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Analytical-numerical approach to analyze forced and parametric vibrations of some pendulum systems

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The parametric oscillations of physical pendulum and forced vibrations of a system with pendulum absorber are analyzed using the approach based on combined application of the concept of nonlinear normal vibration modes, the Rauscher method, and numerical procedures. Frequency responses are obtained.

Key words: *pendulum systems, nonlinear normal vibrationsя, Rauscher method.*

Проведен анализ параметрических колебаний физического маятника и вынужденных колебаний системы с маятниковым гасителем колебаний с применением единого подхода, базирующегося на совместном использовании метода Раушера, метода нелинейных нормальных форм колебаний и численных процедур. Построены амплитудно-частотные характеристики.

Ключевые слова: *маятниковые системы, нелинейные нормальные колебания, метод Раушера.*

Проведено аналіз параметричних коливань фізичного маятника та вимушених коливань системи з маятниковим гасителем коливань із застосуванням єдиного підходу, що базується на спільному використанні методу Раушера, методу нелінійних нормальних форм коливань та чисельних процедур. Побудовано амплітудно-частотні характеристики.

Ключові слова: *маятникові системи, нелінійні нормальні коливання, метод Раушера.*

1. Introduction

It is well-known that resonance forced vibrations of a single-degree-of freedom (DOF) nonlinear systems under small periodic perturbations in the region of main resonance are close to natural vibrations of unperturbed conservative system. This result can be transferred to finite-DOF systems. In the last case the resonance forced vibrations are close to nonlinear normal vibration modes of corresponding finite-DOF conservative systems. Thus, it is appropriate to use the nonlinear normal vibration modes of conservative systems to construct forced resonance vibrations. It permits to consider vibrations with essential amplitudes.

Origins of the nonlinear normal vibrations theory can be found in works by Lyapunov [1] on systems with the first analytical integral. Concept of nonlinear normal vibration modes (NNMs), which is based on construction of trajectories in the system configuration space, is developed in works by Kauderer [2] and Rosenberg [3]. Approach of construction of curvilinear trajectories of NNMs is proposed in publications [4,5]. Principal aspects of the NNMs theory by Kauderer-Rosenberg are presented in books [5,6] and in review [7]. Approach which combines the concept of nonlinear normal vibration and the Rauscher method is used to construct forced resonance vibrations of systems having few degrees of freedom in [8,5,6]. (Initially the Rauscher method was proposed for a nonlinear conservative single-DOF system [9]) Note that the same approach can be used also in construction of parametric vibrations.

Here the approach based on combined use of the concept of NNMs, the Rauscher method and numerical procedures, is used in problem of parametric vibrations of the spring pendulum and in problem of forced resonance vibrations of the system containing a pendulum absorber.

2. Parametric vibrations of a spring pendulum

The model of two-DOF spring pendulum is presented in Fig. 1. Vibrations of the mass m on the linear spring of the length l in unstressed state are considered. Dynamics of the system is described by two generalized coordinates ρ and φ.

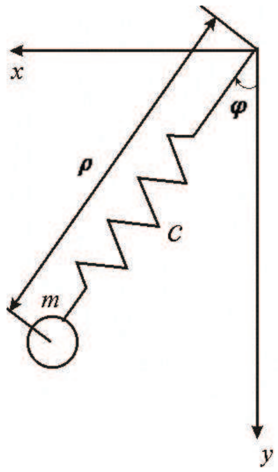

Fig. 1. Spring pendulum

Let the small periodic perturbation applied in vertical direction is considered. Equations of parametric vibrations of the system can be wrote as the following:

$$\begin{cases} \ddot{z} + z = \mu\left(\rho_0\dot{\varphi}^2 + \dfrac{(g+\mu F\cos\Omega t)}{2}\varphi^2\right) + \mu^2\dot{\varphi}^2 z; \\ \rho_0^2\ddot{\varphi} + (g+\mu F\cos\Omega t)\rho_0\varphi = \mu\left(-2\rho_0\ddot{\varphi} - 2\rho_0\dot{\varphi}\dot{z} - (g+\mu F\cos\Omega t)z\varphi\right) + \\ +\mu^2\left(-z^2\ddot{\varphi} - 2z\dot{z}\dot{\varphi} + \dfrac{(g+\mu F\cos\Omega t)\rho_0}{6}\varphi^3\right) + \mu^3\dfrac{(g+\mu F\cos\Omega t)}{6}z\varphi^3, \end{cases} \tag{1}$$

where $\mu z = \rho - \rho_0$, $\rho_0 = l + \dfrac{gm}{c}$ is an extension of the spring in the equilibrium state; μ is a formal small parameter.

