse-Smale systems are structurally stable.

We consider how the atom is given in the form f-graph. The idea of atoms and molecules is applied to trajectory classification Morse-Smale flows on closed two-dimensional surfaces. Various applications of atoms and molecules are in the works [3], [1], [4] developed this approach; bifurcations theory considered in [7].

The main purpose of the research is to explore the topological properties of the family of Morse-Smale vector fields using atoms and molecules; to describe all possible transitions for atoms, the complexity of which is not greater than 3. We proved for orientable graphs in [2], and now explore non-orientable case.

Atoms and f-graphs

Definition 1. *An atom* is is a neighborhood P^2 of a critical fiber (which is defined by the inequality $c-\epsilon \leqslant f \leqslant c+\epsilon$ for sufficiently small ϵ), foliated into level lines of f and considered up to the fiber equivalence. An f-atom is a atom with fixed direction of the function arising.

Definition 2. A finite connected graph Γ is said to be an *f-graph* if the following conditions hold:

1) All vertices have degree 3.

2) Some of the edges of graph Γ are oriented, and each vertex is adjacent to two oriented edges, one of which enters this vertex and other goes out of it. A vertex can be the beginning and the end of the same oriented edge if this edge is a loop.

3) Each non-oriented edge of graph Γ is endowed with a number ± 1.

Definition 3. We call two f-graphs *equivalent* if one of them can be obtained from the other by a sequence of the following operations. It is allowed to change orientation on all the edges of a certain cycle and at the same time, to change all marks on the non-oriented edges incident to this cycle. If both of a non-oriented edges belong to the same cycle, then the mark is not changed. The equivalence classes of f-graphs are called *f-invariants*.

There exists a natural one-to-one correspondence between f-invariants and f-atoms.

We show how the idea of atoms and molecules can be applied to the problem of orbital classification of Morse-Smale flows on closed two-dimensional surfaces.

Vector fields v_1 and v_2 given on closed surfaces M_1 and M_2 are called *topologically orbitally equivalent* if there exists a homeomorphism $h : M_1 \to M_2$ that sends the trajectories of the vector field v_1 to those of v_2 while preserving their natural orientation.

Definition 4. Vector field v on a manifold M is called *structurally stable* if the topological behavior of its trajectories is not changed under small perturbations; i.e., after any sufficiently small perturbation $v \to \widetilde{v}$ the field $\widetilde{v}$ remains orbitally topologically equivalent to v.

Structurally stable vector fields on two-dimensional surfaces are exactly Morse-Smale fields. In the case of two-dimensional surfaces, they can be defined as following way.

A vector field v on a closed two-dimensional surface M^2 is called *a Morse-Smale field* if:

1) v has a finite number of singular points and closed trajectories, and all of them are hyperbolic;

2) there are no trajectories going from a saddle to a saddle;

3) for each trajectory of v, its α-limit ω-limit sets are either a singular point or a closed trajectory, a limit cycle.

We describe the classification of Morse-Smale flows with no closed trajectories. Such flows are called Morse flows.

Morse flows have another natural description. They are exactly gradient-like flows without separatrices going from one saddle to another. A flow is called *gradient-like* if it is topologically orbitally equivalent to the flow *gradf* for some Morse function f and Riemannian metric g_{ij} on the manifold M. Each Morse flow on a two-dimensional surface M^2 can be associated with some f-graph or f-atom, in such a way that the correspondence between f-atoms and topological orbital equivalence classes of Morse flows will be bijective. Let us describe this construction explicitly.

The singular points of a Morse flow can be divided into three types: sources, links, and saddles. Besides, the flow has separatrices that connect sources and sinks with saddles. Each saddle has two incomming and two outgoing separatrices.

Consider a small circle around each source which is transversal to the flow. Choose some orientation on it and mark the points of intersection with the separatrices Consider a graph whose vertices are the marked points and whose edges are of two types. The edges of the first type are the arcs of the circles around sources, the edges of the second type are the curves consisting of two separatricos connecting the pair of vertices. As a result, each vertex is incident to three edges, two of which (edges of the first type) are oriented, but the third (an edge of the second type) is not. This sets the atom of Morse-Smale field.

Atoms of Morse-Smale fields

We consider only the gradient-like Morse-Smale fields, field without closed trajectories.

Let M be a closed two-dimensional surface with a Riemannian metric on it.

