Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
Том 84, 2016, с.31–45

Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics and Mechanics"
Vol. 84, 2016, p. 31–45

УДК 515.12

Semi-classical analysis for proof extinction-property in finite time of solutions for parabolic equations with homogeneous main part and degenerate absorption potential

K. Stiepanova

Simon Kuznets Kharkiv National University of Economics (KhNUE)
ave. Science, 9-a, Kharkiv, 61166, Ukraine
stepanova.ekaterina@hneu.net

We study the behavior of solutions of parabolic equation with double nonlinearity and a degenerate absorption term.
The main topic of interest is the property of finite time extinction, i.e., the solution vanish after finite time.
Keywords: degenerate nonlinear parabolic equation, diffusion–absorption, extinction-property of solutions, semi-classical analysis.

Степанова Е. В. **Полуклассический анализ при доказательстве свойства затухания решения за конечное время для параболических уравнений с однородной главной частью и вырождающимся абсорбционным потенциалом.** Изучается поведение решений параболического уравнения с двойной нелинейностью и вырождающимся потенциалом.
Основной интерес представляет собой свойство затухания решения за конечное время, то есть, зануление решения по истечению конечного времени.
Ключевые слова: вырожденное нелинейное параболическое уравнение, диффузия–абсорбция, свойство затухания решений, полуклассический анализ.
Стєпанова К. В. **Напівкласичний аналіз при доведенні властивості згасання розв'язків за скінчений час для параболічних рівнянь з однорідною головною частиною та вироджуваним абсорбційним потенціалом.** Вивчається поведінка розв'язків параболічних рівнянь з подвійною нелінійністю та вироджуваним потенціалом.
Основний інтерес являє собою властивість згасання розв'зку за скінчений час, тобто, занулення розв'язку за скінчений час.
Ключові слова: вироджене нелінійне параболічне рівняння, дифузія–абсорбція, властивість згасання розв'язків, напівкласичний аналіз.
2000 Mathematics Subject Classification 35A25, 35B40.

© Stiepanova K., 2016

1. Introduction

The theory of quasilinear parabolic equations has been developed since the 50-s of the 19th century. The properties of these equations differ greatly from those of linear equations. These differences were revealed in the scientific papers of the mathematicians: Barenblatt G. I., Oleinic O. A., Kalashnikov A. S., Zhou Yu Lin and others. The properties under consideration are the final velocity of propagation of the support of the solutions, time compact support property and long-time extinction of solutions in finite time and so on. Hundreds of outstanding scientists all over the world closely scrutinize these properties (V. A. Kondratiev, G. A. Iosif'yan, E. V. Radkevich, J. I. Diaz, L. Veron, A. E. Shishkov, B. Helffer, Y. Belaud, M. Fila, D. Andreucci, V. Vespri, A.F. Tedeev and others). The most important aspect of such investigations is the description of structural conditions affecting the appearance and disappearance of various non-linear phenomena.

Our investigations are devoted to the study of the extinction of solutions in finite time to initial-boundary value problems for a wide classes of nonlinear parabolic equations of the second orders with a degenerate absorption potential, whose presence plays a significant role for the mentioned nonlinear phenomenon.

This paper is organized as follows:

(1) Introduction.
(2) Brief history of the problem.
(3) The problem statement.
(4) Main Result.
(5) The proof of main result.
(6) Appendix.
Acknowledgements.
References.

2. Brief history of the problem

As well known the extinction property means that any solution of the mentioned equation vanishes in Ω in a finite time.

The questions of a detailed characterization of the effect of extinction of a solution (estimates of the extinction time, asymptotic behavior near the extinction time, etc.) for various classes of semilinear parabolic equations of the diffusion–absorption type were studied in many works (see, e.g., [1]–[6] and references therein). For example, for the following equation

$$\partial_t u - \Delta u + a_0(x) u^\lambda = 0,$$

the extinction property in a finite time was studied by several authors. In fact this type of equation is a simple model to understand some phenomelogical properties of nonlinear heat conduction.

- It is well-known that in case of non-degenerate absorption potential, i.e.,

$$\text{when} \quad a_0(x) \geq c = const > 0$$

  the solution $u(t,x)$ of parabolic equation of non-stationary diffusion with double nonlinearity and a degenerate absorption term vanishes for

$$t \geq T_0 = \frac{\|u_0\|_{L_\infty}^{1-\lambda}}{c(1-\lambda)},$$

  where u_0 is initial data from Cauchy condition. This fact was proved by J. Diaz, L. Veron, S. Antontsev, S.I. Shmarev (see, for example, works [7] and [8]).