Combination of the NNMs approach and the Rauscher method is used to construct normal modes of parametric vibrations. One considers, first of all, the autonomous system obtained from equations (1) in the zero approximation by the small parameter ($\mu = 0$). In regime of nonlinear normal mode one has $z = z(\varphi)$; and the system under consideration is reduced to the single-DOF system with respect to the variable φ. The nonlinear normal mode can be obtained by power series [4,5]:

$$z = z(\varphi) = z_0 + \mu z_1 + ..., \tag{2}$$

$$\begin{aligned} z_0 &= a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 + a_4\varphi^4 + ...; \\ z_1 &= b_0 + b_1\varphi + b_2\varphi^2 + b_3\varphi^3 + b_4\varphi^4 + ... \end{aligned} \tag{3}$$

One uses a representation of the generalized coordinate φ in the regime of the NNM as Fourier series, namely, $\varphi = A_0 + A_2\cos(2\Omega t) + A_4\cos(4\Omega t) + ...$ We introduce the Fourier series to the obtained previously single-DOF system, saving only three first terms of the series and then using the harmonic balance. One obtains, as a result, a system of three nonlinear algebraic equations with respect to four unknowns (A_0, A_2, A_4, Ω). During the next numerical calculations a value A_2 is given with some step. The system of nonlinear algebraic equations is solved with respect to unknown quantities (A_0, A_2, A_4, Ω) for each value of A_2. One has from here the required parameters (A_0, A_4, Ω). As a result, the first coefficient of the Fourier series will be determined.

One transforms now the Fourier series using known trigonometric formulae, as

$$\begin{aligned} \varphi &= A_0 + A_2\cos(2\Omega t) + A_4\cos(4\Omega t) + ... = \\ &= A_0 + A_2\left(2\cos^2(\Omega t) - 1\right) + A_4\left(8\cos^4(\Omega t) - 8\cos^2(\Omega t) + 1\right) + ... = \\ &= (A_0 - A_2 + A_4) + (2A_2 - 8A_4)\cos^2(\Omega t) + 8A_4\cos^4(\Omega t) + ... \end{aligned} \tag{4}$$

The following expansion can be obtained from the relation (4):

$$\varphi = (A_0 - A_2 + A_4) + (2A_2 - 8A_4)\cos^2(\Omega t) + 8A_4\cos^4(\Omega t) + ... \tag{5}$$

Then, some algebraic transformations permit to invert the expansion (5) and to obtain the following relation:

$$\cos\Omega t = \alpha_0 + \alpha_1\varphi + \alpha_2\varphi^2 + ... \tag{6}$$

On has the external periodic excitation is presented as a function of the generalized coordinate φ in zero approximation by the small parameter. Introducing the expansion (6) to the initial non-autonomous dynamical system (1), one obtains so-called «pseudo-autonomous» dynamical system. Such adduction of the non-autonomous system to the autonomous one corresponds to main idea of the Rauscher method. In obtained autonomous dynamical system the nonlinear normal vibration mode can be anew obtained. It permits to make more precise expansions (2) - (3), and to realize again a transfer from the initial non-autonomous system to the «pseudo-autonomous» one. Thus, the iteration *analytical-numerical procedure* can be used for a construction of forced resonance vibrations, which permits to obtain a solution with good exactness.

Two nonlinear normal vibration modes can be selected in this system: a) localized vibration mode close to longitudinal vibration mode of the pendulum system without the external excitation; in the localized mode amplitudes of rotations are small;
b) coupled vibration mode, when amplitudes of longitudinal vibrations and rotations are comparable.