Let M be a closed two-dimensional surface with a Riemannian metric on it. Consider a Morse function $f : M \to R$ all of whose critical points are located on the same level $f^{-1}(c)$, and consider its gradient flow with respect to the given Riemannian metric. The mapping $f \to \mathrm{grad} f$ establishes a natural one-to- one correspondence between fiber equivalence classes of such functions and orbital topological equivalence classes of Morse flows on M.

This one-to-one correspondence does not depend on the choice of a Riemannian metric on M [5].

Class T_i of vector field represents all vector fields which have no more than "i" saddle points.

Two families X_t and Y_t *topologically equivalent* if there is a homeomorphism $h : [0, 1] \to [0, 1]$ and there is a set of homeomorphisms $\varphi_t : M \to M$, $t \in [0, 1]$, such that they are topological equivalences between the vector fields X_t and vector fields $Y_{h(t)}$.

Let M be n-dimensional manifold with boundary M.

n-dimensional disk H is called *handle of index* λ (or λ *-handle*) if there is a homeomorphism $\varphi : D^\lambda \times D^{n-\lambda} \to H$ such that $\varphi(\partial D^\lambda \times D^{n-\lambda}) = H \cap M \subset \partial M$.

Definition. *Handle decomposition of a closed manifold M is called the decomposition* $M = H_0 \bigcup H_1 \bigcup ... \bigcup H_m$ where $H_0 - n$ -dimensional disk and H_i – handle attached to $M_{i-1} = \bigcup_{j<i} H_j$.

While working with handles we will use the operations of handles slide and reduction [6].

There is one atom of complexity 1, 4 atoms of complexity 2, which correspond 6 f-graphs and 25 atoms of complexity 3, which correspond to 43 f-graphs. They are depicted in Fig.1-3

Deformation of vector fields We say that two atoms are adjacent if one can be obtained from the other using handles reduction and sliding operations.

Theorem 1. *All possible pairs of adjacent atoms of complexity 1–3 are given in the table.*

1	B_{11}	D_{11}, D_{12}
2	C_{11}	$D_{11}, E_{22}, F_{11}, F_{21}$
3	C_{12}	E_{31}, E_{42}, F_{52}
4	C_{21}	$E_{31}, E_{51}, F_{21})$
5	D_{11}	$B_{11}, C_{11}, E_{22}, F_{11}, F_{21}, F_{51}, H_{31}, H_{41}$
6	D_{12}	$B_{11}, H_{31}, G_{42}, G_{52}, G_{62}, G_{72}$
7	D_{21}	$G_{22}, G_{21}, F_{21}, F_{41}$
8	E_{11}	$E_{12}, E_{22}, F_{11}, F_{21}$
9	E_{12}	$E_{11}, E_{21}, F_{11}, F_{21}$
10	E_{21}	E_{12}, F_{11}
11	E_{22}	E_{11}, F_{11}
12	E_{31}	$C_{12}, E_{41}, F_{31}, F_{41}$
13	E_{32}	E_{42}, F_{32}, F_{42}
14	E_{41}	E_{31}, E_{51}, F_{41}
15	E_{42}	$C_{12}, E_{32}, E_{51}, F_{42}, F_{71}$
16	E_{51}	$C_{21}, E_{41}, E_{51}, F_{31}$
17	E_{52}	E_{42}, E_{61}, F_{32}
18	E_{61}	E_{52}, E_{62}, F_{61}
19	E_{62}	E_{61}, F_{62}, F_{71}
20	E_{71}	F_{52}
21	E_{72}	F_{51}
22	F_{11}	$E_{11}, E_{12}, E_{21}, E_{22}, G_{11}, H_{11}$
23	F_{21}	E_{11}, E_{12}
24	F_{31}	$C_{21}, E_{31}, E_{51}, E_{51}, G_{21}, H_{21}$
25	F_{32}	$C_{21}, E_{32}, E_{52}, G_{22}, H_{22}$
26	F_{41}	$D_{21}, E_{31}, E_{41}, G_{21}, G_{31}, H_{21}$
27	F_{42}	$E_{32}, E_{42}, E_{61}, G_{22}, G_{32}, H_{22}$
28	F_{51}	$C_{11}, D_{11}, E_{72}, G_{61}, G_{71}, H_{41}$
29	F_{52}	$C_{12}, E_{71}, G_{61}, G_{71}, H_{42}$
30	F_{61}	$C_{12}, E_{52}, E_{61}, G_{41}, G_{51}, H_{31}$
31	F_{62}	$C_{11}, D_{12}, E_{62}, G_{42}, G_{51}, H_{31}$
32	F_{71}	$D_{12}, E_{42}, E_{62}, G_{41}, G_{42}, H_{31}$
33	G_{11}	F_{11}
34	G_{21}	$D_{21}, F_{11}, F_{31}, F_{41}, H_{21}$
35	G_{22}	$D_{21}, F_{31}, F_{41}, H_{22}$
36	G_{31}	F_{41}
37	G_{32}	F_{42}
38	G_{41}	D_{12}, F_{61}, F_{71}
39	G_{42}	D_{11}, F_{62}, F_{71}
40	G_{51}	$D_{11}, D_{12}, F_{61}, F_{62}, H_{31}$
41	G_{61}	D_{11}, F_{51}, H_{41}
42	G_{62}	D_{12}, F_{52}, H_{42}
43	G_{71}	D_{11}, F_{51}, H_{41}
44	G_{72}	D_{12}, F_{52}, H_{42}
45	H_{11}	F_{11}
46	H_{21}	$D_{21}, F_{31}, F_{41}, G_{21}$
47	H_{22}	$D_{21}, F_{32}, F_{42}, G_{22}$
48	H_{31}	$D_{11}, D_{12}, F_{61}, F_{62}, F_{71}, G_{51}$
49	H_{41}	$D_{11}, F_{51}, G_{61}, G_{71}$
50	H_{42}	$D_{12}, F_{52}, G_{62}, G_{72}$

