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Dynamics of oscillation processes in siphon U-tubes

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The dynamics of oscillation processes in a siphon U-tube is studied for the system of connected vessels filled with homogeneous liquid. The equations and phase paths describing the motion of non-viscous liquid and fountain effects are given, oscillation frequencies are considered. Oscillations are nonlinear in general case, but they turn into linear by setting special parameters of the system. Phase portraits are obtained and their dependences on parameters of the system are analyzed for the linear and non-linear cases. It is shown that the behavior of the deep and shallow water in such a system could be discussed using analogy with the propagation of elastic waves in condensed matter. Some interesting analogies between a siphon U-tube and hydrodynamic, mechanical, electromagnetic phenomena, wave motion are also analyzed.
Keywords: communicating vessels, oscillations of liquid, motion equations, phase paths.

В работе исследована динамика колебательных процессов в сифонной U-образной трубке на основе системы сообщающихся сосудов, заполненных однородной жидкостью. Приведены уравнения и фазовые траектории, описывающие движение жидкости без учета вязкости и фонтанирования, рассмотрены частоты колебаний в исследуемой системе. В общем случае колебания являются нелинейными, но при определенных параметрах системы возможна их линеаризация. Получены и проанализированы в зависимости от параметров системы фазовые портреты в линейном и нелинейном случаях. Показано, что рассмотрение поведения глубокой и мелкой воды в сифонной U-образной трубке может быть проведено по аналогии с распространением упругих волн в конденсированных средах. Также рассмотрены аналогии с гидродинамическими, механическими, электрическими явлениями, волновым движением.
Ключевые слова: сообщающиеся сосуды, колебания жидкости, уравнения движения, фазовые траектории.

У роботі досліджена динаміка коливальних процесів у сифонній U-подібній трубці на основі системи сполучених сосудів, які заповнені однорідною рідиною. Наведені рівняння та фазові траєкторії, які описують рух рідини без врахування в'язкості та фонтанування, розглянуті частоти коливань в досліджуваній системі. В загальному випадку коливання є нелінійними, але при певних значеннях параметрів системи можлива їх лінеарізація. Одержано та проаналізовано в залежності від параметрів системи фазові портрети в лінійному та нелейному випадках. Показано, що випадки мілкої та глибокої води у сифонній U-подібній трубці можуть бути розглянуті по аналогії з розповсюдженням пружних хвиль в конденсованому середовищі. Також розглянуто аналогії з гідродинамічними, механічними, електричними явищами, хвильовим рухом.
Ключові слова: сполучені посудини, коливання рідини, рівняння руху, фазові траєкторії.

Introduction

Studying the properties of motion of a liquid in a siphon U-tube has a long history and is still of an essential interest nowadays. System like communicating vessels has a fundamental and practical importance. Here we can reference work [1] which examined transmission of liquid helium through superleak connecting two vessels in process of heating one of the containers. This process is interesting because a thermomechanical effect [2] in phonon regime take place, which is also discussed in terms of an increase of quantum degerancy in colder compartment. However, the way of changing the quantum degerancy is not uniform; another method to cause it is adiabatic displacement of the wall dividing two compartments of homogeneous quantum fluid. On the thermomecanical effect a well-known process called helium fountain is based: if superfluid helium is heated, the flow of liquid can achieve velocity high enough for a short-time liquid excess the level of vertical vessel [3]. Practical importance of connected vessels is, for example, that using them as a U-tube damper systems can lead to reduction of the vibration amplitude of high buildings; on ships these systems are used to reduce the rolling motion caused by waves [4]. By the principle of communicating vessels water locks of the rivers and channels operate, as well as the level-measuring tubes for water tanks. Siphon U-tubes are also used to determine the volume of a non-magnetic body of a random shape [5]. There is also an interesting example of relations between principles of system of communicating vessels in which one vessel is half-immersed into another: this principle can be used in experimental studies of ^{4}He equilibrium in its liquid/solid state [6]. Other important phenomena in this field are siphon properties of liquid, i. e. the ability of liquid to overcome a certain barrier without external mechanical action [7].