  It is very important to note here, that on the opposite (see papers of M. Cwikel [9], L. Evans, B. Gidas), if we assume that absorption potential is identically equal to zero: $a_0(x) \equiv 0$ for any x from some connected open subset $\omega \subset \Omega$, then there exists solution which never vanish on whole Ω, as any solution $u(x,t)$ of corresponding equation

$$\partial_t u - \Delta u = 0 \ \text{ in } \ \omega \times (0, \infty)$$

  is bounded from below by

$$\sigma \exp(-t\lambda_\omega)\varphi_\omega(x) \ \text{ on } \ \omega \times (0,\infty),$$

  where

$$\sigma = \operatorname{ess\,inf}_{\omega} u_0 > 0,$$

  λ_ω and φ_ω are first eigenvalue and corresponding eigenfunction of $-\Delta$ in $W_0^{1,2}(\omega)$.

  Obviously, that between those two cases there exists a wide class of situations. Thus, an open problem is to find sharp border which distinguish two different properties.

- The paper [10] V.A. Kondratiev and L. Veron must be considered as the first one where the extinction-property in a finite time was systematically investigated for a semilinear parabolic equation in the case of a non-constant strong absorption term, depending both on the media and the temperature

u (i.e. in the case of general potential $a_0 \geq 0$)). They used the fundamental states of the associated Schrödinger operator

$$\mu_n = \inf \left\{ \int_\Omega (|\nabla\psi|^2 + 2^n a_0(x)\psi^2)\, dx \ : \right.$$
$$\left. \psi \in W^{1,2}(\Omega), \quad \int_\Omega \psi^2 dx = 1 \right\}, \quad n \in N,$$

and proved that, if

$$\sum_{n=0}^{\infty} \mu_n^{-1} \ln(\mu_n) < \infty,$$

then Cauchy-Neumann problem for non-stationary diffusion with absorption term with possesses the extinction property.

But, unfortunately, under this form obtained result is not easy to apply.

- Y. Belaud, B. Helffer and L. Veron [11] obtained an explicit sufficient condition in the term of potential $a_0(x)$ which imply that any solution of above equations (with $0 \leq \lambda < 1$) vanishes in finite time. In the work [11] also establish a series of sufficient conditions on $a_0(x)$ which imply that any supersolution with positive initial data does not to vanish identically for any positive t. The method in [11] was based on the so-called semiclassical analysis [12], which uses sharp estimates of the spectrum of some Shrödinger operators and it was also assumed that solution has a certain regularity (as, in particular, in their approach the exact upper estimates of $\|u(t,x)\|_{L_\infty(\Omega)}$) were used). Unfortunately, such an estimate is difficult to obtain or just is unknown for solutions of equations of more general structure than we considered above.
- A. Shishkov and Y. Belaud (see the paper [13]) were the first who investigated the initial-boundary-value problem to mentioned above equation with degenerated absorption potential with the help of two different methods. The first one is a variant of a local energy method (for a radial potential), which uses no "additional" properties of regularity of solutions. And the second one is derive from semiclassical limits of some Shrödinger operators (for any degenerate potential).

So, in this article we consider the behavior of solutions for a much more general class of nonlinear equations which need not satisfy any comparison principle between solutions, namely we study the parabolic equation of non-stationary diffusion with double nonlinearity and a degenerate absorption term:

$$\left(|u|^{q-1}u\right)_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla_x u|^{q-1}\frac{\partial u}{\partial x_i}\right) + a_0(x)|u|^{\lambda-1}u = 0 \quad \text{in} \quad \Omega \times (0,T),$$

where Ω is bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, $0 \in \Omega$, $a_0(x) \geq d_0 \exp\Big(-\frac{\omega(|x|)}{|x|^{q+1}}\Big)$, $x \in \Omega \setminus \{0\}$, $d_0 = const > 0$, $0 \leq \lambda < q$, $\omega(\cdot) \in C([0,+\infty))$, $\omega(0) = 0$, $\omega(\tau) > 0$ when $\tau > 0$. Modifying the semiclassical analysis [13] and [14], we obtain a condition on the function $\omega(\cdot)$ that ensures the extinction.

