Вісник ХНУ, серія «Фізика», вип. 26, 2017. с. 6-27

PACS
UDC

Mathematical bases of the theory of N-point gravitational lenses. Part 1. Elements of algebraic geometry

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In this paper we consider the theory of N-point gravitational lens from the standpoint of classical algebraic geometry. The first section explains the physical statement of the problem and given the conclusion of the basic equation of the gravitational lens. In the second - a brief discussion of the main objects of study in classical algebraic geometry, and justified its application to the theory of N-point gravitational lenses. Then we give the definition of the central concepts of algebraic geometry - and the resultant theorems related. The fourth section shows, a well-known, Bezout theorem on the number of solutions of polynomial equations of the system and its corollary. In our approach, this theorem is needed to study the solutions of the gravitational lens. In the fifth section, we formulate and prove a criterion of irreducibility of polynomials in several variables over the field of complex numbers. We do not know analogues of this criterion for polynomials in several variables over a field of characteristic zero. The final section provides an overview of the solutions of systems of polynomial equations and formulated a number of challenges and problems the solution of which, in our opinion, it is advisable to apply the presented mathematical apparatus.
Keywords

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1. Physical formulation of problem

When the light ray propagates near a point massive object (gravitational lens), its trajectory bends. In case when the minimum distance ξ (see fig.1) on which the ray approaches to the attracting body is much more than its gravitational radius r_g, the true light ray trajectory can be replaced by asymptotes to hyperbola. Then the angle between the asymptotes is defined with the following expression [1,2]

$$\alpha = \frac{2r_g}{\xi} = \frac{4GM}{c^2\xi}, \tag{1.1}$$

where M - mass of point lens, G - gravitation constant, c - speed of light in vacuum.

For the possibility of extension of a one-point gravitational lens to a lens consisting of *N* point masses situated in the lens plane in points with radius vectors $\vec{\xi}_i$ the formula for light-ray deflection angle (1) is written in the vector form

$$\vec{\alpha} = \frac{4GM}{c^2\xi^2}\vec{\xi}\,, \tag{1.2}$$

where the direction of vector $\vec{\alpha}$ coincides with the direction of vector $\vec{\xi}$ (where $|\vec{\alpha}| = \alpha$). As the massive body "attracts" the light ray, it deflects to the body under consideration. Hence, the expression for the deflection is $\vec{\vartheta} = -\vec{\alpha}$.

The formula (2) can be easily written for every M_i mass included in the lens

$$\vec{\alpha}_i = \frac{4GM_i}{c^2\left|\vec{\xi} - \vec{\xi}_i\right|^2}\left(\vec{\xi} - \vec{\xi}_i\right), \tag{1.3}$$

where vector $\Delta\xi_i = \vec{\xi} - \vec{\xi}_i$ is directed from the point of arrangement of *i*-th mass into the point of intersection of the light ray with the lens plane (which is defined by vector $\vec{\xi}$). It is obvious that at $\vec{\xi}_i \to 0$ (point mass is situated in the origin of coordinates) the formula (1.3) turns into (1.2). The full deflection angle will be equal to the vector sum of deflection angles from every *i*-th mass

$$\vec{\alpha} = \sum_{i=1}^{N}\vec{\alpha}_i\,. \tag{1.4}$$

In case of small deflection angles from fig.1 we have

$$\vec{\eta} + D_{ds}\,\vec{\alpha} = \vec{\eta} + \vec{\varsigma}\,. \tag{1.5}$$

On the other hand

$$\vec{\eta} + \vec{\varsigma} = D_s\vec{\beta}\,, \tag{1.6}$$

where $\vec{\beta} = \vec{\xi} / D_d$.

Finally, from (5) and (6) we obtain the equation of gravitational lens [3,4]

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\,\vec{\alpha} \tag{1.7}$$

Specifically, for a one-point lens we have the following lens equation

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\,\frac{4GM}{c^2\xi^2}\vec{\xi}\,. \tag{1.8}$$

At $\vec{\eta} = 0$, i.e. when the light source is situated on the lens axis, we have equation on $\vec{\xi}$

$$\vec{\xi}\left(\frac{D_s}{D_d} - D_{ds}\,\frac{4GM}{c^2\xi^2}\right) = 0\,. \tag{1.9}$$

The solution (9) is usually denoted as ξ_0

$$\xi_0 = \sqrt{\frac{4GM \cdot D_{ds} \cdot D_d}{c^2 \cdot D_s}} \tag{1.10}$$

Thus, if the point source, point lens and observer are situated on the same straight line, the observer will see a circle with radius ξ_0. This radius is usually called Einstein-Chwolson radius [5,6], and the circle itself - Einstein-Chwolson ring. Introducing dimensionless variables

$$\vec{x} \equiv \frac{\vec{\xi}}{\xi_0},\;\; \vec{y} \equiv \frac{D_d}{D_s}\frac{\vec{\eta}}{\xi_0} \tag{1.11}$$

the equation of one-point gravitational lens (8) becomes

$$\vec{y} = \vec{x} - \frac{\vec{x}}{x^2}\,. \tag{1.12}$$

The equation of *N*-point lens (7) can be written as

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\sum_{i=1}^{N}\frac{4GM_i}{c^2\left|\vec{\xi} - \vec{\xi}_i\right|^2}\left(\vec{\xi} - \vec{\xi}_i\right). \tag{1.13}$$

For writing (1.13) in the dimensionless form we take into account that Einstein-Chwolson radius ξ_0 is determined from the full mass of the gravitational lens $M = \sum_i M_i$ (though there is no Einstein-Chwolson ring in case of *N*-point lens).

Rewrite (1.13) in the form

$$\frac{D_d\vec{\eta}}{D_s\xi_0} = \frac{\vec{\xi}}{\xi_0} - \frac{4G D_d D_{ds} M}{\xi_0 D_s c^2}\sum_{i=1}^{N}\frac{M_i / M}{\left|\vec{\xi} - \vec{\xi}_i\right|^2}\left(\vec{\xi} - \vec{\xi}_i\right). \tag{1.14}$$

Let us note that in (1.14) there is ξ_0^2 / ξ_0 expression before the summation symbol. Taking into account (1.11) and also introducing $m \equiv M_i / M$ – dimensionless masses we obtain from (14)

$$\vec{y} = \vec{x} - \xi_0 \sum_{i=1}^{N} \frac{m_i}{\left|\vec{\xi} - \vec{\xi}_i\right|^2}\left(\vec{\xi} - \vec{\xi}_i\right) = \vec{x} - \sum_{i=1}^{N} \frac{m_i}{\left|\vec{\xi} / \xi_0 - \vec{\xi}_i / \xi_0\right|^2}\left(\vec{\xi} / \xi_0 - \vec{\xi}_i / \xi_0\right). \quad (1.15)$$

Finally, the equation of *N*-point gravitational lens becomes

$$\vec{y} = \vec{x} - \sum_i m_i \frac{\vec{x} - \vec{l}_i}{\left|\vec{x} - \vec{l}_i\right|^2}, \quad (1.16)$$

where $\vec{l}_i = \vec{\xi}_i / \xi_0$ - dimensionless radius vectors of point masses included into the lens. It is obvious that $\sum_i m_i = 1$.

For the further analysis let us rewrite the set of equations (1.16) in the coordinate form. Taking into account that the vectors have the following components $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$, $\vec{l}_i = (a_i, b_i)$ we receive the following system

$$\begin{cases} y_1 = x_1 - \sum_{i=1}^{N} m_i \frac{x_1 - a_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} \\ y_2 = x_2 - \sum_{i=1}^{N} m_i \frac{x_2 - b_i}{(x_1 - a_i)^2 + (x_2 - b_i)^2} \end{cases}. \quad (1.17)$$

One of the main tasks of the theory of gravitational lens is the image construction from the specified source. That is, from the known coordinates of the source (y_1, y_2) to find the images coordinates (x_1, x_2). The problem is that in the general case we do not know a constructive or analytic algorithm for the solution of the system (1.17). The availability of such an algorithm would make it possible to apply methods of symbolic programming. Currently, numerical methods similar to the routing method are applied [7-9]. In this paper we develop an approach based on algebraic geometry. This approach makes it possible to construct quasianalytic and, sometimes, analytic algorithms for the solution of a number of problems. The beginning of this approach is initiated in [10].