Our article is devoted to the research of the behavior of liquid forced out of an equilibrium state and left to itself. In other words, we study the free vibrations of liquid in the system of communicating vessels including oscillations, which can lead to realization of the siphon mechanism. At the same time we do not take into account pouring, viscous, and fountain effects. Actually, we examined such motion as oscillations so for describing them the differential equations and phase portraits were used. The equations are nonlinear in general case, but special parameters of the system, for example the equality of the cross sections of vertical vessels, leads to motion described by linear equations. It should be noted that such a system is also analyzed in [8], where the relation between heights of the liquid in two vessels and time is obtained, and the period of oscillation is derived through a transformation formula of the elliptical integral of the second order. The main idea of an article [8] is using the unsteady Bernoulli equation (energy method) for describing the liquid motion in the U-tank. We obtain the characteristics describing the motion of a liquid by means of Euler-Lagrange equations and the law of energy conversation. Of interest are also cases of deep and shallow water; which can be examined by using analogy with propagation of elastic waves in condensed matter. System like siphon U-tube has some analogies with the various physical processes, i.e. there are some relations between motion of liquid in communicating vessel and hydrodynamic, mechanical, electromagnetic phenomena, and a wave motion.

Overview of the system

Here we study the behavior of incompressible liquid in U-tube in Earth's gravity field. Geometry of this system is shown in Fig.1. Index g corresponds to the wide container (tube), index s corresponds to the narrow container (tube), L corresponds to the communicating tube; S_g and S_s are cross sections of the vertical tubes, H_g and H_s are heights of liquid in these tubes. Zero coordinate of Z - axis is assigned to the equilibrium height H_0 of liquid under the upper line of the connecting tube, S_L is the cross section of this tube. The levels of the liquid never reach the height low enough to let the surrounding air enter the connecting tube. Length

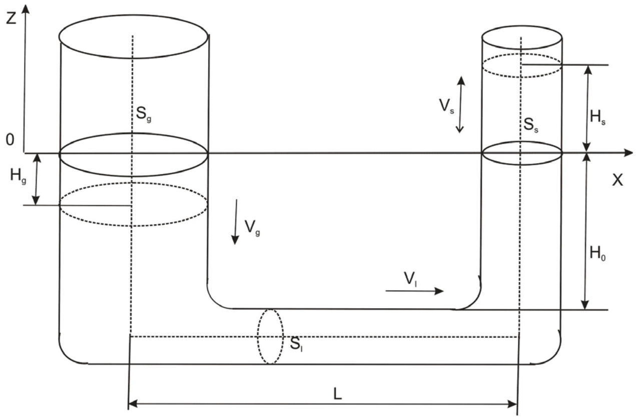

Fig. 1. Geometry of the system.

of the tube L is measured between the axes of vertical vessels, as shown in Fig.1.

In case of the free oscillations in the system disturbed from equilibrium state, motion equations for incompressible liquid could be written as a balance between volumes of liquid in vertical vessels or by using parameters such as cross sections and height of the liquid:

$$S_g H_g = -S_s H_s, \tag{1}$$

and after differentiation we obtain an equation for the speeds of the liquid in the different tubes of the system:

$$S_g v_g = -S_s v_s = -S_L v \tag{2}$$

Here we consider that integer constant is zero. Directions of the speeds v_L и v_s are shown in Fig.1. Flow through the connecting tube S_L does not change the volume of liquid in it. Let us perform calculation of the main physical properties of this system using Lagrangian formalism.

Dynamical equations of the system of communicating vertical vessels

To obtain Lagrange function L=T-U (where T is a kinetic and U is a potential energy) we first find potential energy U when liquid is disturbed from equilibrium state, which can be done by various ways: for example, by heating the liquid or by means of mechanical impact on liquid surface. In the narrow container the height of the liquid is $H_s/2$, and the change of its mass is positive $m_s=\rho\Delta V=\rho S_s H_s>0$. For a large container the height of mass center of the liquid is $H_g/2$, and the change of its mass is negative $m_g = \rho\Delta V = \rho S_g H_g < 0$ because the liquid flows from large to narrow container. So the corresponding potential energies are:

$$U_s = \frac{1}{2}\rho g S_s (H_s)^2 > 0. \tag{3}$$

$$U_g = \frac{1}{2}\rho g S_g (H_g)^2 > 0. \tag{4}$$

Kinetic energies in the respective parts (tubes) of the vessel are:

$$T_s = \frac{1}{2} m_s v_s^2 = \frac{1}{2} \rho S_s (H_0 + H_s) v_s^2, \quad (5)$$

$$T_g = \frac{1}{2} m_g v_g^2 = \frac{1}{2} \rho S_g (H_0 + H_g) v_g^2, \quad (6)$$

$$T_L = \frac{1}{2} m_L v_L^2 = \frac{1}{2} \rho S_L L v_L^2. \quad (7)$$

Introducing the dimensionless parameters significantly simplifies the analysis of liquid behavior in a U-tube:
$$\sigma_{sg} = \frac{S_s}{S_g}, \sigma_{Lg} = \frac{S_L}{S_{g,}}, \sigma_{Ls} = \frac{S_L}{S_s},\ l = \frac{L}{H_0},\ y = \frac{H_s}{H_0}.$$