3. The problem statement

Let Ω is C^1 a bounded connected open set of $\mathbb{R}^N$ ($N \geqslant 1$). The aim of this paper is to investigate the time vanishing properties for energy solutions to initial–boundary value problems for the quasi-linear parabolic equation with neutral diffusion:

$$\begin{cases} \left(|u|^{q-1}u\right)_t - \sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla_x u|^{q-1}\frac{\partial u}{\partial x_i}\right) + a_0(x)|u|^{\lambda-1}u = 0 \text{ in } \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x) \text{ on } \Omega. \end{cases} \tag{3.1}$$

The parameters of the equation satisfy the following relationships: $0 \leq \lambda < q$, the absorption potential $a_0(x)$ is a non-negative continuous function, and $u_0 \in L_{q+1}(\Omega)$. It is assumed also that the origin 0 belongs to Ω and that $a_0(x)$ degenerates at the origin.

Definition 1 *Following [15], an energy solution of problem* (3.1) *is the function*

$$u(t,x) \in L_{q+1,\,loc}\big([0,+\infty); W^1_{q+1}(\Omega)\big)$$

such that:

$$\frac{\partial}{\partial t}\big(|u|^{q-1}u\big) \in L_{\frac{q+1}{q},\,loc}\big([0,+\infty); \big(W^1_{q+1}(\Omega)\big)^*\big), \quad u(0,x) = u_0(x)$$

and satisfying the following integral identity:

$$\int_0^T \langle (|u|^{q-1}u)_t, \varphi\rangle dt + \int_0^T \int_\Omega \left(\sum_{i=1}^N |\nabla_x u|^{q-1}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} + a_0(x)|u|^{\lambda-1}u\varphi\right) dxdt = 0$$

for arbitrary $\varphi(t,x) \in L_{q+1,\,loc}([0,+\infty); W^1_{q+1}(\Omega))$ $\forall T < +\infty$.

Remark 1 *In the integral equality of Definition 1, $\langle\cdot,\cdot\rangle$ stands for the bilinear operation of pairing of elements of the space V and its dual V^*.*

Remark 2 *We note that the existence of an energy (weak) solution of problem* (3.1) *follows from results in [16].*

Definition 2 *If for any solution $u(x,t)$ of the mentioned problem exist $0 < T < +\infty$ such that $u(x,t) = 0$ a.e. in Ω $\forall t \geq T$, then solution vanishes in Ω in a finite time.*

Let for an arbitrary potential $a_0(x)$ from (3.1) exist radial minorant

$$a_0(x) \geq d_0 \exp\Big(- \frac{\omega(|x|)}{|x|^{q+1}}\Big) := a(|x|) \quad \forall\, x \in \Omega, \quad d_0 = const > 0, \tag{3.2}$$

where $\omega(\cdot)$ is a continuous function on $[0, +\infty)$, that is a continuously differentiable on the $(0, +\infty)$, a nondecreasing function. We also suppose that function $\omega(s)$ from condition (3.2) satisfies the conditions:

(A) $\quad \omega(\tau) > 0 \quad \forall \tau > 0,$

(B) $\quad \omega(0) = 0,$

(C) $\quad \omega(\tau) \leq \omega_0 = const < \infty \quad \forall \tau \in \mathbb{R}^1_+$

4. Main Result

The main result of the present work is the following theorem.

Theorem 1 *Let $0 \leq \lambda < q$ in equation from* (3.1)*, initial data $u_0(x) \in L_{q+1}(\Omega)$, function $\omega(\cdot)$ from* (3.2) *satisfy assumptions* (A), (B), (C) *and the main condition of Dini type:*

$$\int\limits_{0+} \frac{\omega(\tau)}{\tau} d\tau < \infty. \tag{4.1}$$

Suppose also that $\omega(\cdot)$ satisfies the following technical condition

$$\omega(\tau) \geq \tau^{q+1-\delta} \qquad \forall\, \tau \in (0, \tau_0), \; 0 < \delta < q. \tag{4.2}$$

Then an arbitrary energy solution $u(x,t)$ of the problem (3.1) *vanishes in finite time.*

Remark 3 *Note, that Theorem 1 is a generalization of the corresponding statement, which was obtained in [13] and coincides with it under $q = 1$.*

Remark 4 *In addition, also note here that result, which was obtained in [17] by a local energy estimates (where the author established a condition of the Dini type for the function $\omega(\cdot)$:*

$$\int\limits_0^c \frac{\omega(\tau)}{\tau} d\tau < \infty,$$

ensuring the extinction of an arbitrary solution in a finite time, was found as well) coincides with Theorem 1 of this article. But on the contrary with [17] the proof here is carried out by using a different technique – in the spirit of paper [13]. As we noticed above the proof of Theorem 1 is based on some variant of the semi-classical analysis, which was developed, particulary in [18, 19, 9, 12, 11, 13].

