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Multi-fractal analysis of the gravitational waves

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Keywords: gravitational wave signal; fractal analysis; multi-fractal analysis; non-linear paradigm.

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Introduction

Undoubtedly, one of the biggest discoveries of the XXI century is the gravitational wave detection. On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational Wave Observatory (LIGO) placed in the United States simultaneously observed a transient gravitational wave signal [1]. A century after the fundamental predictions of Einstein [2, 3] and Schwarzschild [4], the first direct detection of gravitational waves and the first direct observation of a black hole system merging to form a single black hole were reported [1]. In our opinion, by value this discovery can be compared with well-known Hertz's experimental confirmation of the Maxwell's electromagnetic wave existence prediction only. Other hand, the black hole merger discussed is very strong-field and powerful, unique ultra-wideband process [5]. According the so called non-linear paradigm [6], been formulated by one of the authors of this paper in the last 1980th, all processes in open, non-linear, dynamical systems are very complex, non-linear, ultra-wideband, fractal ones. The black hole merger system is one of them. Therefore, it seems to be interesting, actual and useful to check the fractal property existence for the experimental gravitational wave signals, obtained by LIGO [1].

The purpose of the paper is to investigate the fractal and multi-fractal properties of the gravitational wave signals with usage of modern fractal and multi-fractal analysis methods.

Fractal Definition and Fractal Classification

The term 'fractal' (from the Latin 'fractus', meaning 'broken') has been proposed by American mathematician Benoit Mandelbrot in 1975 [7]. Mandelbrot defined a fractal to be a set with Hausdorff dimension strictly greater than its topological dimension [7].

Nevertheless, now there are many different definitions of the fractal introduced in the last forty years by different researchers (see, for example, [8 – 11]), as well as the fractal concept developed rapidly in these years. But on our opinion, the most adequate of them is following one, proposed by K. J. Falconer in 1990 [8]. According to this, when we refer to a set F as a fractal, we will typically have the following in mind.

1. F has a fine structure, i. e. detail on arbitrarily small

scales.

2. F is too irregular to be described in traditional geometrical language, both locally and globally.

3. Often F has some form of self-similarity, perhaps approximate or statistical.

4. Usually, the 'fractal dimension' of F (defined in some way) is greater than its topological dimension D_T.

5. In most cases of interest F is defined in a very simple way, perhaps recursively.

Some later, a self-similarity requirement was generalized and replaced by the self-affinity one [12].

The fractals can be classified in different ways. One hand, all fractals can be separated on mathematical and real, or physical, ones. First of them are a mathematical idealization only, and second of them are really existing natural objects, such as, for example, trees, heaven, mounts et al. The ways of describing of these two fractals types are slightly different [13].

All mathematical fractals can be separated on the deterministic (algebraic and geometric) fractals and stochastic fractals. The properties of self-similarity and self-affinity for the stochastic fractals are considered not in a literal sense, but in a statistical one. It means that fractal properties are shown not by the stochastic object as such, but by its deterministic numerical characteristics [8 – 13].

General difference between mathematical and physical fractals is in following. Strictly speaking, the physical fractals don't satisfy the fractal definition listed above as well as the minimal scale of mathematical fractals vanishes by the definition, but the minimal scale of physical fractal has a finite, non-zero limit. As the result, the self-affine property of physical fractal exists in limited range of scales only [8 – 13]. This is another real difference between the true world (physical fractals) and one of its models (mathematical fractals). Nevertheless, the physical fractals can be divided into deterministic and stochastic ones deter too, as well as their non-stochastic numerical characteristics have namely such properties.

Other hand, there are so called mono-fractals and multi-fractals. To describe a mathematical mono-fractal, it is enough to use the Hausdorff dimension D_H as a fractal dimension D_F [7 – 9]. It is important to point, that only one value of this dimension is able to characterize a mono-fractal as the self-similar (or self-affine) structure. At the same time, to describe a physical mono-fractal, instead of the Hausdorff dimension D_H the Minkowsky dimension D_M is usually applied [14]. All existing algorithms allowing to estimate a fractal dimension D_F of the object investigated, in particular, as a mathematical, as a physical fractal, are included in so called fractal analysis, which can be named more accurately as the mono-fractal analysis too.