And Lagrange function equals to the difference between total kinetic and total potential energies can be written as:

$$\begin{aligned} \mathrm{L} = T - U = -\frac{1}{2} \rho g S_s H_0^2 \{ y^2 (1+\sigma_{sg}) - \\ -\dot{y}^2 [y(1-\sigma_{sg}^2) + (1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}] \}. \end{aligned} \quad (8)$$

Here we introduce the dimensionless time $\tau = t/t_0$, $t_0 = \sqrt{H_0/g}$.

Dynamics of the system is described by Lagrange equation:

$$\begin{aligned} 2\ddot{y}[y(1-\sigma_{sg}) + 1 + \frac{l}{\sigma_{Ls}(1+\sigma_{sg})}] + \\ \dot{y}^2 (1-\sigma_{sg}) + 2y = 0. \end{aligned} \quad (9)$$

Nonlinear parts of motion equation appear because of variation of the height of liquid when it moves in the vessels with different cross sections. It is clear that when $\sigma_{sg}=1$ ($S_g=S_s$) the equation is linear:

$$\ddot{y} + \frac{\sigma_{Ls}}{l} y = 0. \quad (10)$$

So only equality of cross sections of vertical vessels is important for the linearity of the equation (9); the cross section of horizontal connecting tube thus can actually be of an arbitrary value. Equation is linear in a variable y and describes ordinary harmonic oscillations with a frequency (in dimensional parameters):

$$\omega = \sqrt{\frac{g}{H_0 [1 + \frac{L}{2H_0} \frac{S_s}{S_L}]}}. \quad (11)$$

Period of those oscillation can be written using reduced length $l_r = H_0 + L/2\sigma_{Ls} = H_0(1 + l/2\sigma_{Ls})$:

$$= \frac{}{} = 2\pi \sqrt{\frac{H \quad LS_s/\ S_L}{g}} = 2\pi \sqrt{\frac{}{g}}, \quad (12)$$

As we see there takes place analogy with physical pendulum. In case of equality to zero of the parameter *l=0* the reduced length coincides with the equilibrium height of liquid H_0, and the physical pendulum becomes mathematical with oscillation period $T=2\pi(H_0/g)^{1/2}$. The reduced length equals to equilibrium height of liquid in case of $\sigma_{Ls}=\infty$ too as then the component $LS_s/(2S_L)$ is zero. If cross sections of connecting tube and vertical tubes are of the same value ($\sigma_{sg}=1$, $\sigma_{Ls}=1$), the reduced length differs from equilibrium liquid height H_0 on a half of length of connecting tube L. Otherwise at equality of all cross sections reduced length equals to the half of length of a connecting tube. The equation also becomes linear in the case of infinitesimal amplitude of oscillations ($y<<1$); then oscillation frequency is defined by equation (11). If we assume cross sections to be equal within a certain small parameter α ($\sigma_{sg} = 1+\alpha$), we will receive the equation which is nonlinear in general case, but if α is small as well as y, we have in the first order of a smallness in accuracy expression for linear oscillations, but in the second order of smallness we do not anymore obtain the equation like (10).

Phase portraits. Analysis of nonlinear oscillations of the system

Let us consider behavior of the system generally and integrate the equation (9). Without friction the integral of motion is a total energy of the system:

$$\begin{aligned} W = T + U = \frac{1}{2} \rho g S_s H_0^2 [y^2 (1+\sigma_{sg}) + \\ \dot{y}^2 [y(1-\sigma_{sg}^2) + (1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}] = const. \end{aligned} \quad (13)$$

Constant of integration is defined by maximum value of liquid height in narrow vertical vessel $y_0=H_{max}/H_0$ (turning point when y=const), i.e.:

$$\begin{aligned} y^2(1+\sigma_{sg}) + \dot{y}^2 [y(1-\sigma_{sg}^2) + \\ +(1+\sigma_{sg}) + \frac{l}{\sigma_{Ls}}] = y_0^2 (1+\sigma_{sg}). \end{aligned} \quad (14)$$

For studying of phase portraits it is convenient to rewrite the equation (13) into:

$$\dot{y}^2 = \frac{y_0^2 - y^2}{y(1-\sigma_{sg}) + 1 + \frac{l}{\sigma_{Ls}(1+\sigma_{sg})}}. \quad (15)$$

This rearrangement allows to find period of nonlinear oscillations as the integral of this equation [9]. The equation (15) defines a phase portrait of the system generally.