2. Basic objects of study in classical algebraic geometry

The main direction of algebraic geometry is the study of the properties of algebraic varieties over an algebraically closed field. Most often consider affine and projective variety over the field of real or complex numbers. Obvious reason that studying the variety and not a vector space. If varieties properties do not depend on the structure of a vector space, we can, the basic elements of space, regarded as points, and not as a vector. To study the affine n-dimensional space, it is fixed in some basis (particularly selected origin of coordinates). Further, each *S* - family of polynomials *K* rings put in correspondence the set points *V*(*S*) whose coordinates satisfy all polynomials of the set *S*. Obviously, the coordinates of a set of points *V*(*S*) are solutions of the system, which is composed of equations belonging to the family *S*.

It is known that the property of being a polynomial function does not depend on the choice of basis. On this basis, we can speak of polynomial functions as a set of common zeros of *V*(*S*) of functions of the family *S*. The sets that can be represented in the form *V*(*S*), called algebraic sets. On the other hand, any algebraic set can be uniquely represented as the union of a finite number of disjoint algebraic varieties.

Thus, the main object of study of classical algebraic geometry, as well as in a broad sense and modern algebraic geometry, are the set of solutions of algebraic systems, in particular polynomial, equations. This fact gives us the opportunity to apply the techniques of algebraic geometry in the theory of *N*-point gravitational lenses.

In the late 1950s, Alexander Grothendieck gave a schema definition that the concept of an algebraic variety see [26]. This event is considered the beginning of modern algebraic geometry and the end of classical see. [27].

In algebraic geometry formed a number of directions. We mention some of them. Complex Algebraic Geometry. In a separate direction is isolated, and the study of the real points of a complex manifold. This area is called, real algebraic geometry. Complex Algebraic Geometry. In a separate area is isolated, and the study of the real points of a complex manifold. This area is called, real algebraic geometry.

Separate direction of studying features of complex algebraic varieties (including one dimension - Riemann surfaces) and real algebraic varieties. Singularity Theory varieties naturally intersects with algebraic topology.

At the intersection of algebraic geometry and computer algebra we have computational algebraic geometry. Its basic task - creation of algorithms and software for studying the properties of explicit algebraic varieties

The concepts and theorems set forth in this article, mostly apply to sections of algebraic geometry, known as elimination theory and the theory of algebraic curves.

All results are set out in affine coordinates, including those that have been proven by using projective coordinates. We managed to avoid the use of projective coordinates by applying the linear fractional transformations of affine coordinates.

We also paid attention an important concept of modern algebra - basis Gröbner, which can be applied to the study of systems of polynomial equations, in particular, for the construction of an efficient algorithm for answering the question: is finite or infinite number of solutions? We are considering the system contain a small number of equations, that allows you to answer these and similar questions, using other available means, such as a theorem of the resultant and the criterion of irreducible polynomials.

3. The resultant is the central concept of classical algebraic geometry. Fundamental theorems on the resultant

This section contains some well-known, definitions and theorems of classical algebraic geometry. These definitions and theorems presented in a form that meets, in our opinion, the objectives of this article. Theorems are given without proof, but are referenced to the appropriate sources.

The resultant is a central concept of classical algebraic geometry. The current literature [13-15] resultant usually defined as follows:

Definition 3.1 Let K - arbitrary field, $f(x)$ and $g(x)$ - ring of polynomials $K[x]$. The resultant $R(f,g)$ of polynomials $f(x)$ and $g(x)$ is called an element field K, defined by the formula:

$$R(f,g)=a_0^n b_0^m \prod_{i=0}^{i=n}\prod_{j=0}^{j=m}(\alpha_i-\beta_j), \tag{3.1}$$

where α_i,β_j - roots of polynomials $f(x)=\sum_{i=0}^{i=n}a_i x^{n-i}$ and $g(x)=\sum_{j=0}^{j=m}b_j x^{m-j}$, correspondingly with the highest coefficients, a_0,b_0 such that $a_0\neq 0, b_0\neq 0$.

Assume us know the roots of polynomials $f(x)$ and $g(x)$, to calculate their resultant, we can use the formula (3.1). Assume that the coefficients of these polynomials only then to calculate the resultant can use the Sylvester matrix for these polynomials. Sylvester matrix is a block matrix of the two blocks. Each unit has a banded matrix. We have a definition of the Sylvester matrix.

Definition 3.2. Matrix Sylvester for polynomials $f(x)=\sum_{i=0}^{i=n}a_i x^{n-i}$ and $g(x)=\sum_{j=0}^{j=m}b_j x^{m-j}$, we call a square matrix $S=S(f,g)$ of order $n+m$ with elements s_{ij} defined by the formula:

$$s_{ij}=\begin{cases} a_{j-i}, \ f \quad 0\le j-i\le n, \quad i=1,\dots,m, \quad j=1,\dots,n+m\,; \\ b_{j-i+m}, \ f \quad 0\le j-i+m\le n, \quad i=(m+1),\dots,(n+m) \quad j=1,\dots,n+m\,; \\ 0, \quad \text{for others } i,j \end{cases} \tag{3.2}$$

i.e.

$$S(f,g)=[s_{ij}]=\begin{array}{r} \\ n-lines\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ m-lines\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array}\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-1} & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & b_{m-1} & b_m \end{bmatrix} \tag{3.3}$$

Sometimes Sylvester matrix is called the matrix:

$$S(f,g)=[s_j]=\begin{array}{r}n-lines\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.\\ m-lines\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right.\end{array}\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & \cdots & 0 & 0\\ 0 & a_0 & a_1 & a_2 & \cdots & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \cdots\\ 0 & 0 & \cdots & \cdots & \cdots & a_{n-2} & a_{n-1} & a_n\\ 0 & 0 & \cdots & \cdots & \cdots & b_{n-2} & b_{m-1} & b_m\\ 0 & 0 & \cdots & \cdots & b_{n-2} & b_{m-1} & b_m & 0\\ \cdots & \cdots & \iddots & \iddots & \iddots & \iddots & \cdots & \cdots\\ b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 & 0\end{bmatrix}, \tag{3.4}$$

see. e.g. [12], [19], [24].

Sylvester matrix that defined by (3.3) will be denoted by $Sul(f,g)$, see. e.g., [24], and write $Sul(f,g)=S(f,g)$. Sylvester matrix that defined by (3.4) - denote by the $Sul^{>}(f,g)$, and write $Sul^{>}(f,g)=S(f,g)$. Superscript in the designation of the matrix reflects the location of the bands.

The Presentation matrices of Sylvester in various forms have rationale. Many important results are stated in terms of the minors of these matrices.

The determinant of the matrix $Sul^{>}(f,g)$ is different from the determinant of the matrix $Sul(f,g)$ only sign. We have the following relation:

$$\det Sul^{>}(f,g)=(-1)^{[m/2]}\det Sul(f,g), \tag{3.5}$$

where $\left[m/2\right]$ the integer part of number m.

The resultant $R(f,g)$ and matrix of Sylvester $Sul(f,g)$ associated equation.

Theorem 3.1. The resultant $R(f,g)$ of the polynomials f and g is equal to the determinant of Sylvester matrix these polynomials, i.e.

$$R(f,g)=\det Sul(f,g). \tag{3.6}$$

Proof of Theorem 3.1., see. e.g., [13], [14].

Example 3.1. Calculate the resultant of the polynomials: $f_1=x^2-3x+2$ and $f_2=x^2+1$.

Solution. $R(f_1,f_2)=\det Sul(f_1,f_2)=\begin{vmatrix}1 & -3 & 2 & 0\\ 0 & 1 & -3 & 2\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{vmatrix}=10$

Sometimes resultant $R(f,g)$ determine the determinant of the Sylvester matrix $Sul(f,g)$, and equation (3.1) proves, see. e.g. [13], [14].

Have the following

Theorem 3.2. Polynomials f and g have a common root if and only if

$$R(f,g)=0. \tag{3.7}$$

The proof of Theorem 3.2. See, for example, in [19].

Example 3.2. Do polynomials $f_1=x^3-1$ and $f_2=x^2-1$ common roots?

