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Estimating the number of solutions equation of N-point gravitational lenses algebraic geometry methods

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One of the main problems in the study of system of equations of the gravitational lens, is the computation of coordinates from the known position of the source.

In the process of computing finds the solution of equations with two unknowns. The difficulty lies in the fact that, in general, is not known constructive or analytical algorithm for solving systems of polynomial equations In this connection, use numerical methods like the method of tracing.

For the N-point gravitational lenses have a system of polynomial equations. Systems Research is advisable to start with an assessment of the number of solutions. This can be done by methods of algebraic geometry.

Keywords: gravitational lenses, algebraic geometry, Bézout's theorem.

Однією з основних завдань, при дослідженні системи рівнянь гравітаційної лінзи, є обчислення координат зображення за відомим положенням джерела.

У процесі обчислень доводиться знаходити рішення системи рівнянь з двома невідомими. Складність полягає в тому, що в загальному випадку не відомий конструктивний або аналітичний алгоритм для вирішення систем нелінійних рівнянь. У зв'язку з цим вдаються до чисельних методів подібним методу трасування.

У разі N-точкових гравітаційних лінз система рівнянь є полиномиальной. Дослідження такої системи доцільно почати з оцінки числа рішень. Ми проводимо це дослідження методами алгебраїчної геометрії.

Ключові слова: гравітаційні лінзи, алгебраїчна геометрія, теорема Безу.

Одной из основных задач, при исследовании системы уравнений гравитационной линзы, есть вычисление координат изображения по известному положению источника.

В процессе вычислений приходится находить решение системы уравнений с двумя неизвестными. Трудность состоит в том, что в общем случае не известен конструктивный или аналитический алгоритм для решения систем нелинейных уравнений. В связи с этим прибегают к численным методам подобным методу трассировки.

В случае N-точечных гравитационных линз система уравнений является полиномиальной. Исследование такой системы целесообразно начать с оценки числа решений. Мы проводим данное исследование методами алгебраической геометрии.

Ключевые слова: гравитационные линзы, алгебраическая геометрия, теорема Безу.

Introduction

According to the general theory of relativity, the light beam, which passes close to a point source of gravity (gravitational lens) at a distance ξ from it (in case $\xi >> r_g$) is deflected by an angle

$$\vec{\alpha} = \frac{2 \cdot r_g}{\xi^2}\vec{\xi} = \frac{4 \cdot G \cdot M}{c^2 \cdot \xi^2}\vec{\xi}, \qquad (1)$$

where r_g - gravitational radius; M - mass point of the lens; G - gravity constant; c - velocity of light in vacuum.

The detailed derivation of the formula (1) can be found in many classic books [1-3]. For N - point of the gravitational lens, in the case of small tilt angles have the following equation in dimensionless variables [4], [5]:

$$\vec{y} = \vec{x} - \sum_i m_i \frac{\vec{x} - \vec{l}_i}{\left|\vec{x} - \vec{l}_i\right|^2}, \qquad (2)$$

where $\vec{l}_i$ - dimensionless radius vector of point masses outside the lens, and the mass m_i satisfy the relation $\Sigma m_i = 1$.

The Equation (2) in coordinate form has the form of system:

$$\begin{cases} y_1 = x_1 - \sum_{i=1}^{N} m_i \dfrac{x_1 - a_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} \\ y_2 = x_2 - \sum_{i=1}^{N} m_i \dfrac{x_2 - b_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} \end{cases}, \qquad (3)$$

where a_i and b_i are the coordinates of the radius-vector $\vec{l_i}$, i.e. $\vec{l_i} = (a_i, b_i)$.

The right parts the equations of system (3), are rational functions of the variables x_1 and x_2. We transform each equation of the system (3) in a polynomial equation, and we obtain a system of equations:

$$\begin{cases} F_1(x_1, x_2, y_1) = 0 \\ F_2(x_1, x_2, y_2) = 0 \end{cases}, \qquad (4)$$

To study the solutions of the system (4), it will be convenient methods of algebraic geometry.

