UDC 517.958, 517.984.5

# On solution to integral equations with a logarithmic singularity of the kernel on several intervals of integration: elements of the spectral theory

Y.V. Shestopalov, E.V. Chernokozhin
Karlstad University, SE-651 88 Karlstad, Sweden

We consider the semi-inversion of meromorphic integral operator-valued functions defined on one or several intervals of integration. We perform approximate semi-inversion of the operator principal parts and obtain small-parameter expansions of the inverse operators. We show how the developed techniques can be applied to approximate determination of characteristic numbers and solution to boundary value problems for the Helmholtz equation that are reduced to boundary Fredholm integral equations and systems of integral equations of the first kind with a logarithmic singularity of the kernel. The results are applied to the analysis of oscillations in cylindrical slotted resonators.

***Key words: analytical semi-inversion, integral operator-valued function, characteristic number, slotted resonators.***

Розглядається напівобернення мероморфних інтегральних оператор-функцій, заданих на одному або декількох інтервалах інтегрування. Ми виконуємо наближене напівобернення головної частини оператору і отримуємо розклад зворотного оператора за малим параметром. Ми покажемо, яким чином розроблений метод може бути застосований до наближеного визначення характеристичних чисел і рішення рівняння Гельмгольца шляхом зведення до граничних інтегральних рівнянь Фредгольма і систем інтегральних рівнянь першого роду з логарифмічною особливістю в ядрі. Отримані результати додаються до аналізу коливань в циліндричних щілинних резонаторах.

***Ключові слова:*** *оборотне аналітичне напівобернення, інтегральний оператор-функція, характеристичні числа, щілинні резонатори.*

Рассматривается полуобращение мероморфных интегральных оператор-функций, заданных на одном или нескольких интервалах интегрирования. Мы выполняем приближенное полуобращение главной части оператора и получаем разложение обратного оператора по малому параметру. Мы покажем, каким образом разработанный метод может быть применен к приближенному определению характеристических чисел и решению уравнения Гельмгольца путём сведения к граничным интегральным уравнениям Фредгольма и системам интегральных уравнений первого рода с логарифмической особенностью в ядре. Полученные результаты используются для анализа колебаний в цилиндрических щелевых резонаторах.

***Ключевые слова***: *аналитическое полуобращение, интегральная оператор-функция, характеристические числа, щелевые резонаторы.*

## 1. Introduction

We apply the theory of integral operator-valued functions (OVFs) [1] and the methods of analytical semi-inversion [2–4] developing the method of approximate semi-inversion [5–8] to the case of meromorphic integral OVFs with a logarithmic singularity of the kernel defined on several intervals of integration. In fact, approximate inversion can be used when the spectral parameter is sufficiently distant from the singular points because, according to [1], there exists [5–8] a characteristic number (CN) (i.e., the point, at which the invertibility of the OVF is violated) in a neighborhood of a pole of the integral OVF with a logarithmic singularity of the kernel. We take into account the closeness to a simple first-order pole and modify the inversion formulas for this case. We apply the developed techniques [7, 8] of approximate semi-inversion to the analysis of eigenoscillations in cylindrical slotted cavities with narrow slots.

**2. Semi-inversion of integral operators**

Consider the integral OVF with a logarithmic singularity of the kernel

$$K(\lambda)\varphi = \alpha L\varphi + N(\lambda)\varphi \equiv \int_{\Gamma}\left[\alpha\frac{1}{\pi}ln\frac{1}{|t_0 - t|} + N(t_0,t,\lambda)\right]\varphi(t)dt \quad , \qquad t_0 \in \Gamma \, , \tag{1}$$

where $N(t_0,t,\lambda)$ is once continuously differentiable in $\Gamma\times\Gamma$ and a meromorphic function of $\lambda$, so that, in the vicinity of a pole $\lambda_\nu$ of $N(t_0,t,\lambda)$,

$$K(\lambda)\varphi = K_\nu(\lambda)\varphi + \frac{m_\nu}{\lambda_\nu - \lambda}(\varphi,\varphi_\nu)\varphi_\nu, \quad (\varphi,\varphi_\nu) \equiv \int_{\Gamma}\varphi_\nu(t)\varphi(t)dt \, , \tag{2}$$