Solution. $R(f_1, f_2) = \det Sul(f_1, f_2) = \det Sul(x^3 - 1, x^2 - 1) = \begin{vmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{vmatrix} = 0$,

Polynomials $f_1 = x^3 - 1$ and $f_2 = x^2 - 1$ have common roots.

You can determine the number of common roots of polynomials f and, if you use some of the concepts related to the resultant $R(f, g)$.

Definition 3.2. Let M be an arbitrary square matrix of order n. Innor M_k, $1 \le k \le \left[\frac{n}{2}\right]$ (in brackets is the integer part $\frac{n}{2}$), the matrix M is called the matrix obtained from the matrix M by deleting its elements which are in k first and last rows and k the first and last columns. The matrix M is called the matrix of innor, see. [25].

Matrix of innor M is denoted by M_0

Definition 3.3. Let $S = S(f, g)$ Sylvester matrix of the polynomials f and g, it inures will be denoted by S_k..

Definition 3.4. Determinants $\det S_k$ will be denoted by $R^{(k)}$ and name subresultants resultant $R(f, g)$ polynomials f and g.

Have the following

Theorem 3.3. The polynomials $f(x)$ and $g(x)$ have d common roots if and only if

$$R(f, g) = R^{(1)}(f, g) = ... = R^{(d-1)}(f, g) = 0,$$

where d such that $1 \le d \le \min(\deg f(x), \deg g(x))$.

Example 3.3. What are the common roots of polynomials, $f_1 = x^3 - 1$ and $f_2 = x^2 - 1$?

Solution. We calculate subrezultantes $R^{(1)}(f_1, f_2)$, $R^{(2)}(f_1, f_2)$:

$$R^{(1)}(f_1, f_2) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1 \ne 0, \quad R^{(2)}(f_1, f_2) = |-1| \ne 0,$$

Polynomials $f_1 = x^3 - 1$ and $f_2 = x^2 - 1$ have a common root, because the $R(f_1, f_2) = 0$ and $R^{(1)}(f_1, f_2) = 1 \ne 0$ (In the first row subresultants $R(f_1, f_2)$, $R^{(1)}(f_1, f_2)$, $R^{(2)}(f_1, f_2)$ first subresultant, which is not zero, have number 1).

The greatest common divisor $\deg(GCD(f, g)$ of the polynomials $f(x)$ and $g(x)$ can be computed using Euclid's algorithm, see. [12-14], [16], or by using subresultants, see. [12], [19], [25].

A special case of the resultant polynomial is the discriminant.

Definition 3.5. Let K - arbitrary field, $f = f(x)$ - polynomial in the polynomial ring $K[x]$.

The discriminant $D(f)$ of $f = f(x)$ is called an element of K, defined as follows:

$$D(f) = a_0^{2n-2} \prod_{1 \le j < i \le n} (\lambda_i - \lambda_j)^2, \tag{3.8}$$

where $n = \deg f(x)$ - the degree of the polynomial $f(x)$, a_0 - its leading coefficient, $\lambda_1,...,\lambda_n$, its roots, see [13].

Have the following

Theorem 3.4. Let $f = f(x)$ - polynomial in the polynomial ring $K[x]$, f' - its derivative, then for the discriminate $D(f)$ we have the relation:

$$D(f) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_0} R(f, f'). \tag{3.9}$$

The proof of Theorem 3.4 is given in [13].

From Theorem 3.2 follows

Theorem 3.5. The polynomial f has a multiple root if and only if, $D(f) = 0$.

The proof is given in [13-14], [24].

Similarly, as the subresultants some authors define subdiscriminant, see. [25]. Using concepts: innor, subresultants, subdiscriminant etc. we can formulate and prove a number of theorems on the distribution of the roots of polynomials, such as the criterion of Routh-Hurwitz, see. [19].

An important theorem is the theorem on the number of solutions of systems of polynomial equations. Bezout Theorem is a theorem of this kind. Bezout Theorem is discussed in Section 4.

Many allegations of algebraic geometry begins with the assumption of irreducibility (or reducible) polynomial. Under the irreducible polynomial f over K understands the impossibility of its representation in the form of a product of two polynomials f_1 and f_2 nonzero degree over the same field, i.e. $f \neq f_1 \cdot f_2$.

Here for example one of them.

Theorem 3.6. Let f and g are polynomials with coefficients from the field K and the polynomial f is reducible, i.e. $f = f_1 \cdot f_2 =$, where f_1 and f_2 are not polynomials of degree zero over the field K. Then, to the resultant $R(f, g)$, equation holds:

$$R(f, g) = R(f_1, g) R(f_2, g). \tag{3.10}$$

Theorem 3.6 allows us to reduce the number of operations in the calculation of the resultant.

Availability of convenient criterion is irreducible is an effective tool for studying systems of polynomial equations. In Section 5, we formulate and prove a criterion for irreducible polynomials of several variables.

4. Bezout theorem on the number of solutions of a polynomial system of equations

This section contains several important theorems. One them - Bezout theorem on the number of solutions of polynomial equations of the system, see for example [11]. From this theorem follows the basic theorem of algebra. The wording of some theorems, we present only for special cases. This allows us to give them simple proofs.

We introduce the necessary notation.

Let $f = f(x, y)$ be a polynomial in two variables over a field K, $n = \deg_x f(x, y)$ - his degree in the variable x, $m = \deg_y f(x, y)$ - the degree of the variable y and $d = \deg f(x, y)$ - the degree of the set of variables.

For our purposes, as the K field, unless otherwise stated, we choose the field of complex numbers C. A $C[x]$ will denote the field of rational functions of x with coefficients in C.

Let $f = f(x, y)$ and $g = g(x, y)$ - polynomials two variables over the field of complex numbers C, and let the polynomials are defined as follows:

$$\cancel{f} = f(x, y) = \sum_{i,j=0}^{i,j=n;\ i+j \le n} a_{ij} x^i \quad {}^j, \tag{4.1}$$

$$\acute{g} = g(x,y) = \sum_{i,j=0}^{i,j=m;\ i+j\le m} b_{ij} x^i \ \ ^j . \qquad (4.2)$$

If at least one of the senior coefficients, a_{ij}, $i+j=n$, of the polynomial f is not zero, then $\deg f = n$, and if at least one of the senior coefficients, b_{ij}, $i+j=m$, of the polynomial g It is not zero, then $\deg g = m$.

Convenient to describe the theory is notion of eliminate polynomials.

Definition 4.1. Let $f(x,y), g(x,y) \in C[x,y]$ - of two polynomials variables over the field of complex numbers C. The Eliminante of polynomials $f(x,y)$ and $g(x,y)$ is called a polynomial $X(x)$, the variable x, is defined by the equation:

$$X(x) = R_y\left(f(x,y), g(x,y)\right), \qquad (4.3)(4.3)$$

where $R_y\left(f(x,y), g(x,y)\right)$ - resultant polynomial $f(x,y)$ and $g(x,y)$ in the variable y.

Similarly, determined the second eliminante for polynomials of the system $Y(y)$, i.e.:

$$Y(y) = R_x\left(f(x,y), g(x,y)\right). \qquad (4.4)$$

Following [19] the sign $(-1)^{\frac{n(n-1)}{2}}$ in determining both eliminates will be ignored.

Also note that eliminantes is polynomials, and are defined up to a multiplicative constant from the field of coefficients.

Have the following

Theorem 4.1. (Bezout) Let the polynomials f and g are defined by (4.1) and (4.2), respectively. Let their coefficients such that $a_{0n} \ne 0$, $a_{no} \ne 0$, $b_{0m} \ne 0$, $b_{m0} \ne 0$. Then eliminante $X(x)$, such that its degree $\deg X(x) = \deg f(x,y) ?\deg g(x,y) = nm$.

The proof of the theorem see, e.g., [19].

Similar assertion holds for eliminate $Y(y)$, i.e., $\deg Y(y) = nm$.

In order to state the next theorem we recall the definition of an algebraic curve.

Definition 4.2. An algebraic curve f is the set of points (a coordinate space), the coordinates of which satisfy the equation:

$$f(x,y) = 0. \qquad (4.5)$$

Equation (4.5) is called the equation of the curve f (x, y).