Multi-fractal is a fractal, which is not principally allowed to be described with usage of one value of a fractal dimension D_F only. To do this, it is necessary to use a set of fractal dimension values. Such approach is well known as the multi-fractal analysis [15, 16].

Being as natural, as artificial origin, many real signals and processes in nature have fractal properties and, therefore, are the physical fractals [7 – 16]. Using fractal and multi-fractal analyses, these properties having statistical sense can be investigated. The gravitational wave signals listen above are real signals, which require to solve the problem: whether these signals are mono-fractal or multi-fractal or not. If they are, it is necessary to estimate their characteristics. Namely these questions will be answered below.

Fractal Analysis Method

To apply the fractal analysis (more precisely, the mono-fractal analysis) to a real physical signal (or process) investigation, it is necessary [10]:

1) to identify the presence of the self-affinity (or self-similarity) properties of this signal;

2) if they are, to define the scale range (or multiple ranges), in which this happens;

3) using the Minkowski dimension D_M, to estimate the fractal dimension D_F value (or some fractal dimension values, if multiple ranges were found) of the signal investigated.

Oddly enough, but there are many different approximations of the Minkowski dimension D_M, which are usually estimated for the real physical fractal analysis in practice. In particular, there are the cluster dimension D_K [17], the capacity (or box, or fractional) dimension D_C [14], the pointwise dimension D_P and the averaged the pointwise dimension $\left\langle D_P \right\rangle$ [18, 19], the correlation dimension D_G [19], the information dimension D_I [19], the internal (or hidden) dimension D_D [9], the mass dimension D_m [18] and other.

In this paper, we use direct calculation of the capacity dimension D_C and apply another well-known method of the fractal dimension D_F estimation (more precisely, of course, of the Minkowski dimension D_M estimation), which is based on the Hurst exponent H calculation.

Following the Generalized Brownian Motion Model, the Hurst exponent H and fractal dimension D_F are connected with the relation $D_F = 2 - H$ [9].

To estimate the Hurst exponent of the signal $X(t)$, two different ways can be used. Being proposed by H. Hurst in 1951 [20], the first way is the oldest, is known as the Rescaled Range Method or RS-method [9] and is considered as the 'classical' way. The second way is based on the wavelet analysis, namely on the investigation of rate of increasing of mean values of the wavelet coefficient module squares [21].

Then if these dependences obtained in both cases and plotted in the double logarithmic coordinates can be successfully approximated in some scale range with a linear function (for example, with usage of the least square method), the Hurst exponent H in this scale range can obtained. For fractals the Hurst exponent value H should be limited in the range $0 < H < 1$. Otherwise the signal analyzed is appeared to be not self-affine and, therefore, is not fractal [9]. If the condition $0 < H < 1$ was successfully satisfied, then we can believe that the signal investigated has mono-fractal properties in this range. It is quite possible that for the same signal some different scale ranges with different Hurst exponent values will be obtained [9].

Meanwhile, the real physical processes, special being in open, non-linear, dynamical systems [6], are appeared to be non-stationary ones. Moreover, their fractal properties can vary with time too. So, the Hurst exponent H should be estimated for some limited, slide time window $W(t)$, but not for all signal $X(t)$ at once. In this case, the Hurst exponent becomes a function of the time $H = H(t)$ [22]. In our opinion, it is convenient to connect these Hurst exponent values with corresponding time locations of the center of the slide time window $W(t)$ used. Namely such way is applied in this paper.

Some experienced authors (see, for example, [10, 21]) believe, that fractal analysis is closely connected with wavelet analysis, in particular, with continuous wavelet transform (CWT). Therefore, investigations of CWT spectral density function (SDF) of the signal analyzed and of its skeleton are a part of fractal analysis. In our opinion, this is really appeared to be very important and useful addition to usual fractal analysis tools.