Indeed, the main object of study of classical algebraic geometry, as well as in a broad sense and modern algebraic geometry, are the set of solutions of algebraic systems, in particular polynomial, equations. This situation gives us the opportunity to apply the techniques of algebraic geometry in the theory of *N*-point gravitational lenses [6] - [9].

Most of the results that we need stated in terms of homogeneous coordinates to projective curves that are defined in the projective plane.

Therefore, we need, in the beginning, the system transform (4) to a form convenient for evaluation. To transform the system (4) we need the following terms and definitions.

Mathematical definitions and theorems

Before that provide definitions, we describe the projective plane and homogeneous (projective) coordinates. The real projective plane can be thought of as the Euclidean plane with additional points added, which are called points at infinity, and are considered to lie on a new line, the line at infinity. There is a point at infinity corresponding to each direction (numerically given by the slope of a line), informally defined as the limit of a point that moves in that direction away from the origin. Parallel lines in the Euclidean plane are said to intersect at a point at infinity corresponding to their common direction. Given a point (x, y) on the Euclidean plane, for any non-zero real number S, the triple (xS, yS, S) is called a set of homogeneous coordinates for the point. By this definition, multiplying the three homogeneous coordinates by a common, non-zero factor gives a new set of homogeneous coordinates for the same point. In particular, $(1,2)$ is such a system of homogeneous coordinates for the point (x, y). For example, the Cartesian point $(x, y, 1)$ can be represented in homogeneous coordinates as $(1,2.1)$ or $(2,4.2)$. The original Cartesian coordinates are recovered by dividing the first two positions by the third. Thus unlike Cartesian coordinates, a single point can be represented by infinitely many homogeneous coordinates.

Some authors use different notations for homogeneous coordinates which help distinguish them from Cartesian coordinates. The use of colons instead of commas, for example $(x : y : z)$ instead of (x, y, z), emphasizes that the coordinates are to be considered ratios [10]. Square brackets, as in $[x, y, z]$ emphasize that multiple sets of coordinates are associated with a single point [11]. Some authors use a combination of colons and square brackets, as in $[x : y : z]$ [12].

The properties of homogeneous coordinates on the plane:

We define homogeneous projective coordinates for the first points of the projective plane not lying on a straight ∞.

At all points of the projective plane, in addition to lying on a straight ∞ (the line at infinity) are homogeneous coordinates of the projective: three numbers, not both zero.

The basic points for arithmetization projective plane (i.e., the introduction of non-homogeneous projective coordinates) are the origin Systems; ∞_x (infinity on the x-axis), ∞_y (infinity on the y-axis), $(1,1)$ - unit. Obviously, the line at infinity passes through the points ∞_x and ∞_y.

We give a precise definition. Homogeneous coordinates of a point M is said to be three numbers X_1, X_2, X_3, not both zero and such that $X_1/X_3 = x$; $X_2/X_3 = y$, where x and y - projective heterogeneous (affine) position. Homogeneous coordinates of points M_∞ lying on a line, ∞, call three numbers

X_1, X_2, X_3 under conditions:

1. $X_3 = 0$;

2. Of two numbers X_1, X_2 is at least one non-zero;

3. The ratio X_1 / X_2 is equal to $B / (-A)$, where *A* and *B* - coefficients of each line $Ax + By + C = 0$, passing through M_∞.

Let us look at some of the properties of homogeneous coordinates in the projective plane.

1) Each point of the projective plane is the homogeneous coordinates.

2) If X_1, X_2, X_3 - homogeneous coordinates of the point *M*, then sX_1, sX_2, sX_3 (where *s* - any non-zero number), too, are homogeneous coordinates of the point *M*.

3) different points correspond to different attitudes X_1 / X_3; X_2 / X_3 their homogeneous coordinates.

4) Direct $A_1 A_2$ - is the line at infinity - it is in homogeneous coordinates the equation $X_3 = 0$.

5) The axes have their usual equation.

