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Statistical convergence cannot be generated by a single statistical measure

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We demonstrate that statistical convergence cannot be generated by a single statistical measure, thus solving in negative a question from recent paper by Li Xin Cheng, Li Hua Lin, and Xian Geng Zhou.
Keywords: filter convergence; statistical convergence; statistical measure.

Кадець В. **Статистичну збiжнiсть не можна задати однiєю статистичною мiрою.** Ми доводимо, що статистичну збiжнiсть не можна задати однiєю статистичною мiрою, вiдповiдаючи таким чином на питання з нещодавної статтi Лiсiна Ченга, Лiхуа Лiн та Сянгенга Чжоу.
Ключовi слова: збiжнiсть за фiльтром; статистична збiжнiсть; статистична мiра.

Кадец В. **Статистическую сходимость нельзя задать одной статистической мерой.** Мы доказываем, что статистическую сходимость нельзя задать одной статистической мерой, отвечая таким образом на вопрос из недавней статьи Лисина Ченга, Лихуа Лин и Сянгенга Чжоу.
Ключевые слова: сходимость по фильтру; статистическая сходимость; статистическая мера.
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Introduction

This short note is a follow-up of [2], where the reader can find an extensive list of references related to the subject. We do not repeat historic remarks, connections with other mathematical concepts and motivation presented in [2], but for the reader’s convenience we give below precisely those definitions and explanations that are necessary to understand our article.

Recall that a *filter* $\mathcal{F}$ on the set $\mathbb{N}$ of all naturals is a non-empty collection of subsets of $\mathbb{N}$ satisfying the following axioms: $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; and for every $A \in \mathcal{F}$ if $B \supset A$ then $B \in \mathcal{F}$.

A sequence (x_n), $n \in \mathbb{N}$ in a topological space X is said to be $\mathcal{F}$-*convergent* to x if for every neighborhood U of x the set $\{n \in \mathbb{N} : x_n \in U\}$ belongs to $\mathcal{F}$. In particular if one takes as $\mathcal{F}$ the filter of those sets whose complements are finite (the *Fréchet filter*), then $\mathcal{F}$-convergence coincides with the ordinary one.

The natural ordering on the set of filters on $\mathbb{N}$ is defined as follows: $\mathcal{F}_1 \succ \mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. Maximal in this ordering filters are called *ultrafilters*. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the following is true: for every subset $A \subset \mathbb{N}$ that does not belong to $\mathcal{U}$, the complement $\mathbb{N} \setminus A$ belongs to $\mathcal{U}$.

A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *free* if it dominates the Fréchet filter. In this case every ordinary convergent sequence is automatically $\mathcal{F}$-convergent.

For a subset A of naturals its *lower density* is defined as

$$\delta_*(A) := \liminf_n \frac{\#\{k \leq n : k \in A\}}{n}, \tag{1}$$

where # stands for the number of elements of the set. The *upper density* $\delta^*(A)$ is defined in a similar way by substituting lim inf by lim sup in (1). If the ordinary limit in (1) exists, i.e., upper and lower densities coincide, this limit is called *natural density* and is denoted by $\delta(A)$. Remark, that $\delta_*(A) = 1$ if and only if $\delta(A) = 1$, and $\delta^*(A) = 0$ if and only if $\delta(A) = 0$.

A sequence (x_k) in a topological space X is *statistically convergent to* x if for every neighborhood U of x the set $\{k : x_k \in U\}$ has natural density 1. In other words, statistical convergence is the same as convergence with respect to the filter $\mathcal{F}_{st} = \{A \in 2^{\mathbb{N}} : \delta_*(A) = 1\}$.

A non-negative finitely additive measure μ defined on the collection of all subsets of $\mathbb{N}$ is said to be a *statistical measure* if $\mu(\mathbb{N}) = 1$ and $\mu(\{k\}) = 0$ for all $k \in \mathbb{N}$. Evidently, a statistical measure cannot be countably additive. An example of statistical measure is the characteristic function $\mathbf{1}_{\mathcal{U}}$ of a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$: $\mathbf{1}_{\mathcal{U}}(A) = 1$ if $A \in \mathcal{U}$, and $\mathbf{1}_{\mathcal{U}}(A) = 0$ if $A \in 2^{\mathbb{N}} \setminus \mathcal{U}$. More examples can be easily given by combining several statistical measures of above-mentioned type, like $\frac{1}{2}(\mathbf{1}_{\mathcal{U}} + \mathbf{1}_{\mathcal{V}})$, where $\mathcal{U}$ and $\mathcal{V}$ are two different free ultrafilters.