$m_\nu$ is a constant, and $\varphi_\nu$ is a given differentiable function. OVFs $K(\lambda)$ and $K_\nu(\lambda)$ are Fredholm integral OVFs with a logarithmic singularity of the kernel and may have therefore not more than a finite number of CNs in every ball $B_r = \{\lambda : |\lambda| < r\}$ and are invertible at all (regular) points $\lambda$ that differ from CNs. For every $r > 0$ there exists [1] a sufficiently small $w = w(r)$ such that $B_r$ contains only regular points $\lambda$ of the OVF $K(\lambda)$ (and $K_\nu(\lambda)$). Approximate representation for the inverse operator

$$K_\nu^{-1}(\lambda)f = L_1^{-1}f - w^2\, ln\, w L_1^{-1}N_\nu L_1^{-1}f + O\left(w^4\, ln^2\, w\right),$$
$$\Gamma = (a,b) = (d - w, d + w), \tag{3}$$

as a segment of asymptotic series in powers of small parameter $\beta = \left(\frac{1}{\pi}ln\frac{1}{w}\right)^{-1}$ is obtained using the method of approximate semi-inversion developed in [1, 3–8]; here

$$L_1\varphi \equiv \alpha L\varphi + g(\varphi,1)1, \quad g = g(\lambda) = \frac{\alpha}{\beta} + M_0(\lambda) = const \quad (\alpha \neq 0) \quad ,$$
$$M_0(\lambda) = N(d,d,\lambda),$$
$$L_1^{-1}(\lambda,\beta)f = \frac{1}{\alpha}L^{-1}f - \frac{ln2}{\alpha\pi}\left(1 - \beta\frac{ln2}{\pi} + \beta^2\frac{ln2}{\pi}\left(\frac{M_0(\lambda)}{\alpha} + \frac{ln2}{\pi}\right)\right)\left(L^{-1}f,1\right)L^{-1}1$$
$$+ O\left(\beta^3\right).$$

We apply this result to determine CNs of $K(\lambda)$ and show [8] (by the reasoning stated briefly below in Sec. 3) that the CNs are roots of the equation

$$\lambda = \lambda_\nu + m_\nu\left(K_\nu^{-1}(\lambda)\varphi,\varphi_\nu\right) \tag{4}$$

that can be obtained as a segment of an asymptotic series in powers of $\beta$

$$\lambda_\nu^* = \lambda_\nu + \beta \frac{m_\nu |c_\nu|^2}{\alpha} + \beta^2 \frac{m_\nu |c_\nu^2|}{\alpha}\left(\frac{ln 2}{\pi} + \frac{M_{0,\nu}(\lambda_\nu)}{\alpha}\right) + O(\beta^3), \quad c_\nu = \varphi_\nu(d). \quad (5)$$

### 3. The case of *n* intervals

In the case of $n$ integration intervals in (1) $\Gamma = \bigcup_{j=1}^n \Gamma$, $\Gamma_j = (a_j, b_j) = (d_j - w_j, d_j + w_j)$, $j = 1,2, \dots n$, $n \ge 2$, OVF $K(\lambda)$ can be written, in the vicinity of a pole $\lambda_\nu$ of $N(\lambda; t_0, t)$ as a matrix operator using the vector quantities and the inner product

$$f = (\varphi_1, \dots, \varphi_n)^T, \qquad f_\nu = \varphi(1, \dots, 1)^T,$$

$$(f, f_\nu) = \sum_{j=1}^n (\varphi_j, \varphi_\nu)_j = \sum_{j=1}^n \int_{\Gamma_j} \varphi(t)\varphi_\nu(t)dt$$

associated with the set of integration intervals. Then OVF $K(\lambda)$ can be represented as

$$K(\lambda)f = K_\nu(\lambda)f + \frac{m_\nu}{\lambda_\nu - \lambda}(f, f_\nu)f_\nu , \quad (6)$$

where $K_\nu(\lambda)$ is a matrix OVF with the entries $K_{ij,\nu}(\lambda)$, $i, j = 1,2, \dots n$.

Use (6) and consider a local representation of the integral equation $K(\lambda)f = F$ in the vicinity of the chosen pole $\lambda_\nu$

$$K(\lambda)f = K_\nu(\lambda)f + \frac{m_\nu}{\lambda_\nu - \lambda}(f, f_\nu)f_\nu = F , \quad (7)$$

where $F = (F_1, F_2, \dots, F_n)^T$. Applying operator $K_\nu^{-1}(\lambda)$ to both sides of (7), we obtain the equivalent equation

$$f + \frac{m_\nu}{\lambda_\nu - \lambda}(f, f_\nu)K_\nu^{-1}(\lambda)f_\nu = K_\nu^{-1}(\lambda)F . \quad (8)$$

The solution to (8) is uniquely defined if the inner product $(f, f_\nu)$ is uniquely defined. Calculating the inner product of both sides of equation (8) with $f_\nu$, we obtain

$$(f, f_\nu) + \frac{m_\nu}{\lambda_\nu - \lambda}(f, f_\nu)\left(K_\nu^{-1}(\lambda)f_\nu, f_\nu\right) = \left(K_\nu^{-1}(\lambda)F, f_\nu\right) . \quad (9)$$