5. The proof of main result

First, we introduce for $h > 0$ and $\alpha > 0$ the following spectral characteristics:

$$\lambda_1(h) = \inf \left\{ \int_\Omega |\nabla v|^{q+1} + h^{-(q+1)} a(|x|)|v|^{q+1}\, dx : \ v \in W^{1,q+1}(\Omega), \right.$$
$$\left. ||v||_{L_{q+1}(\Omega)} = 1 \right\},$$

and

$$\mu(\alpha) = \lambda_1(\alpha^{\frac{q-\lambda}{q+1}}).$$

We define

$$r(z) = a^{-1}(z)$$

or equivalently

$$z = a(r(z)) \ \text{ and } \ \rho(z) = z(r(z))^{q+1} \ \text{ for } \ z \ \text{ small enough.}$$

Scheme of the proof:

1) The first step in the proof of Theorem 1 is the two-side estimation of ρ^{-1} in a neighbourhood of zero.

2) We will use the following statement for spectral characteristics $\lambda_1(h)$.

Lemma B (Corollaries 2.28, 2.31 in [14]) *Under assumptions $(A)-(C)$ and (4.2), there exist four positives constants C_1, C_2, C_3 and C_4 such that*

$$C_1 h^{-(q+1)} \rho^{-1}(C_2 h^{q+1}) \le \lambda_1(h) \le C_3 h^{-(q+1)} \rho^{-1}(C_4 h^{q+1})$$

for $h > 0$ small enough.

3) Then, due to the two-side estimation of ρ^{-1} from first step of our proof, we continue inequality in Lemma **B** for $h = \alpha^{\frac{q-\lambda}{q+1}}$.

4) Finally, we will check up that the condition from the following Theorem is satisfies:

Theorem BHV (Theorem 2.2 in [11], p. 50). *Under assumptions $(A)-(C)$, if there exists a decreasing sequence (α_n) of positive real numbers such that*

$$\sum_{n=0}^{+\infty} \frac{1}{\mu(\alpha_n)} \left(\ln(\mu(\alpha_n)) + \ln\left(\frac{\alpha_n}{\alpha_{n+1}}\right) + 1 \right) < +\infty,$$

then an arbitrary energy solution $u(x,t)$ of the problem (3.1) vanishes in finite time.

Lemma 1 *Under assumptions (A)-(C) and* (4.2) *there holds*

$$\frac{s}{(1+\gamma)}\ln\left(\frac{d_0}{s}\right)\left[\omega\left(\left(\frac{\omega_0(1+\gamma)}{\ln\left(\frac{d_0}{s}\right)}\right)^{\frac{1}{q+1}}\right)\right]^{-1} \le \rho^{-1}(s) \le$$
$$\le s\ln\left(\frac{d_0}{s}\right)\left[\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{s}\right)}\right)^{\frac{1}{\delta}}\right)\right]^{-1}, \quad (5.1)$$

for arbitrary $\gamma > 0$, for all $s > 0$ small enough.

Proof:

Since ω is a nondecreasing function, from Lemma 4 it follows that

$$\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}}\right) \le \omega(r(z)) \le \omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}\right).$$

Therefore, substituting the definition of $\omega(r)$ (see (6.3)), we obtain

$$\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}}\right) \le (r(z))^{q+1}\ln\left(\frac{d_0}{z}\right) \le \omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}\right),$$

or

$$\frac{1}{\ln\left(\frac{d_0}{z}\right)}\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}}\right) \le (r(z))^{q+1} \le \frac{1}{\ln\left(\frac{d_0}{z}\right)}\omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}\right).$$

It follows the estimate for $\rho(z)$ (as $\rho(z) = z(r(z))^{q+1}$ for z small enough):

$$z\frac{1}{\ln\left(\frac{d_0}{z}\right)}\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}}\right) \le \rho(z) \le z\frac{1}{\ln\left(\frac{d_0}{z}\right)}\omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}\right). \quad (5.2)$$