Multi-Fractal Analysis Method

If the process analyzed is appeared to be multi-fractal, the abilities of the mono-fractal analysis will be quite insufficient. Of course, the application of mono-fractal analysis to the multi-fractal signal investigations can not be completely forbidden, but the results, which would be obtained, will relate to the so called multi-fractal support only [16]. The multi-fractal support is considered as a mono-fractal, which makes the greatest contribution in the multi-fractal considered [16]. May be, it seems to be interesting for the researcher too, but to describe the multi-fractal much more complete, another approach named as the multi-fractal analysis must be used.

There are two basic multi-fractal analysis methods, which are usually applied to the signal analysis. First of them is called as the Wavelet Transform Module Maxima (WTMM) method and is based on the CWT [21]. Being the basic informational characteristics of the multi-fractal analysis, the multi-fractal spectrum $f(\alpha)$ of the signal investigated is connected with the CWT SDF of the signal. Traditional shapes of the multi-fractal spectrum $f(\alpha)$ of the signal are shown on the fig. 1, where two experiment registrations of the gravitational waves discussed above were presented. The α value is known as the Holder exponent (see, for example, [15, 16]).

Suddenly, WTMM method has one significant disadvantage. It doesn't allow to consider the non-stationarity of the signal investigated as well as in this method the signal is investigated at once. At the same time, it is reasonable to predict that all multi-fractal characteristics of a non-stationary signal can significantly vary with time.

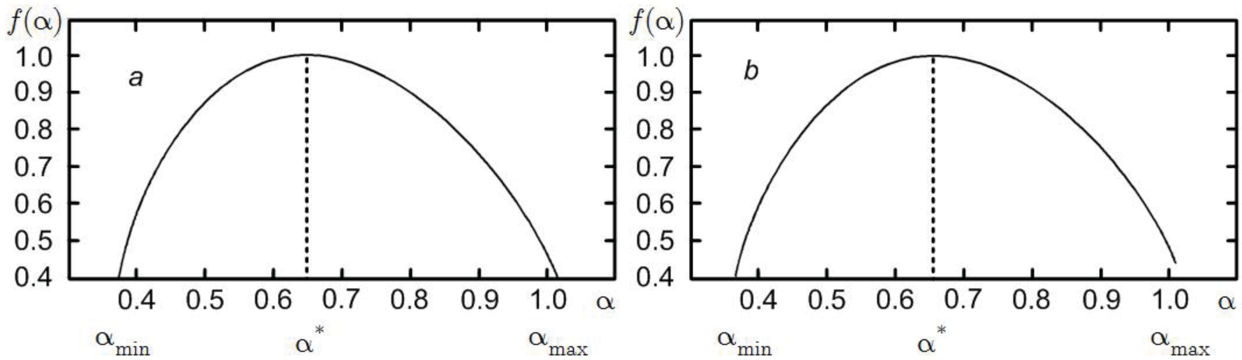

Fig. 1. Multi-fractal spectra $f(\alpha)$ of the gravitational wave signals registered in Hanford (a) and in Livingston (b).

Thus, it is necessary to apply another method, which is free from this disadvantage.

Appearing relatively recently, the second basic multi-fractal analysis method known as the Multi-Fractal Detrended Fluctuation Analysis (MF DFA) is appeared to be convenient to the non-stationary signal investigations in slide time window [23]. As well as the gravitational wave signals are expected to be non-stationary ones, namely MF DFA was chosen in this paper as the main investigation tool.