The polynomial in the left side of the equation (4.5) can be considered over the field of real numbers. In this case, the algebraic curve f is, for example, the curve in the affine plane or in a projective space, i.e. the graph of the function f. If the equation (4.5) is considered over the field of complex numbers, the algebraic curve is a one-dimensional complex manifold in the space $\mathbb{C}^2$. This curve also can be regarded as the graph of a function, or as the Riemann surface of an algebraic function given by equation (4.5).

Have the following

Theorem 4.2. (Bezout) Let the curves are determined by the equations $f(x,y)=0$ and $g(x,y)=0$. If they have more than $nmnm$ points in common, they have a common component, i.e. $\deg GCD(f(x,y), g(x,y)) \ne 0$.

The proof of Theorem 4.2 is given in [15].

Corollary of Theorem 4.2. From Theorem 4.2 it follows that if the system of equations

$$\begin{cases} f\left(x,y\right)=0 \\ g\left(x,y\right)=0 \end{cases} \tag{4.6}$$

has more than nm solutions, the polynomials f and g have a common component.

From Theorem 4.2 obviously follows

Theorem 4.3 Let the polynomials ff and gg have no common components i.e. $deg\mathrm{GCD}\left(f,g\right)=0$, then the number of solutions of (4.6) does not exceed nm .

In [19] are examples showing that this bound is attained.

Theorem 4.4. The polynomials f and g have a common component h , positive degree, i.e., $\deg GCD\left(f,g\right)=deg\left(h\right)\neq 0$, if and only if at least one of eliminante $X\left(x\right)=R_y\left(f,g\right)$ or $Y\left(y\right)=R_x\left(f,g\right)$, is identically zero, or that, too, $R_x\left(f,g\right){}^{2}R_y\left(f,g\right)\equiv 0$.

Proof. Necessity. For definiteness, let eliminante $Y\left(y\right)\equiv 0$, then the polynomials f and g , such that, $R_x\left(f,g\right)\equiv 0$. From equation $R_x\left(f,g\right)\equiv 0$ follows: for any fixed $y=y_0$, holds $R\left(f\left(x,y_0\right),g\left(x,y_0\right)\right)=0$, therefore, the polynomials f and g have common roots. But then it, by virtue of the freedom to choose y , polynomials f and g coincide in an infinite number of points and, according to Theorem 4.2., have a common component.

Sufficiency. Let the polynomials f and g have a common component h , positive degree, i.e. $deg_x\left(h\right)\neq 0$, then $f=f_1h$ and $g=g_1h$. Applying Theorem 1.3., we have:

$$R_x\left(f,g\right)=R_x\left(f_1h\ ,g_1h\right)=R_x\left(f_1\ ,g_1h\right){}^{2}R_x\left(h\ ,g_1h\right)=$$
$$=R_x\left(f_1\ ,g_1\right){}^{2}R_x\left(f_1\ ,h\right){}^{2}R_x\left(h\ ,g_1\right){}^{2}R_x\left(h\ ,h\right)\ .$$

Given the identity $R_x\left(h\ ,h\right)\equiv 0$, we have $R_x\left(f,g\right)\equiv 0$ and $Y\left(y\right)\equiv 0$.

The theorem is proved.

Corollary theorem 4.4. If eliminante $X\left(x\right)$ polynomials $f\left(x,y\right),g\left(x,y\right)$, such that $X\left(x\right)\equiv 0$, *then all of its coefficients is zero, i.e. .:*

$$\frac{d^i}{dx^i}X\left(x\right)=0,\ i=1,2,\dots,m. \tag{4.6}$$

where $m=deg\,X\left(x\right)$.

As $X\left(x\right)=R_y\left(f\left(x,y\right),g\left(x,y\right)\right)\equiv 0$, we have the following relations:

$$\frac{d^i}{dx^i}R_y\left(F\left(x,y\right),f\left(x,y\right)\right)=0,\ i=1,2,\dots,m. \tag{4.7}$$

The criterion of irreducibility

5. The irreducibility criterion for polynomials in several variables

This section is formulated and proved criterion of irreducibility for polynomials of several variables in a weakened form (for polynomials with real or complex coefficients). We are unaware of other criteria for polynomials in several variables over a field of characteristic 0.

We give the necessary definitions.

Definition 5.1. Polynomial *n*-form (or *n*-form in the polynomial basis) of the variables $x_l,\ l=1,\dots k$ over a field K is a formal sum $G=\sum\limits_{0\le l_1+l_2+\ldots+l_k\le n} g_{l_1 l_2\ldots l_k} x_1^{l_1}x_2^{l_2}\ldots x_k^{l_k}$, $n\in\mathbb{N}$ i.e. G - a polynomial of degree n in the variables $x_l, l=1,\dots k$ with coefficients $g_{l_1 l_2\ldots l_k}$ of field K .

In particular, the *1*-form of variable $x_l, l=1,\dots k$ over a field K is a linear form. *2*-form on - quadratic form. The polynomial $f(x,y)$ in two variables x and y, the degree n, with complex coefficients is the n-shape of the variables x and y over the field of complex numbers ℂ. The expression "function will be sought in the form of n-form" is generally understood as a procedure for determining the undetermined coefficients given *n*-form

In the proof of the following criteria will be used

Theorem 5.1. Polynomial in several variables over a field K is identically equal to zero, if and only if all its coefficients are zero, see. [20].

Occurs

Theorem 5.2. Let $F=F(x,y)$ a polynomial in the variables x and y over the field of complex numbers ℂ and $d=\deg F(x,y)$ - its extent, let $n=\left[\frac{d}{2}\right]$ (in brackets is the integer part the number of $\frac{d}{2}$), and let G - n -form in the variables x and y over the field of complex numbers ℂ, i.e.

$$G(x,y)=\sum_{i,j=0}^{0\le i+j\le n} g_{ij}x^i y^j$$

The polynomial F is decomposable the variable of x , if and only if, the system equations

$$\begin{cases} R_x\left(F(x,y),G(x,y)\right)_{|_{y=0}}=0 \\ \dfrac{d^i}{dy^i}R_x\left(F(x,y),G(x,y)\right)_{|_{y=0}}=0 \end{cases},\quad i=1,2,\dots,m;\ \text{where}\ \ m=degR_x(F,G)\ , \qquad (5.1)$$

(The system (5.1) is solved with respect to indeterminate coefficients g_{ij} *n*-form G, as a relatively unknowns) have not zero solution.

Proof. Necessity. Let the polynomial F is decomposable the variable x .

Then F can be expressed as $F=f_1{}^{\mathbf{2}}f_2$, were $f_1=f_1(x,y)$ and $f_2=f_2(x,y)$ such that $deg_y f_1\ne 0$, $deg_y f_1\ne 0$, and $deg_y f_1+deg_y f_2=deg_y F$. From the condition of $degF=d$ implies that $min(degf_1,degf_2)\le\left[\frac{d}{2}\right]$. Let $degf_1\le degf_2$, then $degf_1\le\left[\frac{d}{2}\right]=n$. Let $degf_1=m_1$, then f_1 polynomial can be written as

$$f_1=f_1(x,y)=\sum_{i,j=0}^{0\le i+j\le m_1} c_{ij}x^i y^j,$$

where not all c_{ij} for $1\le i+j\le m_1$ are equal zero.

We denote by

$$b_{ij} = \begin{cases} c_{ij}, & 0 \le i+j \le m_1 \\ 0, & m_1 < i+j \le n \end{cases}, \qquad (5.2)$$

and we show that

$$g_{ij} = b_{ij}, \qquad (5.3)$$

the non-zero solution of system (5.1).

Let $G \equiv f_1$. The polynomials F and f_1, as follows from condition of the theorem, have a common component f_1 and, according to Theorem 4.4 of the resultant $R_x(F, f_1) \equiv 0$. Consequently, the resultant $R_x(F,\ G) \equiv 0$, and then, in accordance with Theorem 5.1, all the coefficients are zero simultaneously. Consequently, there is a system of equations for c_{ij}, and, consequently, relatively, b_{ij}. Thus, the ratio (5.3) is a solution of (5.1).

The resulting solution is not zero, as b_{ij}, that certain system of equations (5.2), are not all zero. The necessity is proved.