By an easy calculation, we have

$$\rho(z)\ln\left(\frac{d_0}{z}\right)\left[\omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}\right)\right]^{-1} \le z \le$$
$$\le \rho(z)\ln\left(\frac{d_0}{z}\right)\left[\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}}\right)\right]^{-1}. \quad (5.3)$$

Let here and further $\rho^{-1}(s) = z$. Substituting $z = \rho^{-1}(s)$ in (5.3) yields

$$s \ln\left(\frac{d_0}{\rho^{-1}(s)}\right)\left[\omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{\rho^{-1}(s)}\right)}\right)^{\frac{1}{q+1}}\right)\right]^{-1} \le \rho^{-1}(s) \le$$

$$\le s \ln\left(\frac{d_0}{\rho^{-1}(s)}\right)\left[\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{\rho^{-1}(s)}\right)}\right)^{\frac{1}{\delta}}\right)\right]^{-1}.$$

In consideration of such fact, that from (5.2), we have for z small enough, $\rho(z) \ge z$, which gives $\rho^{-1}(s) \le s$. Since $\omega(\cdot)$ is a nondecreasing function, due to (6.4), we get

$$s \ln\left(\frac{d_0}{s^{-\frac{1}{1+\gamma}}}\right)\left[\omega\left(\left(\frac{\omega_0}{\ln\left(\frac{d_0}{s^{-\frac{1}{1+\gamma}}}\right)}\right)^{\frac{1}{q+1}}\right)\right]^{-1} \le \rho^{-1}(s) \le$$

$$\le s \ln\left(\frac{d_0}{s}\right)\left[\omega\left(\left(\frac{1}{\ln\left(\frac{d_0}{s}\right)}\right)^{\frac{1}{\delta}}\right)\right]^{-1},$$

which completes the proof. □

Now let us prove the two-sided estimate for $\mu(\alpha)$.

Lemma 2 *Under assumptions* $(A)-(C)$ *and* (4.2)*, there exist positives constants* K_1'', K_2'' *and* K_3'' *such that*

$$K_1'' \ln\left(\frac{1}{\alpha}\right)\left[\omega\left(\frac{K_2''}{\left(\ln\left(\frac{1}{\alpha}\right)\right)^{\frac{1}{q+1}}}\right)\right]^{-1} \le \mu(\alpha) \le K_3'' \ln\left(\frac{1}{\alpha}\right)^{\frac{q+1}{\delta}}. \tag{5.4}$$

for $\alpha > 0$ *small enough.*

Proof: Due to the two-side estimation of ρ^{-1} (5.1) from Lemma 1 (it was the first point of our proof), we continue inequality in Lemma **B** for $h = \alpha^{\frac{q-\lambda}{q+1}}$, and we get

$$K_1 \ln\left(\frac{d_0}{C_2 h^{q+1}}\right)\left[\omega\left(\frac{K_2}{\left(\ln\left(\frac{d_0}{C_2 h^{q+1}}\right)\right)^{\frac{1}{q+1}}}\right)\right]^{-1} \le \lambda_1(h) \le$$

$$\le K_3 \ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\left[\omega\left(\frac{1}{\left(\ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\right)^{\frac{1}{\delta}}}\right)\right]^{-1},$$

and since $\omega^{-1}(r) \le r^{-(q+1-\delta)}$ due to (4.2):

$$\left[\omega\left(\frac{1}{\left(\ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\right)^{\frac{1}{\delta}}}\right)\right]^{-1} \le \left[\frac{1}{\left(\ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\right)^{\frac{1}{\delta}}}\right]^{-(q+1-\delta)}.$$

Let us consider the right side of inequality:

$$\left[\frac{1}{\left(\ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\right)^{\frac{1}{\delta}}}\right]^{-(q+1-\delta)} = \left(\ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\right)^{-\frac{1}{\delta}\cdot -(q+1-\delta)} = \left(\ln\left(\frac{d_0}{C_4 h^{q+1}}\right)\right)^{\frac{q+1-\delta}{\delta}}$$

as a result:

$$K_1 \ln\left(\frac{d_0}{C_2 h^{q+1}}\right)\left[\omega\left(\frac{K_2}{\left(\ln\left(\frac{d_0}{C_2 h^{q+1}}\right)\right)^{\frac{1}{q+1}}}\right)\right]^{-1} \le \lambda_1(h) \le$$