For a given statistical measure μ, a sequence (x_n) in a topological space X is said to be μ-*convergent* to $x \in X$, if $\mu(\{n \in \mathbb{N} : x_n \in U\}) = 1$ for every neighborhood U of x.

For a given non-empty family $\mathcal{S}$ of statistical measures, a sequence (x_n) in a topological space X is said to be $\mathcal{S}$-*convergent* to $x \in X$, if (x_n) is μ-convergent to x for all $\mu \in \mathcal{S}$.

Lingxin Bao and Lixin Cheng [1] remarked that for every free filter $\mathcal{F}$ there is a non-empty family $\mathcal{S}$ of statistical measures such that $\mathcal{S}$-convergence is equivalent to convergence with respect to $\mathcal{F}$ (one can take $\mathcal{S} = \{\mathbf{1}_{\mathcal{U}} : \mathcal{U} \succ \mathcal{F}, \mathcal{U}$ is an ultrafilter$\}$).

In their recent paper [2], Li Xin Cheng, Li Hua Lin, and Xian Geng Zhou performed an extensive study of μ-convergence generated by a single statistical measure μ and presented a number of nice characterizations. But, as they mention in [2, Remark 5.4], for the classical statistical convergence (which was one of motivations of the study) they were not able to determine whether it is equivalent to μ-convergence for a single statistical measure μ. In other words, the question whether there exists a statistical measure μ such that

$$\left\{A \in 2^{\mathbb{N}} : \mu(A) = 1\right\} = \left\{A \in 2^{\mathbb{N}} : \delta_*(A) = 1\right\}$$

remains unsolved. Passing to complements, one reduces the problem to the following one:

Question 1. Does there exist a statistical measure μ such that

$$\left\{A \in 2^{\mathbb{N}} : \mu(A) = 0\right\} = \left\{A \in 2^{\mathbb{N}} : \delta^*(A) = 0\right\}? \tag{2}$$

The aim of this article is to give the negative answer to Question 1.

Some elementary lemmas

Lemma 1 *Let $A \subset \mathbb{N}$ be a set with $\delta^*(A) = 1$. Then, for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there is an $m > n$ such that for $B = A \cap \{n+1, n+2, \dots, m\}$ we have $\frac{\#B}{m} > 1 - \varepsilon$.*

Proof. Due to the definition of $\delta^*(A)$, there is a sequence $m_1 < m_2 < \dots$ such that

$$\lim_k \frac{\#\{j \le m_k : j \in A\}}{m_k} = 1.$$

Then, denoting $B_k = A \cap \{n+1, n+2, \dots, m_k\}$ we obtain that

$$\frac{\#B_k}{m_k} \ge \frac{\#\{j \le m_k : j \in A\}}{m_k} - \frac{n}{m_k} \quad \xrightarrow[k\to\infty]{} \quad 1.$$

Consequently, when k is large enough, m_k and B_k can serve as our m and B respectively.

Lemma 2 *Every set $A \subset \mathbb{N}$ of $\delta^*(A) = 1$ can be represented as a disjoint union of two sets of upper density 1. In other words, there are $A_1, A_2 \subset \mathbb{N}$ with $\delta^*(A_1) = \delta^*(A_2) = 1$ such that $A = A_1 \sqcup A_2$.*

Proof. Using repetitively Lemma 1 we can find $0 = m_1 < m_2 < ...$ and $B_j = A \cap \{m_j + 1, m_j + 2, \dots, m_{j+1}\}$, $j = 1, 2, \dots$ such that $\frac{\#B_j}{m_{j+1}} > 1 - \frac{1}{j}$. It remains to take $A_1 = B_1 \cup B_3 \cup B_5 \cup \dots$, $A_2 = B_2 \cup B_4 \cup B_6 \cup \dots$ Indeed,

$$\delta^*(A_1) \ge \lim_{j\to\infty} \frac{\#\{k \le m_{2j+2} : k \in A_1\}}{m_{2j+2}} \ge \lim_{j\to\infty} \frac{\#B_{2j+1}}{m_{2j+2}} = 1,$$

$$\delta^*(A_2) \ge \lim_{j\to\infty} \frac{\#\{k \le m_{2j+1} : k \in A_2\}}{m_{2j+1}} \ge \lim_{j\to\infty} \frac{\#B_{2j}}{m_{2j+1}} = 1.$$