Phase portraits of the linear oscillations of the system

At equality of vertical tubes cross sections ($\sigma_{sg}=1$) oscillations of the liquid are described by linear equation , and thus in the denominator of expression there is no dependence on coordinate y. In this case phase portrait of

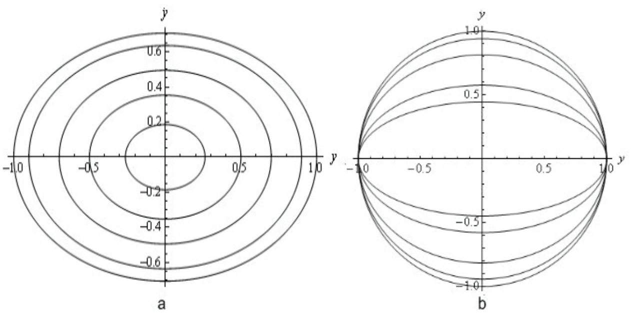

Fig. 2. Evolution of the phase portraits of linear oscillations of the system when: a) connecting tube section is fixed as l/σ_{Ls}=2, parameter y_0=0.2; 0.5; 0.7; 0.9; 1; b) maximum amplitude of oscillations is fixed as y_0=1 while the parameters of the connecting tube l/σ_{Ls}=0; 0.125; 0.5; 2; 4.

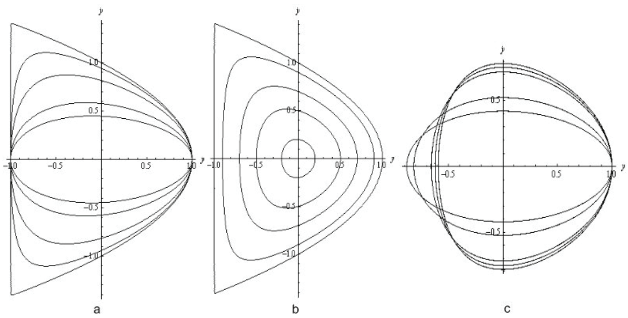

Fig. 3. Evolution of a phase portrait of nonlinear oscillations of the system when: a) maximum amplitude is constant y_0^2=1 and cross sections strongly differ σ_{Ls}=0, the ratio l/σ_{Ls}=0; 0.125; 0.5; 2; 4; b) cross sections strongly differ σ_{sg}=0 and l/σ_{Ls}=0 while the oscillations amplitude y_0=0.2; 0.5; 0.7; 0.9; 1; c) amplitude is small $\dot{y}^2 \to 0$ and the ratio of cross sections is fixed σ_{sg}=0.5, the parameters of the connecting tube l/σ_{Ls}=0; 0.125; 0.5; 2;4.

the system is ellipse with semiaxes $a^2 = y_0^2$, $b^2 = \dfrac{y_0^2}{1+l/\sigma_{Ls}}$ and eccentricity $\varepsilon^2 = \dfrac{l}{(l+\sigma_{Ls})}$:

$$\frac{y^2}{a^2}+\frac{\dot{y}^2}{b^2}=1, \tag{16}$$

Obviously, if we have harmonic oscillations when parameters of the system are fixed, all ellipses are similar for different oscillation amplitudes. Direction of rotation is topological invariant (see Fig. 2).

Interesting dependence on parameter $l/\sigma_{Ls}=LS_s/(S_L H_0)$ follows from the equation (16): in extremely minimum case l/σ_{Ls}=0 phase portrait on plane $y - y_0$ is a circle. This parameter may become zero (or closely near to zero) in two ways: when equilibrium liquid height in vertical vessels much exceeds the length on connecting tube (l = 0), or when two vessels are connected by sealed channel with large cross section ($\sigma_{Ls}=\infty$). The increase in parameter l/σ_{Ls} leads to the flattened phase portrait in Fig. 2 (b), and the decreasing of the frequency of oscillations.