Trajectories of the localized vibration mode of parametric vibrations in the system configuration space are shown in Fig. 4a; trajectories of the mode of coupled vibrations are shown in Fig. 4b. Calculations are made for the next values of the system parameters: $g=9.8$, $l=0.5$, $m=1$, $c=3$, $\mu=0.1$, $f=0.3$, $\varphi_0=0.01$ (for the localized mode) and $g=9.8$, $l=0.5$, $m=0.1$, $c=2$, $\mu=0.1$, $f=3$, $\varphi_0=0.1$ (for the mode pf coupled vibrations). Here red lines correspond to the analytical solution, and blue lines correspond to checking numerical simulation by the Runge-Kutta method, which is made for initial solutions obtained from analytical solution. Numerical calculations confirm good exactness of the analytical results.

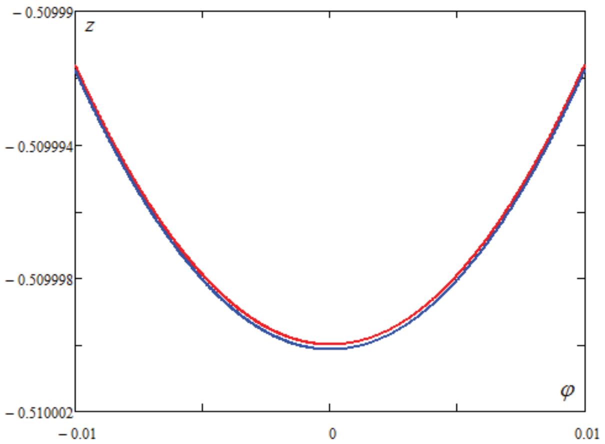

Fig. 2.a

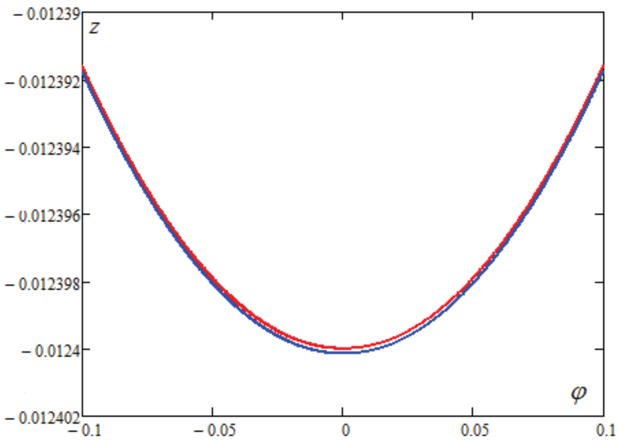

Fig.2.b

Fig. 2. Trajectories of the localized vibration mode of parametric vibrations (Flg.2a) and of the mode of coupled vibrations (Fig.2b). Comparison of analytical and checking numerical solutions.

3. Forced vibrations of the system, which contains a pendulum absorber

The second model under consideration is presented in Fig. 3. Here mechanical subsystem which vibrations must be extinguished is presented as oscillator of the mass m_1 with anchor spring which rigidity coefficient is equal to k. The pendulum absorber of the mass m_2 and of the length l is attached to the linear oscillator.

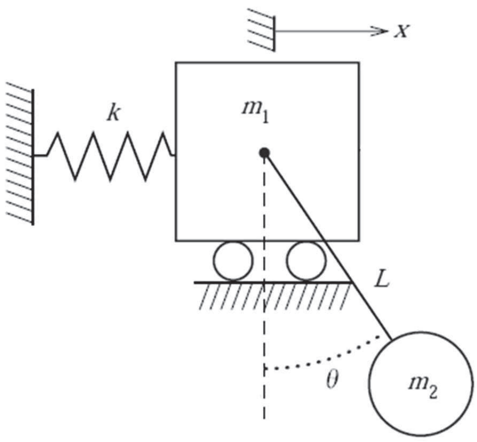

Fig. 3. System containing the pendulum absorber

Motions of the system are described by two generalized coordinates x (displacement of the linear subsystem) and θ (angle of the pendulum absorber).