Sufficiency. We prove first that (5.1) has a zero solution. Indeed, G is not only the form of n-variables x and y, but also linear form for its coefficients. Consequently, we can write: $G(x,y) = G(x,y,g_{ij})$. Because relation holds: $G(x,y,tg_{ij}) = tG(x,y,g_{ij})$, *n*-form *G* is a homogeneous function of its coefficients g_{ij}, and the resultant

$$R_x\left(F(x,y), G(x,y)\right) = R_x\left(F(x,y), G(x,y,g_{ij})\right),$$

is a homogeneous function of the variables $g_{ij} g_{ij}$ degree $d = degFd = degF$. Indeed:

$$R_x\left(F(x,y), G(x,y,tg_{ij})\right) = t^d R_x\left(F(x,y), G(x,y,g_{ij})\right).$$

But then, all the coefficients of resultant, as the polynomial of variable yy, also are homogeneous functions of the variables g_{ij} degree d, and $d \ne 0$. Thus, all of the system (5.3) are homogeneous functions of the variables g_{ij} nonzero of degree d. Homogeneous function nonzero of degree d in the variables g_{ij} is zero, if everyone $g_{ij} = 0$. Consequently, the system (5.1) has a zero solution.

Let the system (5.1), except for the zero solution is still nonzero. Each a nonzero solution of system (5.1) completely determine the undetermined coefficients g_{ij} *n*-form G. Because solution is a nonzero, not all the coefficients g_{ij} zero. Let us denote G_1 n-form G with coefficients g_{ij}, as defined, to some, a non-zero solution of the system (5.1).

We will prove that the resultant $R_x(F, G_1) \equiv 0$. Indeed, the resultant $R_x(F, G_1)$ is a polynomial in the variable y. All coefficients of this polynomial is zero, since, by assumption, we have the system of equations (5.1). Consequently, polynomial, he same resultant $R_x(F, G_1)$ is identically zero. From the identical vanishing of the resultant, follows that the polynomials, F, and G_1, in accordance with Theorem 4.4, have a general component for the variable x. Consequently, the polynomial F a decomposable to the variable x.

The theorem is proved.

From Theorem 5.2 follows

Theorem 5.3 (The reducibility criterion). Let $F = F(x,y)$ polynomial in the variables x and y over the field of complex numbers ℂ and $d = \deg F(x,y)$ - its degree, let $n = \left[\frac{d}{2}\right]$ and G is an n-form in the variables x and

y over the field of complex numbers ℂ, i.e.

$$G(x,y)=\sum_{i,j=0}^{0\le i+j\le n} g_{ij}x^i y^j$$

The Polynomial, F is reducible if and only if at least one of the systems of equations

$$\begin{cases} R_x\ \left(F(x,y),G(x,y)\right)_{|_{y=0}} = 0 \\ \dfrac{d^i}{dy^i}R_x\left(F(x,y),G(x,y)\right)_{|_{y=0}} = 0 \end{cases},\quad i=1,2,\dots,m; \tag{5.4}$$

where $m = degR_x(F,G)$,

$$\begin{cases} R_y\ \left(F(x,y),G(x,y)\right)_{|_{x=0}} = 0 \\ \dfrac{d^i}{dx^i}R_y\left(F(x,y),G(x,y)\right)_{|_{x=0}} = 0 \end{cases},\quad i=1,2,\dots,h; \tag{5.5}$$

where $h = degR_x(F,G)$.

(System is considered relatively of indeterminate coefficients g_{ij} *n*-form G, as a relatively unknowns) has a nonzero solution.

It has the assertion, converse to the opposite assertion of the previous theorem

Theorem 5.4 (The irreducibility criterion). Let $F = F(x,y)$ polynomial in the variables x and y over the field of complex numbers ℂ and $d = \deg F(x,y)$ - its degree, let $n = \left[\dfrac{d}{2}\right]$ and G is an n-form in the variables x and y over the field of complex numbers ℂ, i.e.

$$G(x,y)=\sum_{i,j=0}^{0\le i+j\le n} g_{ij}x^i y^j$$

The Polynomial, F is reducible if and only if for each of the two variables $x_l,\ l=1,2$, the systems of equations

$$\left\{ \frac{d^i}{dx_l^i}R_{x_l}\left(F(x,y),G(x,y)\right)=0\ ,\quad i=1,2,\dots,m; \right. \tag{5.6}$$

where $m = degR_{x_l}(F,G)$, (systems is considered relatively of indeterminate coefficients g_{ij} *n*-form G, as a relatively unknowns, and in equations of systems, the variables with index is not equal *l*, we equated to zero) have only the zero solution.

Corollary of Theorem 5.4. To determine the irreducibility of polynomials in two variables it is sufficiently rational operations over a field of its coefficients. If a polynomial in two variables, over the field of complex numbers ℂ reducible, then to determine it's of the irreducible components, of the rational operations, generally speaking, is not enough. To determine its irreducible components is enough to add a method for calculating of the roots for polynomials of degree n in one variable.

From the theorems proved above follows:

Theorem 5.5 (The irreducibility criterion for polynomials in several variables).

Let KK - the field of real or complex numbers, $F = F(x_1,x_2,\dots,x_k) = F(\overline{x})$ - polynomial in k variables $x_1,x_2,\dots,x_k$, with coefficients from K and $d = \deg F(\overline{x})$ - its degree, let $n = \left[\dfrac{d}{2}\right]$ ((in brackets is the integer part the number of $\dfrac{d}{2}$) and let G - *n*-form in k variables $x_1,x_2,\dots,x_k$, i.e.

$$G = G(\overline{x}) = \sum_{0 \le l_1 + l_2 + \ldots + l_k \le n} g_{l_1 l_2 \ldots l_k} x_1^{l_1} x_2^{l_2} \ldots x_k^{l_k} \ , \ n \in \mathbb{N}$$

Let

$$R_t = \sum_{0 \le r_1 + r_2 + \ldots + r_k \le m_t} r_{r_1 r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2}, \ldots x_k^{r_k} \ , \ n \in \mathbb{N},$$

Lexical-graphical representation in increasing powers of the resultant $R_{x_t}\left(F(\overline{x}), G(\overline{x})\right) = R_t$, in the alphabet of variables $x_1, x_2, \ldots, x_{t-1}, x_{t+1}, \ldots, x_k$, and let $m_t = degR_t$ - its degree.

The Polynomial, F is irreducible (in the expansion of $K[x_1, x_2, \ldots, x_k]$ field coefficients), if, and only if, for any of the variables $x_t,\ t = 1, \ldots, k$ system of equations

$$\left\{ r_{r_1 r_2, \ldots, r_k} = 0 \ , \quad i = 1, 2, \ldots, m_t; \quad 1 \le r_1 + r_2 + \ldots + r_k \le m_t, \right. \qquad (5.7)$$

where $m_t = degR_t$ (system is being considered relatively undetermined coefficients $g_{l_1 l_2 \ldots l_k}$ n-form G, as unknowns) has only the zero solution.

In the statement of the theorem, we consider that $k = 2, 3, \ldots$ for $k = 1$ the question of the reducibility of polynomials is solved in accordance with the fundamental theorem of algebra and its consequences.

Corollary of Theorem 5.5. To determine the irreducible polynomial in several variables it is sufficiently rational operations over a field of its coefficients.

The following examples illustrate the criteria.

In the examples, polynomial n - form G is denoted by $\hat{O}(x, y)$.

Example 5.1. Prove irreducible polynomial $F(x, y) = x^2 + y^2 - 1$.