Lemma 3 *Let $A_n \subset \mathbb{N}$ form a decreasing sequence of sets $A_1 \supset A_2 \supset \dots$ with $\delta^*(A_n) = 1$. Then there is a set $B \subset \mathbb{N}$ with $\delta^*(B) = 1$ such that $\#(B \setminus A_n) < \infty$ for every $n \in \mathbb{N}$.*

Proof. Again, using repetitively Lemma 1 we can find $m_1 < m_2 < \dots$ and $B_j = A_j \cap \{m_j + 1, m_j + 2, \dots, m_{j+1}\}$, $j = 1, 2, \dots$ such that $\frac{\#B_j}{m_{j+1}} > 1 - \frac{1}{j}$. Then $B = \bigcup_{j \in \mathbb{N}} B_j$ is the set we are looking for.

Two sets $A, B \subset \mathbb{N}$ are said to be *almost disjoint*, if $\#(A \cap B) < \infty$.

Lemma 4 *Let μ be a statistical measure, and let $A_\gamma \subset \mathbb{N}$, $\gamma \in \Gamma$ be a collection of pairwise almost disjoint subsets with $\mu(A_\gamma) > 0$ for all $\gamma \in \Gamma$. Then Γ is at most countable.*

Proof. First, remark that since $\mu(A) = 0$ for every finite set A, the finite-additivity formula $\mu\left(\bigcup_{k=1}^n D_k\right) = \sum_{k=1}^n \mu(D_k)$ remains true for every finite collection of pairwise almost disjoint subsets. Now, for every $n \in \mathbb{N}$ denote $\Gamma_n = \{\gamma \in \Gamma : \mu(A_\gamma) > \frac{1}{n}\}$. Then for every finite subset $E \subset \Gamma_n$ we have

$$\#E < n \sum_{\gamma \in E} \mu(A_\gamma) = n\mu\Big(\bigcup_{\gamma \in E} A_\gamma\Big) \le n\mu(\mathbb{N}) = n.$$

Consequently, $\#\Gamma_n < n$. Since $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$, Γ is at most countable.

The main result

Theorem 1 *For any statistical measure μ, μ-convergence is not equivalent to the standard statistical convergence.*

Proof. Assume contrary that there is a statistical measure μ such that the identity (2) holds true. Then, every subset $A \subset \mathbb{N}$ with $\delta^*(A) > 0$ has $\mu(A) > 0$. Consequently, according to Lemma 4, there is no uncountable pairwise almost disjoint collection of sets of positive upper density. So, in order to get the desired contradiction, it is sufficient to build an uncountable collection of pairwise almost disjoint subsets of $\mathbb{N}$ of positive upper density. We will do this even with $\delta^*(B_\theta) = 1$ for all members B_θ of that uncountable collection.

In order to do this, let us construct a tree of subsets of upper density one A_1, A_2, $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, $A_{2,2}$, $A_{1,1,1}$, etc. as follows. At first, using Lemma 2 split $\mathbb{N} = A_1 \sqcup A_2$ in such a way that $\delta^*(A_1) = \delta^*(A_2) = 1$. Then, using the same lemma, split each of them: $A_1 = A_{1,1} \sqcup A_{1,2}$, $A_2 = A_{2,1} \sqcup A_{2,2}$ with $\delta^*(A_{1,1}) = \delta^*(A_{1,2}) = \delta^*(A_{2,1}) = \delta^*(A_{2,2}) = 1$. Next, split each of these four sets in two new sets, etc.

For every sequence $\theta = (\theta_1, \theta_2, \dots) \in \{1, 2\}^{\mathbb{N}}$ the corresponding branch

$$A_{\theta_1}, A_{\theta_1,\theta_2}, A_{\theta_1,\theta_2,\theta_3}, \dots$$

is a decreasing sequence of sets of upper density one, hence Lemma 3 comes in play. Namely, there is a set $B_\theta \subset \mathbb{N}$ with $\delta^*(B_\theta) = 1$ such that $\#(B_\theta \setminus A_{\theta_1,\dots,\theta_n}) < \infty$ for every $n \in \mathbb{N}$. This collection $\{B_\theta : \theta \in \{1,2\}^{\mathbb{N}}\}$ is uncountable (in fact, of continuum cardinality) and pairwise almost disjoint, which completes the proof.

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