Resolving (9) with respect to $(f, f_\nu)$ and substituting the result into (8), we obtain the local representation of the inverse $K(\lambda)^{-1}$ in the vicinity of the pole $\lambda_\nu$:

$$f = K_\nu^{-1}(\lambda)F - \frac{m_\nu\left(K_\nu^{-1}(\lambda)F, f_\nu\right)}{\lambda_\nu - \lambda + m_\nu\left(K_\nu^{-1}(\lambda)f_\nu, f_\nu\right)} K_\nu^{-1}(\lambda)f_\nu \,. \qquad (10)$$

The quantity $(f, f_\nu)$ is uniquely defined for any $F$ if and only if the denominator of the fraction in (10) is not equal to zero. The zeros of the denominator are the points at which $K(\lambda)$ is not invertible. Since the matrix integral operator $K(\lambda)$ is a Fredholm and holomorphic OVF, these points are its CNs. Thus, CNs of $K(\lambda)$ are the roots of the equation

$$\lambda = \lambda_\nu + m_\nu\left(K_\nu^{-1}(\lambda)f_\nu, f_\nu\right) \quad . \qquad (11)$$

For sufficiently small $w$, we prove using the contraction principle, that there exists a root $\lambda_\nu^*$ of equation (11) which can be obtained as a segment of asymptotic series in powers of $\beta$ ; the resulting expressions are similar to (5).

**4. Slotted rectangular resonators**

The cross section of a slotted resonator by the plane $x_3 = 0$ (in the Cartesian coordinate system $(x_1, x_2, x_3)$) is formed by two rectangular domains

$$\Omega^1 = \{r = (x_1, x_2) : \; 0 < x_1 < a_1; 0 < x_2 < b_1\},$$
$$\Omega^2 = \{r : \; 0 < x_1 < a_2; -b_2 < x_2 < 0\}$$

with the common part of the boundary $\partial\Omega^1 \cap \partial\Omega^2 = \{r : \; x_2 = 0, 0 \le x_1 \le min(a_1, a_2)\}$ containing one or several slots $\Gamma$. The permittivity $\varepsilon = \varepsilon_i(r), i = 1,2; r \in \Omega^i$. The squared eigenfrequencies of this cylindrical resonator are eigenvalues of the boundary eigenvalue problem ($M = \left(\partial\Omega^1 \setminus \Gamma\right) \cup \left(\partial\Omega^2 \setminus \Gamma\right)$)

$$\Delta u(r) + \lambda\varepsilon u(r) = 0, \quad r \in \Omega = \Omega^1 \cup \Omega^2 \,, \qquad (12)$$

$$\left.\frac{\partial u}{\partial n}\right|_M = 0, \quad \left[u^1 - u^2\right]_\Gamma = 0, \quad \left.\left[\frac{1}{\varepsilon_1}\frac{\partial u^1}{\partial x_2} - \frac{1}{\varepsilon_2}\frac{\partial u^2}{\partial x_2}\right]\right|_\Gamma = 0 \,. \qquad (13)$$

The Fredholm property and existence of generalized and classical solutions to the corresponding inhomogeneous problem are proved in [5—8]. Therefore, there may exist only isolated eigenvalues of (12), (13). According to [5], this problem has a real eigenvalue between every two neighboring points $\mu_{nm}^{(i)} = \frac{\pi^2}{\varepsilon_i}\left(\frac{n^2}{a_i^2} + \frac{m^2}{b_i^2}\right)$, $n, m = 0,1,...,\quad i = 1,2$ . These eigenvalues coincide with CNs of the integral OVF $K(\lambda)$ with a logarithmic singularity that enters the integral equation $K(\lambda)\varphi = 0$ to which this problem is reduced and can be calculated from (11) as segments of asymptotic series in characteristic small parameter $\beta$ .

**5. Numerical**

Present some typical results of calculations obtained using the technique reported in this study for the rectangular slotted resonator shown in Fig. 1.

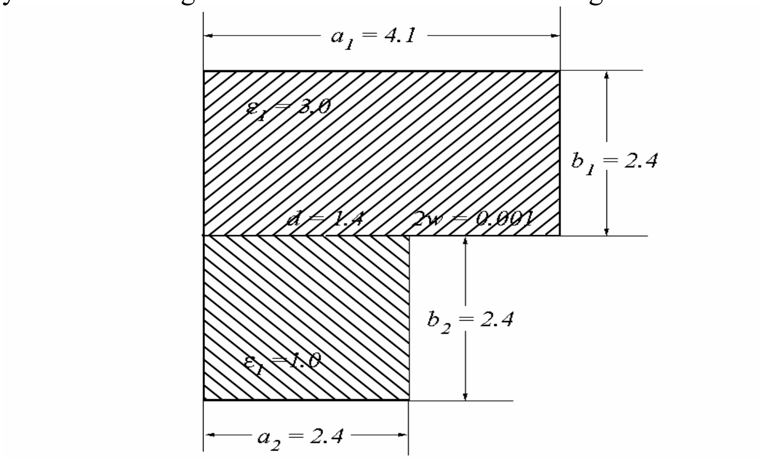


*Fig. 1. Cross-section of the cylindrical slotted resonator formed by two rectangular domains connected through a slot in the common wall. Numerical values of the geometric and material parameters used in computations are also shown.*