6) If the equation of the line $Ax + By + C = 0$ substitute homogeneous coordinates of a point *M*, lying on the straight line ($X_1 / X_3 = x$; $X_2 / X_3 = y$), then we get: $AX_1 + BX_2 + CX_3 = 0$, linear equation in a homogeneous form (there is no free member).

7) The equation of any curve in homogeneous coordinates is a homogeneous equation, and its degree is called the degree of the curve.

8) A polynomial $g(x, y)$ of degree *k* can be turned into a homogeneous polynomial by replacing *x* with X_1 / X_3, *y* with $/$, and multiplying by X_3^k, in other words by defining.

The resulting function *P* is a polynomial so it makes sense to extend its domain to triples where $X_3 = 0$. The equation $P(X_1, X_2, X_3) = 0$ can then be thought of as the homogeneous form of $g(x, y)$, and it defines the same curve when restricted to the Euclidean plane.

Transform the system (3) into the equation system in homogeneous coordinates. We introduce the notation of homogeneous coordinates.

Let $x_1 = \frac{X_1}{X_0}$, $x_2 = \frac{X_2}{X_0}$. We have a system of equations in homogeneous coordinates:

$$\begin{cases} y_1 = \dfrac{X_1}{X_0} - \sum\limits_{i=1}^{N} m_i \dfrac{\dfrac{X_1}{X_0} - a_i}{\left(\dfrac{X_1}{X_0} - a_i\right)^2 + \left(\dfrac{X_2}{X_0} - b_i\right)^2} \\ y_2 = \dfrac{X_2}{X_0} - \sum\limits_{i=1}^{N} m_i \dfrac{x_2 - b_i}{\left(\dfrac{X_1}{X_0} - a_i\right)^2 + \left(\dfrac{X_2}{X_0} - b_i\right)^2} \end{cases}. \tag{5}$$

We transform the system (5)

$$\begin{cases} y_1 = \dfrac{X_1}{X_0} - \sum\limits_{i=1}^{N} m_i \dfrac{\left(X_1 - a_i X_0\right) X_0}{\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2} \\ y_2 = \dfrac{X_2}{X_0} - \sum\limits_{i=1}^{N} m_i \dfrac{\left(X_2 - b_i X_0\right) X_0}{\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2} \end{cases} . \tag{6}$$

We transform the system (6) to form: $\begin{cases} F_1\left(X_0, X_1, X_2, y_1\right) = 0 \\ F_2\left(X_0, X_1, X_2, y_2\right) = 0 \end{cases}$

$$\begin{cases} F_1\left(X_0, X_1, X_2, y_1\right) = 0 \\ F_2\left(X_0, X_1, X_2, y_2\right) = 0 \end{cases} \tag{7}$$

where $F_1 \quad F_1\left(X_0, X_1, X_2, y_1\right)$ and $F_2 = F_2\left(X_0, X_1, X_2, y_2\right)$ - homogeneous polynomials.

In equations of system (6), under the sign of the amount we give to a common denominator. We have:

$$\begin{cases} y_1 = \dfrac{X_1}{X_0} - \dfrac{X_0 \sum\limits_{j=1}^{N} m_j \left(X_1 - a_j X_0\right) \left\{ \prod\limits_{i=1, i\neq j}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right] \right\}}{\prod\limits_{i=1}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right]} \\ y_2 = \dfrac{X_2}{X_0} - \dfrac{X_0 \sum\limits_{j=1}^{N} m_j \left(X_2 - b_j X_0\right) \left\{ \prod\limits_{i=1, i\neq j}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right] \right\}}{\prod\limits_{i=1}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right]} \end{cases} \tag{8}$$

Let denominator through:

$$L = L\left(X_0, X_1, X_2\right) = \prod_{i=1}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right].$$

Transforming further we have:

$$\begin{cases} y_1 = \dfrac{X_1}{X_0} - \dfrac{X_0 \sum\limits_{j=1}^{N} m_j \left(X_1 - a_j X_0\right) \left\{ \prod\limits_{i=1, i\neq j}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right] \right\}}{L\left(X_0, X_1, X_2\right)} \\ y_2 = \dfrac{X_2}{X_0} - \dfrac{X_0 \sum\limits_{j=1}^{N} m_j \left(X_2 - b_j X_0\right) \left\{ \prod\limits_{i=1, i\neq j}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right] \right\}}{L\left(X_0, X_1, X_2\right)} \end{cases}$$