Phase portraits of nonlinear oscillations

In case of the maximum amplitude $y_0 = H_{smax}/H_0 = 1$ at big difference in sections of vertical tubes (σ_{sg}=0), and on condition l/σ_{Ls}=0 the equation is reduced to:

$$\dot{y}^2 = 1 - y. \tag{17}$$

On the phase plane it is equation of the parabola (Fig. 3 (a)) closed by a vertical segment. Analyzing the equation it is possible to see that at increase in length of a connecting tube the parabola is flattened and turns into an ellipse. The graphic analysis of fluctuations for the case l/σ_{Ls}=0 is also given in [4] where two phase paths in the limiting cases are presented: when the cross sections of vertical tubes are identical and when they are much different. When amplitude changes within 0<y_0<1 (displacement from $-y_0$ to y_0) in the system takes place the following evolution of phase paths: semiaxes of ellipses at $y_0 \to 0$ increase with a growth of amplitudes, elipses being strongly deformed in the foots reached by a fluid column; angles of paths on phase plane at an approximation to a parabolic form are sharped (Fig. 3 (b)).

Taking into account a numerical factor the behavior of the system in cases σ_{sg}=1 and y_0<<1 coincide.

When there is small difference between cross sections, the phase paths are ellipses, and in the process of decreasing of the parameter they are imposed at each other, almost merging into one curve. Further, if the oscillation amplitude is small (y<<1), we have the linear description of liquid oscillations in vessels with a frequency . Assuming the amplitude to be a small, but such that we neglect only an item contain square of differential from y, we obtain a nonlinear equation. In Fig. 3 (c) change of phase paths in process of increase in parameter l/σ_{Ls} is the following: elliptic paths are flattened to an ordinate axis having one generic point.

Oscillation frequencies in the system of connected vessels

For studying nonlinear oscillations of the system and analyzing oscillation period we find the integral of equation:

$$t - t_0 = \sqrt{(1-\sigma_{sg})}\int dy\sqrt{\frac{y+q}{y_0^2 - y^2}}, \tag{18}$$

Where t_0 is the constant of integration, and $q = \dfrac{1}{1-\sigma_{sg}} + \dfrac{l}{\sigma_{Ls}\left(1-\sigma_{sg}^2\right)}$.

If limits of integration are substituted:

$$t=\sqrt{(1-\sigma_{sg})}\int_{-y_0}^{y_0}\sqrt{\frac{y+q}{y_0^2-y^2}}dy=\sqrt{1-\sigma_{sg}}\times$$
$$\times 2\left(E(\varphi,k)\sqrt{y_0+q}-\sqrt{\frac{(y_0-y)(y+y_0)}{y+q}}\right), \tag{19}$$

where E(φ, k) is an elliptical integral of a second type and introduced denominations are:

$$\sin\phi=\sqrt{\frac{(y_0+q)(y+y_0)}{2y_0(y+q)}},\ k=\sqrt{\frac{2y_0}{y_0+q}}.$$

The half-period of oscillations is a time span of the oscillations amplitude changing from its minimum to maximum point (from $-y_0$ to y_0). We see that if we substitute the limits into an expression (19), the second term becomes zero both at minimum and maximum value. Furthermore we see that in the case of $y = -y_0$, sinφ = 0 and thus, φ = 0, and if $y = y_0$ then sinφ = 1, and thus φ = π/2. Because E(0,k) = 0, then oscillation frequency is:

$$\omega=\pi\Big/\left(2\sqrt{\left(1-\sigma_{sg}\right)\left(y_0+q\right)}E\left(\frac{\pi}{2},k\right)\right) \tag{20}$$

The influence cased by initial amplitude y_0 and by parameters of the system l/σ_{Ls} and σ_{sg} is shown in Fig. 4 (a) and in Fig. 4 (b). Fig. 4 (a) presents the dependence of oscillation amplitude on initial amplitude and parameter l/σ_{Ls} when the relation between cross sections is constant. It is obvious that oscillation amplitude increases with initial amplitude growth and the exponential law describes decreases with growth of parameter l/σLs, and dependence of frequency on parameter l/σLs. In Fig. 4 (b) there is a dependence of the oscillation frequency on initial amplitude y_0 and a parameter l/σ_{Ls}. As we can see here, oscillation frequency changes with an increase of an initial amplitude y_0 as an exponential function and with an increase of relation between cross sections σ_{sg} as a logarithmic function. To the linear cases of the oscillations lines of constant frequency