Solution. The degree of the polynomial $F(x, y)$, $deg\left(F(x, y)\right) = deg(x^2 + y^2 - 1) = 2$, the degree of the polynomial $\hat{O}(x, y)$, $\deg\left(\hat{O}(x, y)\right) = \left[\frac{\deg\left(F(x, y)\right)}{2}\right] = \left[\frac{2}{2}\right] = 1$. Dividers polynomial $F(x, y)$ we will be sought in the form of a 1-form $\hat{O}(x, y) = ax + by + c$. We calculate the resultant $\hat{R}_x = R_x\left(F(x, y), \hat{O}(x, y)\right)$:

$$\hat{R}_x = R_x\left(F(x, y), \hat{O}(x, y)\right) = R_x\left(x^2 + y^2 - 1,\ ax + by + c\right) =$$

$$= \det \begin{bmatrix} 1 & 0 & y^2 - 1 \\ a & by + c & 0 \\ 0 & a & by + c \end{bmatrix} = (by + c)^2 + a^2\left(y^2 - 1\right) = \left(a^2 + b^2\right) y^2 + 2bcy - a^2 + c^2 .$$

From the assumption the decomposition of the polynomial $F(x, y)$ according to Theorem 5.4, we have:

$R_x \equiv 0$, and

$$\left(a^2 + b^2\right) y^2 + 2bcy - a^2 + c^2 \equiv 0 . \qquad (5.8)$$

According to Theorem 5.1, all the coefficients of the polynomial on the left side of the equation (5.8) are equal to zero. We have a system of equations:

$$\begin{cases} -\tilde{a}^2 + {}^2 = 0 \\ 2bc = 0 \\ a^2 + b^2 = 0 \end{cases} . \tag{5.9}$$

The system (5.9) decomposes into two systems:

$$\begin{cases} -\tilde{a}^2 + {}^2 = 0 \\ b = 0 \\ a^2 + b^2 = 0 \end{cases} , \qquad a \begin{cases} -\tilde{n}^2 + {}^2 = 0 \\ c = 0 \\ a^2 + b^2 = 0 \end{cases} .$$

Each of the systems has a unique solution: $a = 0, b = 0, c = 0$, and this solution is zero.

Thus, the system (5.9) has only the zero solution. Consequently, according to Theorem 5.2, the polynomial $x^2 + y^2 - 1$ irreducible variable x .

Because of the symmetry of variables in the polynomial $x^2 + y^2 - 1,$ if in the resultant $R_x R_x$ replaced variable, the resultant R_y , the result, obviously, does not change. Is therefore, the polynomial $x^2 + y^2 - 1$ irreducible and other variable and, therefore non-trivial irreducible.

Example 5.2. Prove: the polynomial $F(x, y) = y^3 - 3yx - 2x^2$ is irreducible to the variable y .

Solution. The degree of the polynomial $F(x, y)$, $deg\left(F(x, y)\right) = deg(y^3 - 3yx - 2x^2) = 3$, the degree of the polynomial $\Phi(x,y)\Phi(x,y)$, $\hat{d}eg\left(\ y(\ ,\ )\right) = \left[\dfrac{\deg\left(F(x,y)\right)}{2}\right] = \left[\dfrac{3}{2}\right] = 1$. Dividers polynomial $F(x,y)$ $F(x,y)$ we will be sought in the form $\hat{O}(x, y) = ax + by + c$.

We calculate $\hat{R}_y = R_y\left(F(x, y),\ (\ ,\ )\right)$, and have:

$$R_y = R_y(y^3 - 3yx - 2x^2, \alpha x + \beta y + \gamma) =$$

$$= \begin{vmatrix} 1 & 0 & -3x & -2x^2 \\ \alpha & \beta x + \gamma & 0 & 0 \\ 0 & \alpha & \beta x + \gamma & 0 \\ 0 & 0 & \alpha & \beta x + \gamma \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3x & -2x^2 \\ 0 & \beta x + \gamma & 3x\alpha & -2x^2\alpha \\ 0 & \alpha & \beta x + \gamma & 0 \\ 0 & 0 & \alpha & \beta x + \gamma \end{vmatrix} =$$

$$= \left(\beta x + \gamma\right)^3 - 2x^2\alpha^3 - 3x\alpha^2\left(\beta x + \gamma\right) =$$

$$= \beta^3 x^3 + \left(3\beta^2\gamma - 2\alpha^3 - 3\alpha^2\beta\right)x^2 + \left(3\beta^2\gamma - 3\alpha^2\gamma\right) + \gamma^3 ,$$

and then we have:

$$\beta^3 x^3 + \left(3\beta^2\gamma - 2\alpha^3 - 3\alpha^2\beta\right)x^2 + \left(3\beta^2\gamma - 3\alpha^2\gamma\right) + \gamma^3 \equiv 0 . \tag{5.10}$$

Equating the coefficients of equation (5.10) to zero, we have a system of equations to determine the α , β , γ .

$$\begin{cases} \beta^3 = 0 \\ 3\beta^2\gamma - 2\alpha^3 - 3\alpha^2\beta = 0 \\ 3\beta^2\gamma - 3\alpha^2\gamma = 0 \\ \gamma^3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases} \tag{5.11}$$

Thus, the system (5.11) has only the zero solution, and therefore a polynomial $y^3-3yx-2x^2$ is irreducible to the variable y .

Example 5.3. Explore: will be whether the polynomial irreducible?

$$F(x,y)=x^3-x^2+(y^2-1)x-y^2+1.$$

Solution. The degree of the polynomial $F(x,y)$,

$$deg\left(F(x,y)\right)=deg\left(x^3-x^2+(y^2-1)x-y^2+1\right)=3 ,$$

the degree of the polynomial $\hat{O}(x,y)$, $deg\left(\hat{O}(x,y)\right)=\left[\frac{\deg\left(F(x,y)\right)}{2}\right]=\left[\frac{3}{2}\right]=1$. Dividers polynomial $F(x,y)$ we will be sought in the form $\hat{O}(x,y)=ax+by+c$.

We calculate the resultant $R_x=R_x\left(F(x,y),\hat{O}(x,y)\right)$.

$$R_x=R_x\left(F(x,y),\hat{O}(x,y)\right)=R_x\left(x^3-x^2+(y^2-1)x-y^2+1,\ ax+by+c\right)=$$

$$=det\begin{bmatrix}1 & -1 & y^2-1 & -y^2+1\\ a & by+c & 0 & 0\\ 0 & a & by+c & 0\\ 0 & 0 & a & by+c\end{bmatrix}=$$

$$=det\begin{bmatrix}by+c & 0 & 0\\ a & by+c & 0\\ 0 & a & by+c\end{bmatrix}-a\,det\begin{bmatrix}-1 & y^2-1 & -y^2+1\\ a & by+c & 0\\ 0 & a & by+c\end{bmatrix}=$$

$$=(by+c)^3+a\left((by+c)^2+a^2(y^2-1)+a(by+c)(y^2-1)\right)=$$

$$=y^3\left(a^2b+b^3\right)+y^2\left(a^3+a^2c+ab^2+3bc^2\right)+y\left(-a^2b+2abc+3bc^2\right)+$$

$$+\left(-a^3-a^2c+ac^2+c^3\right).$$

From the assumption reducibility of a polynomial $F(x,y)$ according to Theorem 5.2, we have:

$R_x\equiv 0$, and

$$y^3\left(a^2b+b^3\right)+y^2\left(a^3+a^2c+ab^2+3bc^2\right)+y\left(-a^2b+2abc+3bc^2\right)+$$

$$+\left(-a^3-a^2c+ac^2+c^3\right)\equiv 0 \qquad (5.12)$$

According to Theorem 5.1, all the coefficients of the polynomial on the left side of the equation (5.12) equal to zero. We have a system of equations:

$$\begin{cases}-a^3-a^2c+ac^2+c^3=0\\ -a^2b+2abc+3bc^2=0\\ a^3+a^2c+ab^2+3bc^2=0\\ a^2b+b^3=0\end{cases}, \qquad (5.13)$$

and transforming the system (5.13), have:

$$\begin{cases} \left(a^2-c^2\right)\left(a+c\right) = 0 \ = 0 \\ b\left(-a^2+2ac+3c^2\right) \quad = 0 \\ a^3+a^2c+ab^2+3bc^2 = 0 \\ b\left(a^2+b^2\right) = 0 \end{cases}.$$

The system decomposes into eight sub-systems:

$$\begin{cases} a=c \\ b \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases}, \quad \begin{cases} a=c \\ b \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ a^2+b^2=0 \end{cases},$$

$$\begin{cases} a=c \\ -a^2+2ac+3c^2 \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases}, \quad \begin{cases} a=c \\ -a^2+2ac+3c^2 \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ a^2+b^2=0 \end{cases},$$

$$\begin{cases} a=-c \\ b \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases}, \quad \begin{cases} a=-c \\ b \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ a^2+b^2=0 \end{cases},$$

$$\begin{cases} a=-c \\ -a^2+2ac+3c^2 \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases}, \quad \begin{cases} a=-c \\ -a^2+2ac+3c^2 \ = 0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ a^2+b^2=0 \end{cases},$$