$$\le K_3 \ln\left(\frac{d_0}{C_4 h^{q+1}}\right) \cdot \ln\left(\frac{d_0}{C_4 h^{q+1}}\right)^{\frac{q+1-\delta}{\delta}},$$

finally, we have

$$K_1 \ln\left(\frac{d_0}{C_2 h^{q+1}}\right)\left[\omega\left(\frac{K_2}{\left(\ln\left(\frac{d_0}{C_2 h^{q+1}}\right)\right)^{\frac{1}{q+1}}}\right)\right]^{-1} \le \lambda_1(h) \le$$

$$\le K_3 \ln\left(\frac{d_0}{C_4 h^{q+1}}\right)^{1+\frac{q+1-\delta}{\delta}},$$

which leads to

$$K_1' \ln\left(\frac{1}{h}\right)\left[\omega\left(\frac{K_2'}{\left(\ln\left(\frac{1}{h}\right)\right)^{\frac{1}{q+1}}}\right)\right]^{-1} \le \lambda_1(h) \le K_3' \ln\left(\frac{1}{h}\right)^{\frac{q+1}{\delta}}. \tag{5.5}$$

The real number α is defined by

$$h = \alpha^{\frac{q-\lambda}{q+1}}$$

and thus we complete the proof. □

Lemma 3 *Under* $(A)-(C)$ *with* (4.2), *if*

$$\sum_{n=n_0}^{+\infty} \frac{\omega\left(\frac{1}{(n\ln n)^{\frac{1}{q+1}}}\right)}{n} < +\infty, \tag{5.6}$$

then all solutions of (3.1) *vanish in a finite time. Moreover,*

$$\sum_{n=n_0}^{+\infty} \frac{\omega\left(\frac{1}{(n\ln n)^{\frac{1}{q+1}}}\right)}{n} < +\infty \iff \int_{0+} \frac{\omega(x)}{x}\,dx < +\infty. \tag{5.7}$$

Proof:

From Theorem BHV, if (α_n) is a decreasing sequence of positive real numbers and

$$\sum_{n=n_0}^{+\infty} \frac{\omega\left(\frac{C_2''}{\left(\ln\left(\frac{1}{\alpha_n}\right)\right)^{\frac{1}{q+1}}}\right)}{\ln\left(\frac{1}{\alpha_n}\right)} \left[\ln\left(\ln\left(\frac{1}{\alpha_n}\right)\right) + \ln\left(\frac{\alpha_n}{\alpha_{n+1}}\right) + 1\right] < +\infty,$$

then all the solutions of (3.1) vanish in a finite time.

Let $\alpha_n = n^{-Kn}$ for some $K > 0$, then:

$$\ln\left(\frac{1}{\alpha_n}\right) = Kn\ln n,$$

$$\ln\left(\ln\left(\frac{1}{\alpha_n}\right)\right) \sim \ln n,$$

because

$$\ln\left(\ln\left(\frac{1}{\alpha_n}\right)\right) \sim \ln(Kn\ln n) = \ln n + \ln\ln n + \ln K << \ln n + \ln n + const \sim \ln n,$$

and

$$\ln\left(\frac{\alpha_n}{\alpha_{n+1}}\right) = -Kn\ln n + K(n+1)\ln(n+1) =$$

$$= Kn\ln(\frac{n+1}{n}) + K\ln(n+1) \sim Kn\frac{1}{n} + K\ln n \sim K\ln n,$$

and, obviously,

$$1 = o(\ln n),$$

which leads us to (5.6).

Let's show that (5.7) is true. In fact, it is easy to see that

$$\sum_{n=n_0}^{+\infty} \frac{\omega\left(\frac{1}{(n\ln n)^{\frac{1}{q+1}}}\right)}{n} < +\infty \iff \int_{n_0}^{+\infty} \frac{\omega\left(\frac{1}{(t\ln t)^{\frac{1}{q+1}}}\right)}{t}\, dt < +\infty.$$