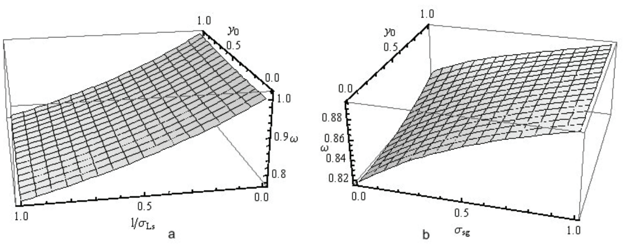

Fig. 4. The graph of dependence of an oscillation frequency: a) at fixed ratio of cross sections σ_{sg}=0.5 of vertical tubes, dependence on the initial amplitude y_0 and parameter l/σ_{Ls}; b) at fixed parameter l/σ_{Ls}=0.5, dependence on the initial amplitude y_0 and ratio of cross sections σ_{sg}.

correspond. For example, if k = 0 (i. e. y_0 = 0) E(π/2,k) = π/2 and then frequency is:

$$\omega=1\Big/\sqrt{\frac{H_0}{g}}\sqrt{1+\frac{S_sS_g}{H_0S_L(S_g+S_s)}}. \tag{21}$$

In addition, k becomes zero at q→∞ (i. e. at σ_{sg} = 1), and then we receive the expression for frequency which is totally coinciding with a formula (11).

If the cross sections of vertical tubes considerably differ, the oscillations period does not depend on a cross section of a vertical tube with a large diameter.

$$\frac{T}{2}=\sqrt{\frac{H_0}{g}}2\sqrt{y_0+1+\frac{LS_s}{S_LH_0}}E\left(\frac{\pi}{2},\sqrt{\frac{2y_0}{y_0+1+\frac{LS_s}{H_0S_L}}}\right). \tag{22}$$

Analogies

Analogy with electric current. In case of σ_{sg}=1 the equation takes the following form:

$$\dot{y}^2=\frac{y_0^2-y^2}{1+\frac{l}{\sigma_{Ls}}}=\frac{(y_0^2-y^2)\frac{2H_0}{S_s}}{\frac{2H_0}{S_s}+\frac{L}{S_L}}. \tag{23}$$

After redesignation $y_0^2-y^2\to\varepsilon$, $\dot{y}^2\to U$, $L/S_L\to r$, $2H_0/S\to R$, the equation transforms into an U=εR/(R+r), which is similar to an expression for the electric field strength, where r is an internal resistance, R is a load resistance, ε is an electromotive force. Instantaneous value of a deviation of level y from equilibrium level y=0 defines a potential energy of the system and is the reason of flow of a liquid. Redesignation corresponds to that from the total energy of all system (vertical containers and a connecting tube);the potential energy of vertical vessels is neglected. The kinetic energy of vertical vessels and a connecting tube exactly corresponds to electromotive force of a source of electric current. In the work [10] it is also discussed how to map the U-tube vessel to an electric circuit.

Mechanical analogies. As it was already considered above, in case of equality of cross section of vertical vessels we obtain the frequency period similar to small frequency period of a physical pendulum $T=2\pi((h+r^2/h)/g)^{1/2}$, where H_0 corresponds to the distance from the suspension point to a pendulum center of gravity h, and $LS_s/(2S_L)$ corresponds to the moment of inertia concerning the axis passing through a center of gravity. And if parameter l/σ_{Ls} equals to zero too, we obtain oscillation period of mathematical pendulum. Same way we can obtain analogy with oscillations of a spring pendulum $T=2\pi(m/\kappa)^{1/2}$ in case of equality of all cross sections in the system if we consider that to the weight m corresponds the value $\rho S(H_0+L/2)$, and to the stiffness there corresponds value κ = ρgS. In case of the maximal possible amplitude and condition $l/\sigma_{Ls} = 0$, when the equation describing motion has a form (17) we see that dimensionless oscillation period is T/2=√2. Respectively, dimensional period is:

$$T/2 = \sqrt{2H_0/g}. \tag{24}$$

Here we observe the analogy to periodic vertical motion of the ball which elastically bounces off a horizontal surface, where H_0 corresponds to the height from which the ball was dropped. The phase path for such a ball motion has the similar shape to the path in Fig. 3 (a) for $l/\sigma_{Ls} = 0$. Fig. 5 shows the time dependence in this case, it is set of parabolic segments (in linear case we have a sinusoidal dependence).