For the first, we have:

$$\begin{cases} a=c \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=c \\ a^3+a^2c=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=c \\ a^2\left(a+c\right)=0 \\ b=0 \end{cases} \Rightarrow$$

The system decomposes into two subsystems, deciding that we have:

$$\begin{cases} a=c \\ a^2=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

$$\begin{cases} a=c \\ a+c=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=c \\ a=-c \\ b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

System has only the zero solution.
For the system:

$$\begin{cases} a=c \\ -a^2+2ac+3c^2 \quad =0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} \begin{cases} a=c \\ -a^2+2ac+3c^2=0 \\ a^3+a^2c=0 \end{cases} \\ b=0 \end{cases} \Rightarrow \begin{cases} \begin{cases} a=c \\ 4a^2=0 \\ 2a^3=0 \end{cases} \\ b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}.$$

System has only the zero solution.
For the system:

$$\begin{cases} a=-c \\ b \ =0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=-c \\ b=0 \\ a^2\left(a+c\right)=0 \end{cases}$$

The system decomposes into two subsystems:

$$\begin{cases} a=-c \\ b=0 \\ a^2=0 \end{cases} \quad ; \quad \begin{cases} a=-c \\ b=0 \\ a+c=0 \end{cases}.$$

System have a solution: $a=-c,\ b=0$.

Solution of the system corresponds to the divisor $\hat{O}\left(x,y\right)=a\left(x-1\right)$, где $a\neq 0$.

For the system:

$$\begin{cases} a=-c \\ -a^2+2ac+3c^2 \quad =0 \\ a^3+a^2c+ab^2+3bc^2=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=-c \\ -a^2+2ac+3c^2 \quad =0 \\ a^3+a^2c=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=-c \\ b=0 \end{cases},$$

we have this same solution.

Similarly, we find solutions to other systems. They have only the trivial solution.

Finally, we have one nonzero solution: $a=-c, b=0$, which determines the divisor $\left(x-1\right)$.

Thus polynomial $F\left(x,y\right)$ is decomposable:

$x^3-x^2+(y^2-1)x-y^2+1=\left(x-1\right)\left(x^2+y^2-1\right)$, and $F\left(x,y\right)=\left(x-1\right)\left(x^2+y^2-1\right)$

This example illustrates the case: if the divisor (even trivial) includes a variable, then on it the polynomial is reducible. In this case, by the resultant of the variable produces a system of equations this is non-zero solution.

Example 5.4. Explore: will be whether the polynomial irreducible to the variable y ?

$$F\left(x,y\right)=x^3-x^2+(y^2-1)x-y^2+1 \ .$$

Solution. The degree of the polynomial F (x, y)

$$deg\left(F\left(x,y\right)\right)=\deg\left(x^3-x^2+(y^2-1\right)x-y^2+1)=3 \ ,$$

the degree of the polynomial $\hat{O}\left(x,y\right)$, $\deg\left(\hat{O}\left(x,y\right)\right)=\left[\dfrac{\deg\left(F\left(x,y\right)\right)}{2}\right]=\left[\dfrac{3}{2}\right]=1$. Dividers polynomial

$F(x,y)$ we will be sought in the form $\hat{O}(x,y) = ax + by + c$.

We calculate the resultant $R_x = R_x\left(F(x,y), \Phi(x,y)\right)$.

$$R_y = R_y\left(F(x,y), \Phi(x,y)\right) = R_y\left((x-1)y^2 + (x-1)(x^2-1),\ ax + by + c\right) =$$

$$= det\begin{bmatrix} (x-1) & 0 & (x-1)(x^2-1) \\ b & ax+c & 0 \\ 0 & b & ax+c \end{bmatrix} =$$

$$= (x-1)\left((ax+c)^2 + b^2(x^2-1)\right) = (x-1)\left(\left(a^2+b^2\right)x^2 + 2acx - b^2\right) = 0$$

We have: $\left(a^2+b^2\right)x^2 + 2acx - b^2 = 0$ where we have a system of equations:

$$\begin{cases} -b^2 = 0 \\ 2ac = 0 \\ a^2 + b^2 = 0 \end{cases} \Rightarrow \begin{cases} b^2 = 0 \\ 2ac = 0 \\ a^2 = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}.$$

The system has only the zero solution, so the polynomial $F(x,y)$ is irreducible variable y .

Example 5.5. Investigated for an irreducible polynomial

$$F(x,y) = x^2y^2 - 2x^2 - y^2 + 2$$

with respect to x .

Solution. The degree of the polynomial $F(x,y)$,

$$deg\left(F(x,y)\right) = deg_x\left(F(x,y)\right) = \deg_x\left((y^2-2)x^2 - (y^2-2)\right) = 2 \ ,$$

the degree of the polynomial $\hat{O}(x,y)$, $\deg \Phi(x,y) = \left[\dfrac{\deg\left(F(x,y)\right)}{2}\right] = \left[\dfrac{2}{2}\right] = 1$. Dividers polynomial $F(x,y)$ we will be sought in the form $\hat{O}(x,y) = ax + by + c$.

We calculate the resultant $R_x = R_x\left(F(x,y), \Phi(x,y)\right)$:

$$R_x = R_x\left(F(x,y), \Phi(x,y)\right) = R_x\left((y^2-2)x^2 - (y^2-2),\ ax + by + c\right) =$$

$$= (y^2-2)\left(b^2y^2 - 2bcy + c^2 - a^2\right) =$$

$$= b^2y^4 - 2bcy^3 + (c^2 - a^2)y^2 - 2b^2y^2 + 4by - 2c^2 + 2a^2 =$$

$$b^2y^4 - 2bcy^3 + (c^2 - 2b^2 - a^2)y^2 + 4by - 2c^2 + 2a^2 \equiv 0$$

According to Theorem 5.1, all the coefficients of the polynomial on the left side of the equation equal to zero. We have a system of equations:

$$\begin{cases} b^2 = 0 \\ -2bc = 0 \\ c^2 - 2b^2 - a^2 = 0 \\ 4b = 0 \\ -2c^2 + 2a^2 \end{cases} \Rightarrow \begin{cases} b = 0 \\ bc = 0 \\ c^2 - 2b^2 - a^2 = 0 \\ c^2 - a^2 = 0 \end{cases} \Rightarrow \begin{cases} b = 0 \\ c^2 - a^2 = 0 \end{cases} \Rightarrow \begin{cases} b = 0 \\ c = \pm a \end{cases}.$$

The system has two non-zero solutions $(a,0,a)$ and $(a,0,-a)$, which correspond to the divisors

$\hat{O}_1(x,y) = ax + a = a(x+1)$ and $\hat{O}_2(x,y) = ax - a = a(x-1)$.

Is therefore, the polynomial F (x, y) reducible with respect to x, and

$$F(x,y) = (x-1)(x+1)(y^2-2).$$

6. General scheme of solution of systems of polynomial equations

When solving systems of polynomial equations (in particular, the equations of N-point gravitational lenses) we believe the best the following algorithm:

1. Check each equation of the system, whether a it is reducible. If it is reducible, the system decomposes into subsystems, each of which we are solving separate.

2. Select one of the equations and one unknown therein. Exclude this unknown of other equations, if possible. If you cannot perform a simple substitution, then eliminate the unknown using the resultant. Exclude selected variable from the other equations, if it is there.

Go to the new system of equations, which will consist of selected equations and calculated resultants, equal to zero. This system of equations is equivalent to the initial one.

However, if the system of two equations with two unknowns to a system of equations of the two resultants is equal to zero, then this transition, generally speaking, is not equivalent. The resulting solution in such a transition cannot be solutions of the original system of equations, see. E.g. [19].

4. The built in system of claim 3, consider a subsystem consisting of resultants, equal to zero. The subsystem will contain at least one equation and one less unknown. None of the resultants cannot be identically zero (except in the case when the equations are the same), because each polynomial is irreducible on the right side of equations. In the case that one of the matching equations remove from the system. The subsystem will contain less at least one equation and one unknown.

To this subsystem apply the process from 1-4.