Let's consider the basic idea of the MF DFA method [23]. Basing on the signal multi-fractal spectrum $F(\alpha)$ analysis (the multi-fractal spectrum of the whole signal was denoted above as $f(\alpha)$) and the slide time window $W(t)$ application, the time dependences of location (minimal $\alpha_{\min}(t)$ and maximal $\alpha_{\max}(t)$ values of α) and of width ($\Delta\alpha(t)$, $\Delta\alpha = \alpha_{\max} - \alpha_{\min}$) of the multi-fractal spectrum can be obtained. Special attention should be paid to the location α^* of the multi-fractal spectrum maximum, given by the requirement $F(\alpha^*) = \max_{\alpha} F(\alpha)$. The α^* value is called as the generalized Hurst exponent as well as for mono-fractal signal we have $\Delta\alpha = 0$ and $\alpha^* = H$. The generalized Hurst exponent α^* describes a multi-fractal support of the signal analyzed. It's fractal dimension is given by relation $D_F = 2 - \alpha^*$ [24].

Analysis Results

Let's start with results of mono-fractal analysis of the gravitational wave signals discussed above. At the fig. 2 these signal registrations obtained in Hanford (fig. 2, a) and in Livingston (fig. 2, c) are shown. One count on the dimensionless time axis corresponds to 21 ms, thus, the whole registration duration is 210 ms. One count on the strain axis is equal to $5 \cdot 10^{-22}$. All calculations described below were performed with usage of the FracLab Toolbox [26] and some original software been developed by authors of this paper.

Capacity dimension D_C of the whole first signal (Hanford) is appeared to be $D_C \approx 1{,}45 \pm 0{,}10$ in range of the dimensionless time $t = 0{,}156 - 5$. The result for second signal (Livingston) is appeared to be $D_C \approx 1{,}44 \pm 0{,}10$ in the same range. Corresponding bounds for Hurst exponent H are $H \approx 0{,}55 \pm 0{,}10$ and $H \approx 0{,}56 \pm 0{,}10$.

Indirect indication on possibility of the fractal property existence for the signals analyzed is given by the CWT skeletons, which have characteristics fork-like looks (fig. 2, b, d) (for CWT SDF calculation the Morlet wavelet was applied). Moreover, on the fig. 2, b, d the fork-like looks of skeletons are excellent seen in the range $T \approx 0{,}1 - 2$, where T is dimensionless period of the signal, which is used in CWT. This results match well with ones obtained during capacity dimension D_C estimation.

Let's consider the results of the multi-fractal analysis. First, WTMM method application should be described. To obtain the CWT SDF of the signals analyzed, the Daubechie's wavelet of fours order (db4) was used. The