Now, let

$$x = (t\ln t)^{-\frac{1}{q+1}},$$

then

$$dx = -\frac{1}{(q+1)(t\ln t)^{\frac{1}{q+1}}} \cdot \frac{\ln\, t+1}{\ln t} \cdot \frac{dt}{t},$$

since

$$\frac{ln\, t}{lnt+1} \to 1 \ \text{ as } \ t \to +\infty,$$

hence

$$\int_{n_0}^{+\infty} \frac{\omega\left(\frac{1}{(t\ln t)^{\frac{1}{q+1}}}\right)}{t}\, dt \ \text{ is finite } \ \text{if and only if } \ (q+1)\int_0^c \frac{\omega(x)}{x}\, dx < +\infty,$$

which completes the proof of our main Theorem. □

6. Appendix

Lemma 4 *Let the function* $\omega(\cdot)$ *from* (3.2) *satisfy (A)–(C) and technical condition* (4.2)*. Then for* $z = a(r(z)) > 0$ *small enough it is true the following estimate:*

$$\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}} \le r(z) \le \left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}. \tag{6.1}$$

Proof:

Starting from condition (4.2) ($\omega(\tau) \ge \tau^{q+1-\delta}\ \forall\, \tau \in (0, \tau_0),\ 0 < \delta < q$) and just using assumption (C) on the function $\omega(\cdot)$ ($\omega(\tau) \le \omega_0 = const < \infty\ \forall \tau \in \mathbb{R}^1_+$) we easily arrive at

$$r^{q+1-\delta} \le \omega(r) \le \omega_0. \tag{6.2}$$

Since for $z = a(r(z))$, from (3.2):

$$a(r(z)) = d_0 \exp\Big(-\frac{\omega(r(z))}{(r(z))^{q+1}}\Big), \quad d_0 = const > 0,$$

it follows that
$$\frac{z}{d_0} = \exp\Big(- \frac{\omega(r(z))}{(r(z))^{q+1}}\Big),$$
then
$$\ln\left(\frac{d_0}{z}\right) = -\frac{\omega(r(z))}{(r(z))^{q+1}},$$
which equivalently
$$\ln\left(\frac{d_0}{z}\right) = \frac{\omega(r(z))}{(r(z))^{q+1}}.$$

So, we conclude that
$$\omega(r(z)) = (r(z))^{q+1}\ln\left(\frac{d_0}{z}\right). \tag{6.3}$$
Due to (6.2) and (6.3) it is easy to see the relationship
$$r(z)^{q+1-\delta} \leq (r(z))^{q+1}\ln\left(\frac{d_0}{z}\right) \leq \omega_0,$$
which completes the proof of Lemma 4, because the last one means:
$$\frac{r(z)^{q+1-\delta}}{\ln\left(\frac{d_0}{z}\right)} \leq (r(z))^{q+1}$$
and
$$(r(z))^{q+1}\ln\left(\frac{d_0}{z}\right) \leq \omega_0.$$
This fact give us
$$\frac{1}{\ln\left(\frac{d_0}{z}\right)} \leq (r(z))^{\delta}$$
and
$$(r(z))^{q+1} \leq \frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)} \qquad \text{respectively.}$$

Hence,
$$\left(\frac{1}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{\delta}} \leq r(z) \quad \text{and} \quad r(z) \leq \left(\frac{\omega_0}{\ln\left(\frac{d_0}{z}\right)}\right)^{\frac{1}{q+1}}.$$
□

Lemma 5 *Let the function* $\omega(\cdot)$ *from* (3.2) *satisfy (A)–(C) and technical condition* (4.2). *Then the following inequality hold for any* $\gamma = cont > 0$*:*
$$\rho^{-1}(s) \leq s^{\frac{1}{1+\gamma}}. \tag{6.4}$$

Proof: By using (6.1) and $\rho(z) = z(r(z))^{q+1}$,

$$\rho(z) \geq z \left(\frac{1}{\ln\left(\frac{d_0}{z}\right)} \right)^{\frac{q+1}{\delta}} \Longleftrightarrow \frac{1}{\rho(z)} \leq \frac{1}{z} \left(\ln\left(\frac{d_0}{z}\right) \right)^{\frac{q+1}{\delta}},$$

or equivalently,

$$\ln\left(\frac{1}{\rho(z)}\right) \leq \ln\left(\frac{1}{z}\right) + \frac{q+1}{\delta} \ln\left(\ln\left(\frac{d_0}{z}\right)\right).$$

Due to

$$\ln(\ln(z^{-1})) << \ln z^{-1} \quad \text{for} \quad z \quad \text{small enough},$$

we obtain

$$\ln\left(\frac{1}{\rho(z)}\right) \leq (1+\gamma) \ln\left(\frac{1}{z}\right) \Longleftrightarrow \rho(z) \geq z^{1+\gamma} \Longrightarrow \rho^{-1}(s) \leq s^{\frac{1}{1+\gamma}}, \qquad (6.5)$$

which completes the proof of Lemma 5. □

Acknowledgements.
This work was financial supported in part by Akhiezer Fund.
The author thank V.I. Korobov whose critical revision of the paper allows to improve it essentially.
The author is very grateful to V.A. Gorkavyy and V.A. Rybalko for useful discussions and valuable comments.