Proof. We obtain B_{11} from D_{11} by glueing up

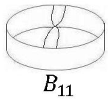

Fig. 1: non-orientable f-graph of complexity 1

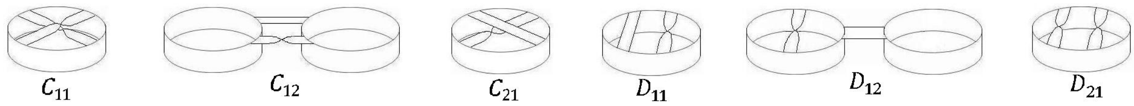

Fig. 2: non-orientable f -graphs of complexity 2

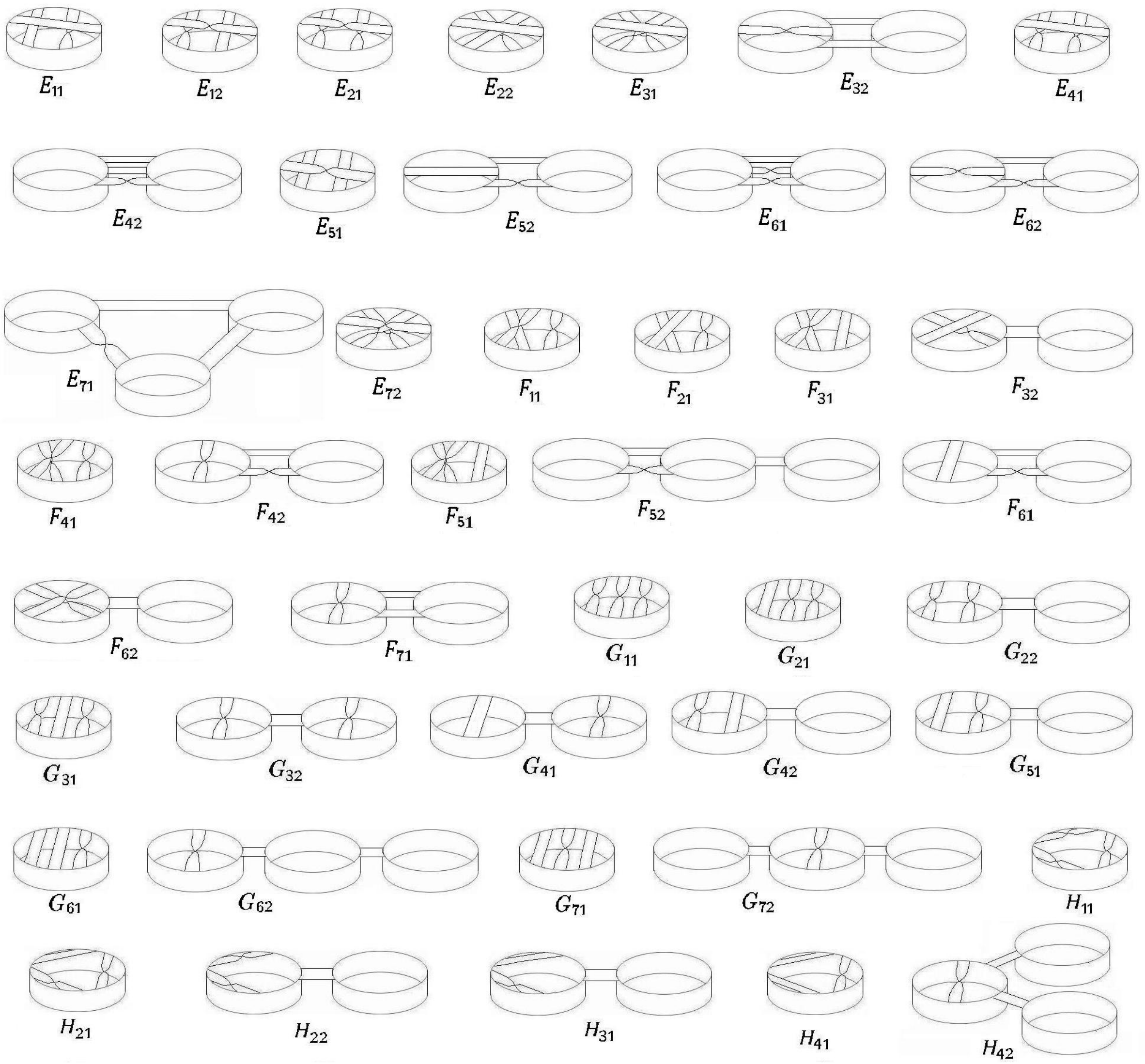

Fig. 3: non-orientable f -graphs of complexity 3

a handle with a disk. Similarly glueing up a handle with a disk we obtain

D_{11} from G_{61}, G_{71}, H_{41};
D_{12} from H_{31}, G_{51};
C_{11} from F_{51},
C_{12} from F_{61}.

We obtain B_{11} from D_{12} by applying a property of handles reduction. From C_{21} to D_{21} one can get by sliding the handle. We obtain D_{21} from G_{21} by glueing up a oriented handle with a disk. From G_{42} to D_{11} we can get by reducing the handle. Using the property of reducing handles we obtain

C_{11} from F_{62},
D_{21} from G_{22},
D_{12} from G_{62},
D_{12} from G_{72},
D_{21} from H_{22},
D_{11} from H_{32},
D_{12} from H_{42}.

Slide handles there is a transition from C_{11} to D_{11}, E_{11} to E_{12}, E_{22}, F_{11} and F_{21}. Similarly sliding the handle we obtain

G_{21}, F_{31}, F_{41} from H_{21};
E_{11}, E_{12} from F_{21};
D_{12}, F_{52}, H_{42} from G_{72}.

Thus, carrying out similar reasoning with the rest of the f-graphs, we will receive all pairs of adjacent elements.

Theorem 2. *If two families are topologically equivalent in the class of T_i, then they set the same sequence of adjacent graphs.*

Proof. Each t corresponds to the graph, except for a finite number of t_k, in which there is a change from one element to another, t_k corresponds to the edges connecting these graphs. Then since $Y_{h(t)}$ is topologically equivalent to X_t, the same f-graphs correspond to them in case when $t = t_k$. Consequently, the corresponding sequences have the same row and adjacent edges, i.e. they coinside.

Conducting considerations in reverse order, we get the following theorem:

Theorem 3. *Two fields of class T_i can be connected by path (family of vector fields) in the class of T_i if there exists a sequence of related graphs connecting corresponding atoms.*

Let the number of movements from one atom to another be n. Let's number possible movement from one atom to the other by integers. These numbers will be called equipment. The sequence of adjacent graphs, each pair of which is specified by equipment, we will call equipped.

Theorem 4. *If two families correspond to identical equipped sequences, then they are topologically equivalent.*

Proof. We can depict the "n"multiple adjacent graphs, then each feature sequence will correspond to one family of vector fields. In case of the vector field deformation, the following situations are possible: 1) topological type of a vector field does not change, then this family is given by Fomenko atom; 2) the bifurcation happens, so the topological type changes. In this case, the deformation is given by a pair of atoms. Then depict all such possible bifurcations. If several bifurcations occur consistently, then they set a sequence of adjacent atoms.

Conclusions

The topological properties of family of vector fields were investigated. All the possible transitions for non-orientable atoms , complexity of which is not greater than 3, was described.

The received results can be used while studying contact structures on 3 -dimensional manifolds, namely, when Morse function is given on manifold. If there are no critical levels between two regular levels of this function, then there is a family of vector fields on the surface, which are Morse-Smale fields in general.

In addition, the results will be of interest in the study of deformation of Morse functions that are undergoing similar processes of reducing critical points, but which differ from the flows so that the addition of handles replaced by changing the order of the critical points.

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