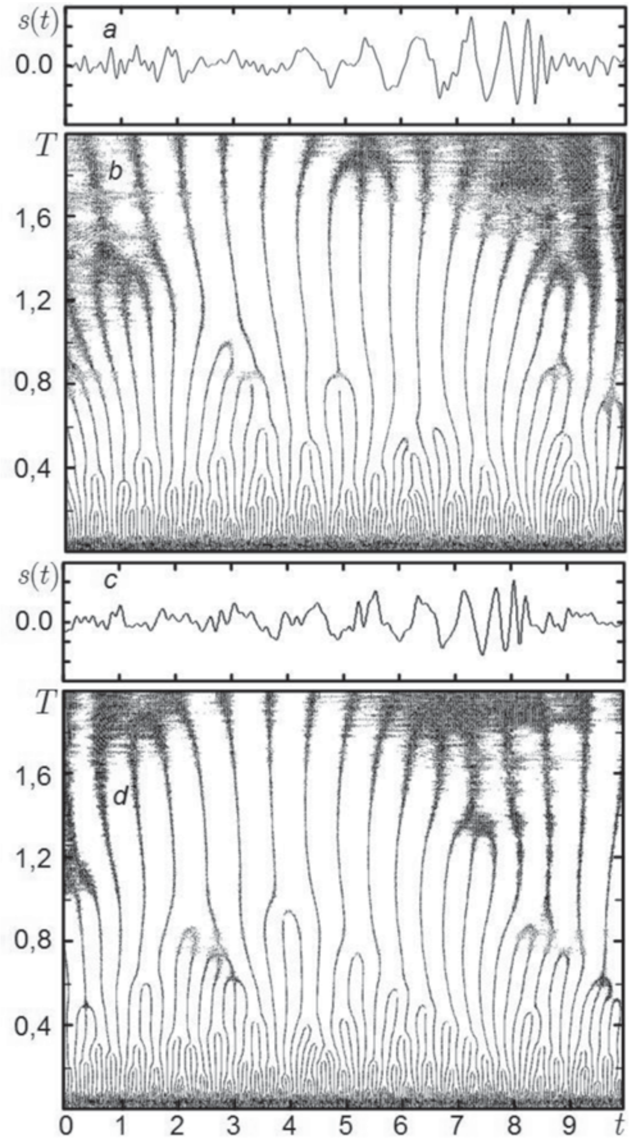

Fig. 2. CWT SDF skeleton analysis results. Signals in time domain: a – Hanford, c – Livingston, CWT skeletons: b – Hanford, d – Livingston.

multi-fractal spectra $f(\alpha)$ of the signal investigated are at the fig. 1. It was found that for the signal registered in Hanford the minimal value of the Holder exponent is $\alpha_{\min} = 0.38$, its maximal value is $\alpha_{\max} = 1.03$, the multi-fractal spectrum width is $\Delta\alpha = 0.65$ and the generalized Hurst exponent is $\alpha^* = 0.65$. For the signal obtained in Livingston we have $\alpha_{\min} = 0.36$, $\alpha_{\max} = 1.01$, $\Delta\alpha = 0.65$ and $\alpha^* = 0.66$ correspondently. These two value sets almost don't differ from each other.

Now let's discuss the results of MF DFA application. All time-dependent values in MF DFA (fig. 3, fig. 4) and additionally the Hurst exponent $H(t)$, which is a part of the fractal analysis method, were calculated with usage of the slide window $W(t)$ with dimensionless width $\Delta t = 3,67$. It is important to point, that each specific value obtained for given window location in time domain was assigned to the position of the window center. The existence of the empty spaces to the right and left of the graphs (fig. 3, b – f, fig. 4, b – f) is explained namely by this reason.

It was found the following. In both cases (as for Hanford, as for Livingston) there are steady tendencies to increase with time for all four multi-fractal functions ($\alpha_{\min}(t)$, $\alpha_{\max}(t)$, $\Delta\alpha(t)$ and $\alpha^*(t)$), which result in a rather sharp decrease. For the Hurst (fig. 3, f, fig. 4, f) exponent there is a weak tendency to increase only. In both cases, the fractality condition for the generalized Hurst exponent ($0 < \alpha^*(t) < 1$) is well satisfied only for $t \le 4,5$. For the Hurst exponent this condition ($0 < H(t) < 1$) is satisfied sometimes in bounds $2 \le t \le 6$.