Equations of motion of the model in the presence of the small external periodic action are the following:

$$\begin{cases} \left(m_1+\varepsilon m_2\right)\ddot{x}+\varepsilon m_2 l\ddot{\theta}\left(1-\dfrac{\theta^2}{2}\right)-\varepsilon m_2 l\dot{\theta}^2\left(\theta-\dfrac{\theta^3}{6}\right)+kx=\varepsilon F\cos\left(\Omega t\right); \\ \ddot{x}\left(1-\dfrac{\theta^2}{2}\right)+l\ddot{\theta}+g\left(\theta-\dfrac{\theta^3}{6}\right)=0. \end{cases} \tag{7}$$

Here ε is the formal small parameter.

Two nonlinear normal vibration modes can be selected in this system: a) localized vibration mode when vibration amplitudes of the linear subsystem are essentially smaller than amplitudes of the pendulum; b) coupled vibration mode, when amplitudes of the linear oscillator and of the pendulum are comparable. The first vibration mode is appropriate for absorption of vibrations of the linear subsystem.

Trajectories of the localized mode of forced vibrations (Fig.4a) and of the mode of coupled vibrations (Fig. 4b) are constructed using approach described in the preceding Section. This approach joints the nonlinear normal mode concept and the Rauscher method. Calculations are made for the following parameters of the system: $m_1=1$, $m_2=0.1$, $l=1$, $k=5$, $\varepsilon=0.1$, $f=0.1$. Here red lines correspond to the analytical solution, and the blue lines correspond to checking numerical simulation by the

Runge-Kutta method, which is made for initial solutions obtained from analytical solution. Numerical calculations confirm good exactness of the analytical results.

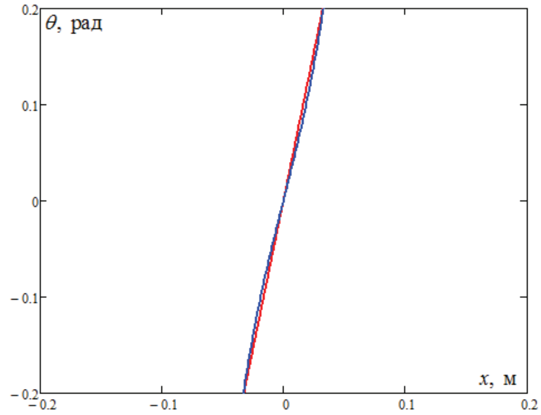

Fig. 4a.

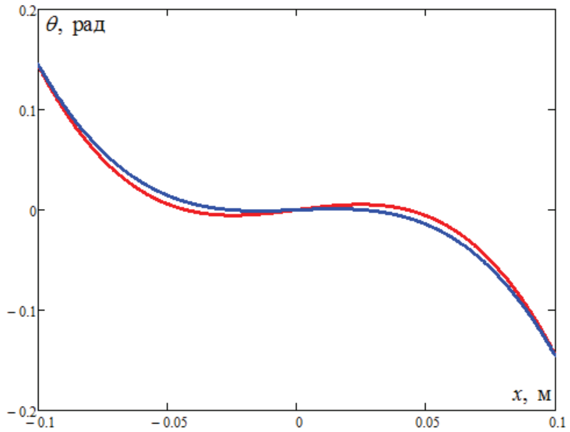

Fig. 4b.

Fig. 4. Trajectories of the localized mode of forced vibrations (Fig.4a) and of the mode of coupled vibrations (Fig.4b) for the system having the pendulum absorber.

Construction of frequency responses is made by the harmonic balance method. In correspondence with this method the variables x and θ are presented in the form of the following sum of harmonics: $x = A_1 \cos(\Omega t) + A_2 \sin(\Omega t)$; $\theta = B_1 \cos(\Omega t) + B_2 \sin(\Omega t)$. The frequency responses are shown in Fig. 5 for the vibration mode of coupled vibrations, and in Fig. 6 for the localized vibration mode. It can see that in regime of the localized vibration mode vibrations of the linear subsystem are essentially smaller than ones of the pendulum absorber; the vibration energy concentrates in the absorber. So, this regime is appropriate for a quenching of vibrations of the linear subsystem.