We transform the equation to a polynomial form.

$$\begin{cases} \left(X_1 - y_1 X_0\right) L\left(X_0, X_1, X_2\right) - X_0^2 \sum\limits_{j=1}^{N} m_j \left(X_1 - a_j X_0\right) \left\{ \prod\limits_{i=1, i\neq j}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right] \right\} = 0 \\ \left(X_2 - y_2 X_0\right) L\left(X_0, X_1, X_2\right) - X_0^2 \sum\limits_{j=1}^{N} m_j \left(X_2 - b_j X_0\right) \left\{ \prod\limits_{i=1, i\neq j}^{N} \left[\left(X_1 - a_i X_0\right)^2 + \left(X_2 - b_i X_0\right)^2 \right] \right\} = 0 \end{cases} \tag{9}$$

Note that the polynomial in the left side of the first (second) equation is homogeneous.

For the degrees of these polynomials we have:

$$deg\left\{(X_1 - y_1X_0)L(X_0, X_1, X_2) - X_0^2\sum_{j=1}^{N} m_j(X_1 - a_jX_0)\left\{\prod_{i=1, i\neq j}^{N}\left[(X_1 - a_iX_0)^2 + (X_2 - b_iX_0)^2\right]\right\}\right\} = 2N+1,$$

$$deg\left\{(X_2 - y_2X_0)L(X_0, X_1, X_2) - X_0^2\sum_{j=1}^{N} m_j(X_2 - b_jX_0)\left\{\prod_{i=1, i\neq j}^{N}\left[(X_1 - a_iX_0)^2 + (X_2 - b_iX_0)^2\right]\right\}\right\} = 2N+1.$$

It's obvious that:

$$deg\, F_1 = deg\, F_2 = 2N+1\,. \qquad (10)$$

Estimating the number of solutions of homogeneous polynomials

Let the number of solutions of (7), of course. The case in which the number of solutions endlessly, it is found out quite simple, and does not need to move to the projective coordinates, see. e.g., [6].

To estimate the number of solutions of homogeneous polynomials (7) we apply the following Bézout's theorem, see. e.g., [10], which says: that if *C* and *D* are plane curves of degrees $deg\, C = m$, $deg\, D = n$, then the number of points of intersection of *C* and *D* is $m \cdot n$, provided that

(i) the field is algebraically closed;

(ii) points of intersection are counted with the right multiplicities;

(iii) we work in to take right account of intersections 'at infinity'.

See for example [10].

Note that the equations $F_1(X_0, X_1, X_2, y_1) = 0$ and $F_2(X_0, X_1, X_2, y_2) = 0$ define two curves in the projective space, and considered over an algebraically closed field of complex numbers. Thus we are in the conditions of application of the Bézout's theorem.

Conclusions

- Number of complex solutions of equations (7), taking into account the multiplicity, in the projective plane P^2 equal $(2N+1)^2$;

-The number of real solutions of the system of equations (7), taking into account the multiplicity equal to $(2N+1)^2 - 2n$ in the projective plane P^2, where *n* - a natural number, that is less than or equal to *N*, that is $n \leq N$. The last relation occurs because the systems of equations (7), although considered over the field of complex numbers, but have real coefficients. The fact, that in such equations, complex roots come in pairs.

-number $(2N+1)^2 - 2n$ - odd number. The number of real solutions of the system of equations (7), taking into account the multiplicity, in the affine plane A^2, is also equal to this number, if the curves do not intersect at infinity.

Note that each real solution of system has an obvious physical meaning: the coordinates of the point - image plane of the lens. Our results, about the oddness of the number of images are in good agreement with the theorem proved in [4], [5]. At the same time, our approach is based on the methods of algebraic geometry.

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