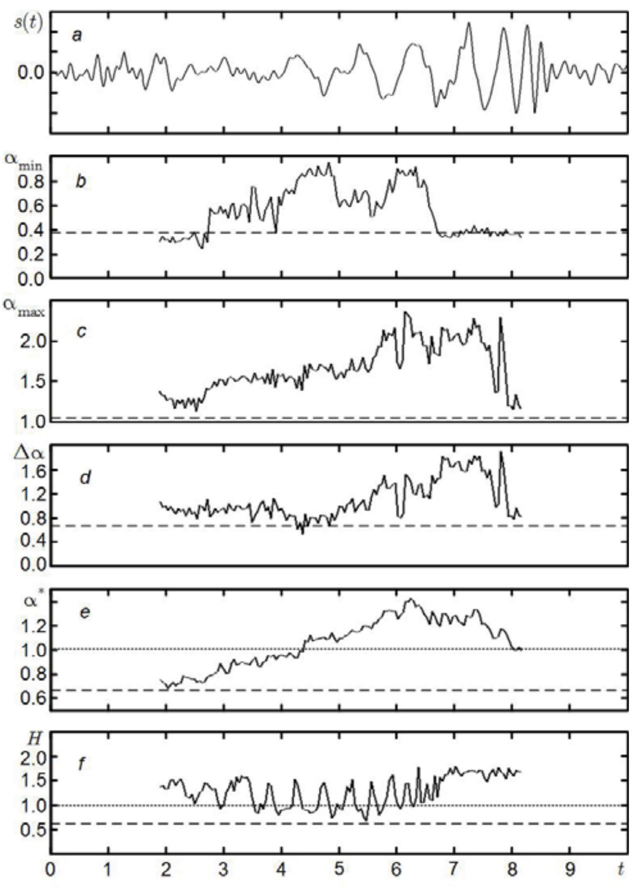

Fig. 3. MF DFA results for gravitational wave signal obtained in Hanford: a – signal in time domain, b – $\alpha_{\min} = \alpha_{\min}(t)$, c – $\alpha_{\max} = \alpha_{\max}(t)$, d – $\Delta\alpha = \Delta\alpha(t)$, e – $\alpha^* = \alpha^*(t)$, f – $H = H(t)$. Dashed lines denote results of the WTMM method, dotted lines indicate upper bound of the value for fractals.

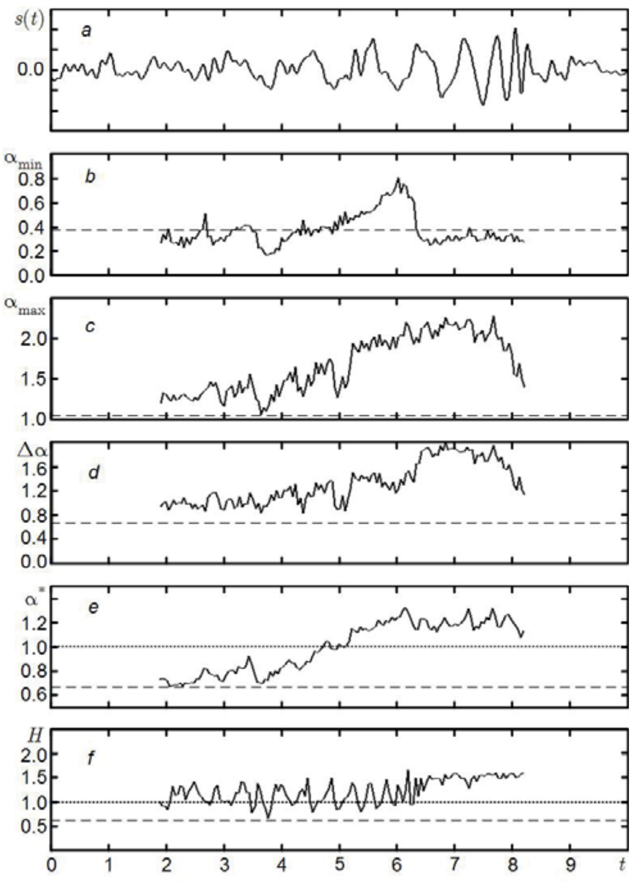

Fig. 4. The same as previous figure for gravitational wave signal registered in Livingston.

Discussion

In one of the previous works of authors [5], it was found that gravitational waves generated by a binary black hole merger were appeared to be a unique natural ultra-wideband (UWB) process with changing mean frequency.

The first gravitational wave registration (Hanford) contains the UWB process with changing mean frequency, which has the duration approximately $\tau \approx 130$ ms, the period band $T \approx 4-30$ ms, the dynamic frequency bandwidth changing from 0.4 to 0.9, the signal mean frequency rising with hyperbolic law, and the signal energy distribution with maximum at $T_0 \approx 20$ ms.

The second gravitational wave registration (Livingston) contains the UWB process with changing mean frequency, which has the duration approximately $\tau \approx 120$ ms, the period band $T \approx 4-30$ ms, the dynamic frequency bandwidth changing from 0.5 to 0.8, the signal mean frequency rising with hyperbolic law, and the signal energy distribution with maximum at $T_0 \approx 20$ ms.