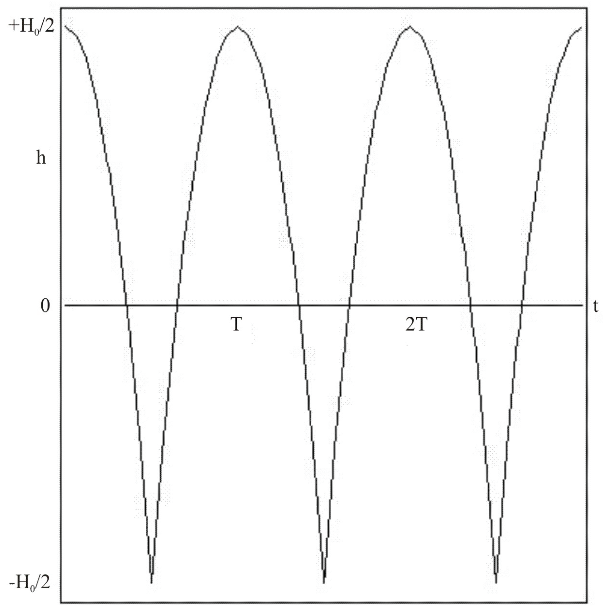

Fig. 5. The height-vs-time dependence in narrow vessel in extremely nonlinear case according to equations (17) and (24).

Analogies with wave motion. If all cross sections of the tubes in the system are equal, an analogy also could be considered with the propagation of elastic waves in condensed matter. Using accordance for stiffness κ = ρgS (on the assumption that vertical tubes and horizontal has the identical cross sections) we can rewrite expression for Young's modulus as:

$$E = \frac{\kappa(H_0 + \frac{L}{2})}{S_L} = \rho g(H_0 + \frac{L}{2}). \tag{25}$$

Consider longitudinal waves we see that phase velocity of the running waves is:

$$v_{\approx} = \sqrt{\frac{E}{\rho}} \to v_{\approx} = \sqrt{g\left(H_0 + \frac{L}{2}\right)} \tag{26}$$

Here we regard oscillations in U-tube as standing wave with the wavelength $\lambda_{st} = \lambda/2 = L$ where L is the distance between vertical containers (walls). This standing wave is a superposition of the running wave and reflected wave with wavelength λ.

An expression for group velocity of running waves is the following:

$$v_{gr} = v_f - \lambda\frac{\partial v_f}{\partial\lambda} = \sqrt{g\left(H_0 + \lambda/4\right)} - \frac{\lambda}{8}\frac{g}{\sqrt{g\left(H_0 + \lambda/4\right)}}. \tag{27}$$

Let us analyze the limiting cases, i.e. we will consider deep and shallow waters. As the criterion of definition of the deep and shallow water, we will take a ratio of length of a connecting tube and a equilibrium height of the liquid. If we consider that for the deep water the length of a connecting tube considerably exceeds liquid column height in a vertical vessel L/2 >> H_0, expression for phase speed has an appearance:

$$v_f = \sqrt{\lambda g/4}. \tag{28}$$

That practically coincides with an expression v_f for gravity waves in deep water [11], and the group velocity of those is:

$${}_{gr} = \sqrt{\quad /4} - \frac{}{8}\frac{\sqrt{\quad}}{/4} = \frac{}{2}. \tag{29}$$

i.e. group velocity is twice less than the phase one.

Now consider a case, when the length of a connecting tube is much smaller than the liquid column height in a vertical container, L/2 << H_0. Then the speed of a longitudinal wave equals to:

$$v_{\approx} = v_f \to \sqrt{gH_0} \tag{30}$$

Thereby we received phase velocity for gravity waves in shallow water. This speed is dispersion-free ($v_f = v_{gr}$).

When carrying out analogy to wave motion and consideration of deep and shallow water, an interesting question is: what exactly in the system of communicating vessels is defined as the limits of deep and shallow water? The criteria of definition of deep and shallow water can be chosen not only as a ratio of length of a connecting tube and equilibrium height, but also as a ratio of diameter of a vertical tube and equilibrium height of the liquid level in vertical containers. In a general case we obtain the nonlinear equation which solution will be expressed through elliptical integral and will have the form of a common solution of the nonlinear equation. Therefore we will focus on the less complicated case of the identical cross sections of vertical tubes.