We consider the influence of the width of the upper rectangular domain (parameter $a_1$) on the structure of $H_{11}^1$- and $H_{20}^1$-types of oscillations in the resonator with and without a slot. Fig. 2 shows the dependences of the eigenvalues $\lambda_{11}^{(1)}$, $\lambda_{20}^{(1)}$ of the rectangular cylindrical resonator on the width of domain $\Omega^1$ in the *absence* of the slot. The curves have the point of intersection when the width of the domain $a_1 = a^* \approx 4.156$. The eigenvalues at this point are $\lambda_{11}^{(1)}\left(a^*\right) = 0.7616$, $\lambda_{20}^{(1)}\left(a^*\right) = 0.7618$.

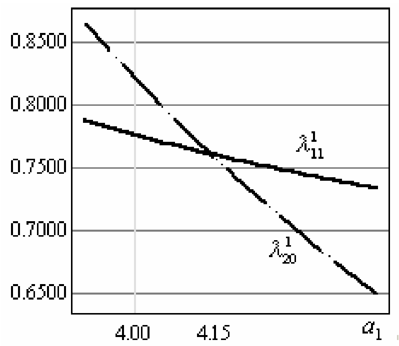


*Fig. 2. Dependence eigenvalues on $a_1$ in the absence of the slot.*

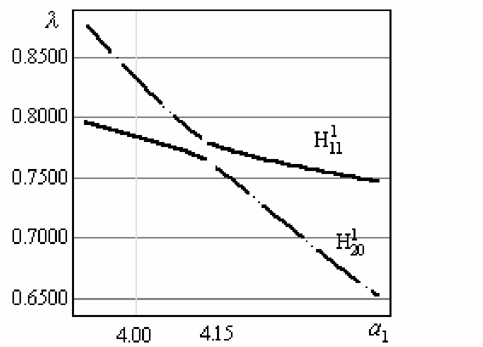


*Fig. 3. Dependence eigenvalues of the problem H on $a_1$.*

Fig. 3 shows the dependences of eigenvalues of problem $H$ (that is, in the presence of the slot) of the considered types of oscillations on parameter $a_1$. It can be seen that the eigenvalue curves no longer merge near the point $a^*$ and the exchange of the oscillation types occurs. This process is called intertype interaction of oscillations. For more details we refer to papers [7, 8].

**6. Conclusion. Acknowledgements**

We have constructed approximate semi-inversion of an integral operator on several integration intervals with a logarithmic singularity of the meromorphic kernel in a vicinity of singularities and show how this technique can be used for determining characteristic numbers.

This work is supported by the Visby project of the Swedish Institute.

REFERENCES

1. Y. Shestopalov, Y. Smirnov, and E. Chernokozhin. Logarithmic Integral Equation in Electromagnetics. – Utrecht: VSP, 2000. – 117 p.
2. V. Shestopalov and Y. Shestopalov. Spectral Theory and Excitation of Open Structures. – London: Peter Peregrinus, 1996. – 399 p.
3. Y. Shestopalov and E. Chernokozhin. Mathematical Methods for the Study of Wave Scattering by Open Cylindrical Structures. // J. Comm. Tech. Elec. - 1997. - V. 42. - pp. 1211-1223.
4. Y. Shestopalov and E. Chernokozhin. Resonant and Nonresonant Diffraction by Open Image-type Slotted Structures. // IEEE Trans. Antennas Propag. - 2001. - V. 49. - pp. 793-801.
5. Y. Shestopalov and O. Kotik. Interaction of Oscillations in Slotted Resonators and its Application to Microwave Imaging. // J. Electromag. Waves Appl. - 2003. - V. 17. - pp. 291-311.
6. Y. Shestopalov and O. Kotik. Eigenoscillations of Rectangular Slotted Resonators. // Vestnik MGU. - 1999. - V. 15. - pp. 4-14.
7. Y. Shestopalov and N. Kotik. Interaction and Propagation of Waves in Slotted Waveguides. // New J. Phys. - 2002. - V. 4. - pp. 40.1-40.16.
8. Y. Shestopalov, N. Kotik, and Y. Okuno. Oscillations in Slotted Resonators with Several Slots: Application of Approximate Semi-Inversion. // Progress in Electromagnetics Research. - 2002. - V. PIER-39. - pp. 193-247.

Надійшла – 13.03.2013.