REFERENCES

1. Bandle C., Stakgold I. The formation of the dead core in parabolic reaction-diffusion problems // Trans. Amer. Math. Soc., 1984. – V. 286, **1**. – P. 275–293.
2. Chen Xu-Yan, Matano H., Mimura M. Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption // J. Reine Angew. Math., 1995. – V. 459, **1**. – P. 1–36.
3. Friedman A., Herrero M.A. Extinction properties of semilinear heat equations with strong absorption // J. Math. Anal. Appl., 1987. – V. 124, **2**. – P. 530–546.
4. Knerr B.F. The behavior of the support of solutions of the equation of nonlinear heat conduction with absorption in one dimension // Trans. Amer. Math. Soc., 1979. – V. 249, **2**. – P. 409–424.
5. Payne L.E., Improperly Posed Problems in Partial Differential Equations. – SIAM, Philadelphia, 1975. – 62 p.
6. Straughan B. Instability, Nonexistence and Weighted Energy Methods in Fluid Dynamics and Related Theories. – Pitman, London, 1982. – 169 p.

7. Diaz J., Veron L. Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. // Trans. Amer. Math. Soc., 1985. – **290:2**. – P. 787–814.
8. Antontsev S., Diaz J., Shmarev S.I. The Support Shrinking Properties for Solutions of Quasilinear Parabolic Equations with Strong Absorption Terms. // Annales de la Faculte des Sciences de Toulouse Math., 1995. – **6:4**. – P. 5–30.
9. Cwickel M. Weak type estimates for singular value and the number of bound states of Schrödinger operator // Ann. Math., 1977. – **106**. – P. 93–100.
10. Kondratiev V.A., Véron L. Asymptotic behaviour of solutions of some nonlinear parabolic or elliptic equations. // Asymptotic Analysis., 1997. – **14**. – P. 117-156.
11. Belaud Y. Helffer B., Véron L. Long-time vanishing properties of solutions of sublinear parabolic equations and semi-classical limit of Schrцdinger operator // Ann. Inst. Henri Poincarré Anal. nonlinear, 2001. – V. 1, **18**. – P. 43–68.
12. Helffer B. Semi-classical analysis for the Schrödinger operator and applications. – Lecture Notes in Math. 1336, Springer-Verlag, 1988. – 107 p.
13. Belaud Y., Shishkov A. Long-time extinction of solutions of some semilinear parabolic equations // J. Differ. Equat., 2007. – **238**. – P. 64–86.
14. Belaud Y. Asymptotic estimates for a variational problem involving a quasilinear operator in the semi-classical limit // Annals of global analysis and geometry, 2004. – **26**. – P. 271 – 313.
15. Alt H.W., Luckhaus S. Quasilinear elliptic-parabolic differential equations // Math. Z., 1983. – V. 183, **3**. – P. 311–341.
16. Bernis F. Existence results for doubly nonlinear higher order parabolic equations on unbounded domain // Math. Am., 1988. – V. 279, **3**. – P. 373–394.
17. Stiepanova K.V. Extinction of solutions for parabolic equations with double nonlinearity and a degenerate absorption potential // Ukrainian Mathematical Journal, 2014. – V. 66, **1**. – P. 89–107.
18. Rosenblyum G. V. Distribution of the discrete spectrum of singular differential operators // Doklady Akad. Nauk USSR, 1972. – **202**. – P. 1012–1015.
19. Lieb E.H., Thirring W. Inequalities for the moments of the eigenvalues of the Schrцdinger Hamiltonian and their relations to Sobolev Inequalities // In Studies in Math. Phys., essay in honour of V. Bargmann, Princeton Univ. Press, 1976. – P. 203–237.

Article history: Received: 27 September 2016; Final form: 28 November 2016; Accepted: 12 December 2016.