Taking into account that results and comparing them with present ones, one can assert the following. Both signals analyzed have really fractal structure. This is well confirmed by the results of application as of the mono-fractal analysis, as of the multi-fractal analysis. There is no too significant difference between the results obtained for two signals registered in Hanford and in Livingston. Analyzing the whole both signals, it is important to point, that the value of the generalized Hurst exponent ($\alpha^* \approx 0,65$) is in well agreement with the estimations of the ($H \approx 0,55 \pm 0,10$) obtained with mono-fractal analysis. But the signals analyzed were appeared to be multi-fractal. This is good shown at the fig. 1. Therefore, the values of the Hurst exponent and of the generalized Hurst exponent describe the multi-fractal support only. Based on the estimation of H , one can assume, that multi-fractal support may be partially related to additive white Gaussian noise, which has $H = 0,5$.

But these were the results of whole signal analysis. Meanwhile, as it was pointed above, the signals are appeared to be significantly non-stationary and this fact should be taken into account. The answer was obtained in bounds of the MF DFA application.

Based on the time dependences, it was found that both signals analyzed can be considered as fractal ones approximately in the range of dimensionless time $t \in [0;6]$, where the condition $0 < \alpha^*(t) < 1$ is well satisfied. It is important to point, that on the fig. 4, e, seems, the narrower range $t \in [2;4]$ is observed. But this range should be extended to the one described above, as well as the width of the slide window applied for these calculation is appeared to be no less than $\Delta t = 4$. This limitation is caused by the MF DFA method peculiarities and by the size of the experimental data vectors used by the authors of the paper. In the range $t \in [0;6]$ there is approximately a half of the UWB process with changing mean frequency. Second half of them is appeared to be non-fractal. But whether this fractal component is a part of the gravitational wave signal or is a noise having quite different physical origin, suddenly, this question remains unanswered now.

Thus, the results obtained in the paper is good consistent with non-linear paradigm. Been generated by extremely powerful, open, non-linear, dynamical system, the gravitational waves were appeared to be a unique UWB process with significant complex, non-stationary multi-fractal structure. Suddenly, it remains unknown whether they are a true fractal UWB (FUWB) processes or a UWB processes registered on pretense of the additive multi-fractal noise, which had quite another physical origin. To solve this problem in the future, new observations and investigations are needed.

Conclusions

1. The transient gravitational wave signals generated by a black hole system merging to form a single black and received in Hanford and Livingston were appeared to be multi-fractal ones.

2. Being the unique natural UWB processes with changing mean frequency, they had complex, non-stationary multi-fractal structure.

3. Mono-fractal analysis shows, that capacity dimension D_C of the multi-fractal support of the signals analyzed was appeared to be $D_C \approx 1,45 \pm 0,10$ in range of the dimensionless time $t = 0,156 - 5$ (Hanford) and $D_C \approx 1,44 \pm 0,10$ in the same range (Livingston).

4. Using the classic multi-fractal analysis (WTMM method), it was obtained, that $\alpha_{\min} = 0.36 - 0.38$, $\alpha_{\max} = 1.01 - 1.03$, $\Delta\alpha = 0.65$ and $\alpha^* = 0.65 - 0.66$ for both signals investigated. Therefore, both signals are multi-fractal ones as whole.

5. With MF DFA application, the signals investigated were shown to be strongly non-stationary ones, including

their multi-fractal numerical characteristics. It was found, that in both cases the fractality condition for the generalized Hurst exponent ($0 < \alpha^*(t) < 1$) is well satisfied only for $t \leq 4,5$. For the Hurst exponent this condition ($0 < H(t) < 1$) is satisfied sometimes in bounds $2 \leq t \leq 6$.

6. To solve the problem whether the signals investigated are a true FUWB processes or a UWB processes registered on pretense of the additive multi-fractal noise, which had quite another physical origin, the new observations and investigations are needed.

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