For the deep water the equilibrium height of liquid in both vertical tubes well exceeds their diameters, and, respectively, the cross sections are $H_0 >> \sqrt{S_g}$ и $H_0 >> \sqrt{S_s}$, and then the period and frequency are:

$$T/2 = \sqrt{H_0/g}\pi, \;\; \omega \;\; = g/H_0. \tag{31}$$

Having made redesignation $\kappa = H_0^{-1}$, we obtain the ratio

$\omega^2 = g\kappa$, that has an appearance similar to a dispersion ratio for deep water $\omega^2 = gk$ [10], where k is a wave vector.

For shallow water ($H_0 << \sqrt{S_g}$, $H_0 << \sqrt{S_s}$) it is visible that frequency is:

$$\omega = \sqrt{\frac{g}{LS_s/2S_L}}. \tag{32}$$

The dispersion ratio for shallow water from [11] has an appearance $\omega^2 = g\kappa h/\chi$, where $\chi = \lambda/(2\pi)$ is a reduced wavelength. For the obtained expression we observe the following analogies: as well as for deep water a wave vector is $\kappa = 1/H_0$, to reduced length of a wave there corresponds value $\chi = LS_s/2S_L$. Respectively, expressions for phase velocity for deep and shallow water are:

$$v_f = \sqrt{g\frac{L}{2}\frac{S_s}{S_L}}, \text{ and } v_f = \sqrt{gH_0}. \tag{33}$$

It is interesting that both approximations for deep and shallow water will well be coordinated among themselves. If in the formula (28) as length of a standing wave we take the value LS_s/S_L (i.e. considering that the cross sections of vertical containers differ from the cross section of the connecting tube), then for deep water we receive expression that exactly coincides with the one for deep water, equation (33). Equality of phase velocities (30) and (33) for shallow water is apparent.

Conclusions

1. Oscillations of liquid in the system of communicating vessels are studied. Frequencies and periods of free oscillation are discribed for the general case through elliptical integral.

2. Phase portraits are plotted for the general case of oscillations when different parameters of the system are changed, i. e. dependence of liquid motion on one parameter of the system when other parameters are fixed. When oscillations are linear, phase paths are elliptical, while in the case of nonlinear oscillations we have parabolas closed by a vertical segment. The same kind of phase portraits also correspond to the vertical motion of a ball elastically bouncing off a horizontal surface, which suggests analysis of the strong analogies between the various mechanical motions and the oscillations of the liquid in the communicating vessels.

3. Analogies are considered with some diverse physical systems: mathematical, spring and physical pendulums, an electric current. We have also studied an interesting analogy with wave propagation in condensed matter which arises when we consider the cases of deep and shallow water. Notice two different ways to approach the problem: after obtaining expressions for the velocities, we can reveal the connection between oscillations in the U-tube and elastic waves, or otherwise we can first analyze the cases of deep and shallow water and then turn to elastic waves.

4. In this work the internal friction (viscosity) is neglected, which must have an influence on real liquids' motion. If we consider viscosity, the general motion equations would be changed, and that will lead to the results different from obtained here for the frequency of oscillations. It is of interest to further analyze the general expression for oscillations frequency because in the current work analysis is presented for the more simple case of the identical cross sections of two vessels. It is also prospective to consider the siphon mechanism within this system and study the fountain effect caused by special initial conditions. Using discussed system of communicating vessels such extraordinary phenomena as superfluid flow (for example, liquid helium or diluted quantum gas) and supersolid (^{4}He) could be experimentally studied.

1. D. J. Papoular, G. Ferrari, L. P. Pitaevskii, and S. Stringari. Phys. Rev. Lett., 109, 084501 (2012).
2. Gérard A. Maugin. Mechanics Research Communications, 69, 79 (2015).
3. D.R. Tilley, J Tilley. Superfluidity and Superconductivity, CRC Press (1990), 240 p.
4. M. J. Smith, J. J. Kobine, F. A. Davidson. Proc. R. Soc. A, 464, 905 (2008).
5. S. Upadhyay. Phys. Educ., 23, 21 (2006).
6. S. Sasaki, R. Ishiguro, F. Caupin, H. J. Maris, S. Balibar. Science, 313, 1098 (2006).
7. C. Gianino. Phys. Educ., 42, 488 (2007).
8. S-I Hong, Resonance, 16, 451 (2011).
9. Gradstein I. S., Ryzhik I. M. Table of Integrals, Series, and Products, The state. publishing house physical - a mat. literatures, M. (1963), p. 247.
10. S-I Hong, Phys. Educ., 45, 323 (2010).
11. Frank S. Crawford. Waves Berkeley Physics course, Nauka, Moscow (1974), 527 p.