5. After a finite number of steps we obtain the resultants to be polynomials in one variable. Imagine every one of them as a product of polynomials with multiple roots. Each of these polynomials we associate a polynomial with non-multiple roots, which is equal to the original roots of the polynomial

6. Thus, it remains to find the roots of the polynomial on the condition that they are all different. Note that up to this point, we only had a finite number of rational operations. Therefore, before this stage of the algorithm can be considered as constructive and analytical. Such an algorithm can be implemented using a symbolic programming. We can use the packages of general purpose applications such as REDUCE, MACSYMA, MATHEMATICA, MAPLE, AXIOM, MuPAD, algorithmic basis of which are operations on polynomials and rational functions.

7. The Calculation of the roots of a polynomial in one variable is a standard procedure. It can be performed with any precision. Note that the final procedure of the algorithm is, the computation of roots of polynomials. Therefore, the accuracy of calculation of the roots of one polynomial will not affect the accuracy of calculating the other.

Below are a few examples of solutions of systems of polynomial equations.

Example 6.1. Solve the system of equations

$$\begin{cases} f(x,y) = 4x^2 - 7xy + y^2 + 13x - 2y - 3 = 0 \\ g(x,y) = 9x^2 - 14xy + y^2 + 28x - 4y - 5 = 0 \end{cases}, \tag{6.1}$$

Solution. Compose the elimination$X(x) = R_y(f(x,y\ ), g(x,y))X(x) = R_y(f(x,y\ ), g(x,y))$:

$$f(x,y) = y^2 + (-7x-2)y + (4x^2+13x-3) = 0 \ ,$$

$$g(x,y) = y^2 + (-14x-4) + (9x^2+28x-5) = 0 \ ,$$

$$X(x)=\begin{vmatrix} 1 & -7x-2 & 4x^2+13x-3 & 0 \\ 0 & 1 & -7x-2 & 4x^2+13x-3 \\ 0 & 1 & -14x-4 & 9x^2+28x-5 \\ 1 & -14x-4 & 9x^2+28x-5 & 0 \end{vmatrix}=24\left(x^4-x^3-4x^2+4x\right).$$

We find eliminante roots $X(x)$. We have: $x_1=0$, $x_2=1$, $x_3=2$, $x_4=-2$.

Each found the root of the substitute in (6.1). For $x_1=0$, we have:

$$\begin{cases} f(x,y)=y^2-2y-3=0 \\ g(x,y)=y^2-4y-5=0 \end{cases} \tag{6.2}$$

The equations (6.2) have a common root $y_1=0$.

Similarly, substituting the other found in the roots of the system (6.1). We have the solution of the system: ($1,2);(2,3);(0,-1);(-2,1)$. By the theorem of Bezout (the number of solutions of polynomial equations) solutions must be four as $degf(x,y)=2$ and $degg(x,y)=2$. Consequently, we find all solutions.

Example 6.2. Solve the system of equations

$$\begin{cases} f(x,y)=x^2-2xy+y^2-1=0 \\ g(x,y)=x^2-y^2+2x+1=0 \end{cases}. \tag{6.3}$$

Solution. Both eliminanty $X(x)\equiv 0$, $Y(y)\equiv 0$, hence the equations system (6.3) have a common component. Really:

$$\begin{cases} x^2-2xy+y^2-1=0 \\ x^2-y^2+2x+1=0 \end{cases} \Rightarrow \begin{cases} (x-y)^2-1=0 \\ (x+1)^2-y^2=0 \end{cases} \Rightarrow \begin{cases} (x-y+1)(x-y-1)=0 \\ (x-y+1)(x+y+1)=0 \end{cases},$$

and the system (6.3) decomposes into four systems::

$$\begin{cases} x-y+1=0 \\ x-y+1=0 \end{cases}, \begin{cases} x-y+1=0 \\ x+y+1=0 \end{cases}, \begin{cases} x-y-1=0 \\ x-y+1=0 \end{cases}, \begin{cases} x-y-1=0 \\ x+y+1=0 \end{cases}.$$

The first system has an infinite number of solutions: $(\alpha,\alpha+1)$ for any $\alpha\in\mathbb{C}$. The second - a unique solution: (-1.0), which satisfies the first system. The third system solution in affine coordinates is not. A fourth system has a unique solution (0, -1).

Note that instead of four systems can be considered one - fourth and one equation $x-y+1=0$. In both cases, the set of solutions (in affine coordinates) obviously coincides.

The geometric meaning of the set of solutions (in affine coordinates) the following: algebraic curves $f(x,y)=0$ and $g(x,y)=0$ have a common branch – direct $y=x+1$ and additionally point (0, -1)

In [19] there is an interesting example in which there is a "false" solution.

Example 6.3 Solve the system of equations:

$$\begin{cases} f(x,y)=xy-1=0 \\ g(x,y)=x^2y+x-2=0 \end{cases}, \tag{6.4}$$

Solution. Each of the system eliminant

$$Y(y)=-2y(y-1),\quad Y(x)=2x(x-1)$$

It has "extra" root, the result generated by the "false" solution $(0,0)$.

There is given the following explanation:

«The reason for the effect is the same as in the previous example (in the previous example: In the explanation of the fact of the appearance of "extra" roots eliminants draw attention to the fact that at x = 0 degree polynomials f and g are reduced, and this effect is manifested in the construction eliminant as a determinant of the matrix».

And later in [19] it is proposed: to monitor such cases it is necessary to check the suspicious values of the variables, i.e., those that reduce the extent of the original equations.

From our point of view a simple reason for the emergence of "false" solution is in technology and computing is clear: no system of equations is equivalent to the system eliminant are equal to zero.

To find out of deeper reasons we pay attention to the number of solutions of (6.4). By Theorem Bezout them, counting multiplicities, must be greater than one (the system is not linear), and we have only one $x=1$, $y=1$.

It is natural to ask: where are the other solutions?

Some solutions may not belong to $\mathbb{C}^2$ and should be sought in the space $\overline{\mathbb{C}}^2$. Such decisions are easy to find if we use a fractional-linear transformation of variables, such as the inversion of one of the variables: $y=\frac{1}{t}$. After the system transformation (6.4) we have the system of equations:

$$\begin{cases} x-t=0 \\ x^2+xt-2t=0 \end{cases}.$$

Where we have solutions for Bezout's theorem, there must be two solutions, and these solutions $x=0$, $t=0$ and $x=1$, $t=1$. By making the inverse transformation, we obtain the solution: $x=0, y=\infty$. Similarly, we obtain another solution of the system (6.4): $x=\infty, y=0$. Thus, the system of equations (6.4) has a $\overline{\mathbb{C}}^2$, at least three different solutions.

The solution (0.0) is the solution of system of equations consisting of eliminant equal to zero. Each of eliminant, is a polynomial in one variable. Decision of such a system will have a direct multiplication of sets of their roots. In this set included (0,0), and one zero belongs to the improper initial decision system, and the second the other.

Note also that the system of equations (6.3) (. See example 6.2) has another solution $\overline{\mathbb{C}}^2$, namely:

$$x=\infty, y=\infty .$$

Research the solutions of (6.4) and (6.3), can be carried out differently, for example, if you go to the homogeneous coordinates, but the transition to the homogeneous coordinates is beyond the scope of this paper.

7. Stakes and challenges. Solved and set

The above theorems and algorithms allow to solve a number of problems in the theory of N-point gravitational lenses, namely,

- Finding the source of the images in the plane gravitational lens (the problem is reduced to finding the solutions of the lens equation - the variety is an algebraic set), see [10].
- Finding the of extended images in the source plane (the problem is reduced to finding the one-dimensional submanifolds of a manifold is an algebraic set);
- The distribution of images in plane gravitational lens(the problem reduces to the problem of the distribution of the roots of a polynomial in one variable);
- Calculation of critical curves and their research;
- Computation of caustics and their research;
- Calculation of the light curves and their research;
- Study of the lens equation, including the study of its fixed points, multiple points, and other local features;
- The study of harmonic component of the lens equation and its complexification;
-Calculating the type multiplicity lens equations of (the problem is reduced to the special case the solved problem, see [17]);
- Classification of the point of gravitational lenses on the basis of features of the lens equation (arithmetic classification).

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