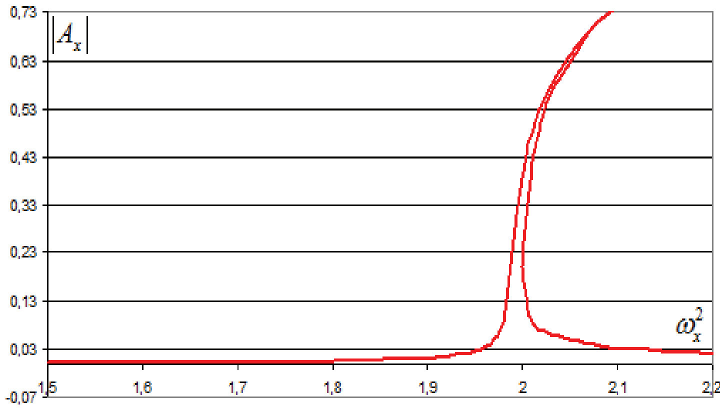

Fig. 5a.

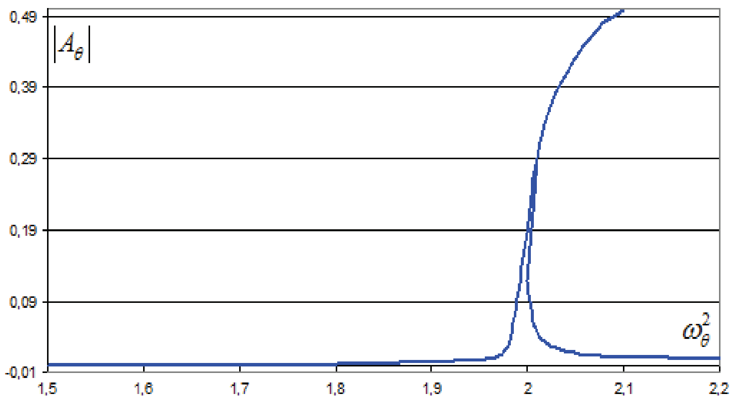

Fig. 5b.

Fig. 5. Frequency responses of the mode of coupled vibrations for the linear subsystem (Fig. 5a) and for the pendulum absorber (Fig. 5b)

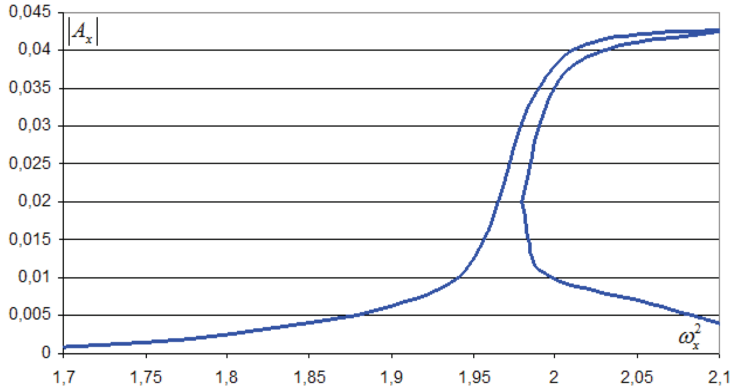

Fig. 6a

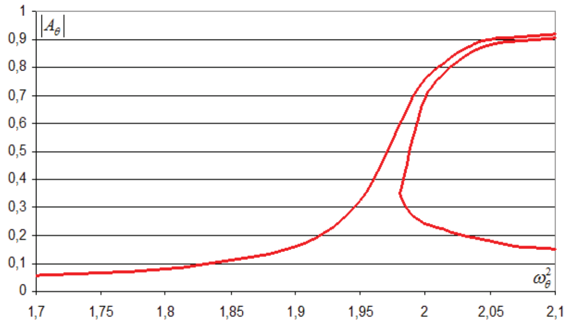

Fig. 6b

Fig. 6. Frequency responses of the localized vibration mode for the linear subsystem (Fig. 6a) and for the pendulum absorber (Fig. 6b)

4. Conclusions

New analytical-numerical approach to analyze parametric and forced vibrations of some pendulum systems is proposed. This approach is based on concept of nonlinear normal vibration modes, the Rauscher method and some numerical procedure. The proposed approach permits to construct trajectories of parametric and forced vibration modes in the system configuration space. Frequency responses are constructed too.

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