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Almost automorphic derivative of an almost automorphic function

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In this article are obtained conditions when the derivative of a continuous almost automorphic (an asymptotically almost automorphic, an almost periodic, an asymptotically almost periodic) function remains a continuous almost automorphic (an asymptotically almost automorphic, an almost periodic, an asymptotically almost periodic) function, respectively.
Keywords: derivative, an almost automorphic, an asymptotically almost automorphic function.

Дімітрова-Бурлаєнко С. Д. **Майже автоморфна похідна майже автоморфної функції.** У цій статті отримані умови, в яких похідна неперервної майже автоморфної (асимптотично майже автоморфної, майже періодичної, асимптотично майже періодичної) функції залишається неперервною майже автоморфною (асимптотично майже автоморфною, майже періодичною, асимптотично майже періодичною) функцією, відповідно.
Ключові слова: похідна, майже автоморфна, асимптотично майже автоморфна функція.

Димитрова-Бурлаенко С. Д. **Почти автоморфная производная почти автоморфной функции.** В этой статье получены условия, при которых производная непрерывной почти автоморфной (асимптотически почти автоморфной, почти периодической, асимптотически почти периодической) функции остается непрерывной почти автоморфной (асимптотически почти автоморфной, почти периодической, асимптотически почти периодической) функцией, соответственно.
Ключевые слова: производная, почти автоморфная, асимптотически почти автоморфная функция.

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1. Introduction

In papers published earlier ([4],[5]) in proofs of almost automorphy of the derivative of almost automorphic function used its uniform continuity. In this paper, we consider an alternative, weaker conditions in which almost automorphic preserved under differentiation.

In this article some results are obtained for the almost automorphic (a.a), almost periodic (a.p.), respectively asymptotically almost automorphic (a.a.a.) asymptotically almost periodic (a.a.p) function. The results can be divided into two groups: The first is based on the condition $[f_a(t)]\prime = [f\prime]_a(t)$, and the second on the uniform continuity of the derivative in a neighborhood, it generated an almost periodic function. Just it shows that differentiation does not change the structure of the asymptotic functions. Derivative asymptotically almost periodic (almost automorphic) function, as well as the original function of the sum of the derivative is almost periodic (almost automorphic) function and the derivative term, converging to zero at infinity.

Questions about the differentiation of Levitan almost periodic functions are described earlier in [9]. They are based on the proposition 4, below.

2. Basic definitions

All studied functions are defined on the real axis and take values in a separable Freshet space Y. Topology of the space is given by increasing the counting system of semi-norms $p_s(y)$, $p_s(y) \leq p_{s+1}(y)$,$s = 1, 2, 3, ...$,$y \in Y$. The metric is specified using quasi-norm

$$\| y - z \| = \sum_{s=1}^{\infty} \frac{p_s(y-z)}{[1 + p_s(y-z)]2^s}, y, z \in Y.$$

We will give some definitions for better clarity of the presentation. Many of them were introduced for numerical functions, generalized for abstract functions with values in Banach spaces. In the article all definitions are given for the Freshet spaces.

Let given a number sequence $a = \{a_n\}_{n=1}^{\infty}$. We extract a subsequence $\{a_{n_k}\}_{k=1}^{\infty} \in \{a_n\}_{n=1}^{\infty}$ for which the sequence $\{f(t + a_{n_k})\}_{k=1}^{\infty}$ converges pointwise to a function, i.e. $\lim_{k\to\infty} f(t + a_{n_k}) = f_a(t)$. In the future, we will assume that from the equality $f_a(t) = g_a(t)$, follows the equality $\lim_{k\to\infty} f(t + a_{n_k}) = \lim_{k\to\infty} g(t+a_{n_k})$, $\forall t \in \mathbb{R}$. That is, the equality achieved for the same subsequence $\{a_{n_k}\}_{k=1}^{\infty} \in \{a_n\}_{n=1}^{\infty}$.

The function $f(t)$ is compact if the closure of the set of values of the function $f(t)$ is compact in Y.

Definition 1.([1],[2]) The sequence of functions $\{f_n(t)\}_{n=1}^{\infty}$, $f_n(t) : \mathbb{R} \to Y$ converges quasi uniformly to the function $f(t) : \mathbb{R} \to Y$, if it converges pointwise to the function $f(t)$ and for each $\varepsilon > 0$ and for each index K exist an index M ($K \leq M$), such that

$$\min_{K\leq n\leq M} \| f_n(t) - f(t) \| < \varepsilon, \forall t \in \mathbb{R}.$$

Definition 2.([3],[6]) The sequence of functions $\{f_n(t)\}_{n=1}^{\infty}$ converges to $f(t)$ almost uniformly, if it quasi uniformly converges to $f(t)$ on $\mathbb{R}$ with each of its subsequences.

The term "almost uniformly"is proposed by G. Fichtenholz. For almost periodic functions will use the criterion of Bochner.

Definition 3.[5] A continuous function $f(t)$ is called almost periodic if the family of functions $\{f(t+h_i)\}_{i=1}^{\infty}$ $(-\infty < h_i < \infty)$ is compact in the sense of uniform convergence on the whole real axis.

Definition 4.([11],[12],[4],[7]) A continuous function $f(t) : \mathbb{R} \to Y$ is called (continuous) almost automorphic if for any sequence $\{x'_n\}_{n=1}^{\infty} \in \mathbb{R}$ is an subsequence $\{x_n\}_{n=1}^{\infty} \subset \{x'_n\}_{n=1}^{\infty}$ and exist a (continuous) function $g(t)$, so that

$$\lim_n f(t+x_n) = g(t), \forall t \in \mathbb{R}$$

$$\lim_m g(t-x_m) = f(t), \forall t \in \mathbb{R}.$$

Definition 5.[4] A continuous function $f(t) : [0;+\infty) \to Y$ is called asymptotically almost automorphic /asymptotically almost periodic/, if it can be represented in the form $f(t) = g(t) + w(t)$, where $g(t)$ is a.a. /a.p./ function on the line,$w(t) : [0;+\infty) \to Y$ - a continuous function, which has a limit $\lim_{t\to\infty} w(t) = 0$.

Definition 6.([8],[11],[12]) The set E is relatively dense on the group $\mathbb{R}$, if there are q elements $c_1, c_2, ..., c_q$ such that

$$\mathbb{R} = \bigcup_{i=1}^{q} (c_i + E).$$

For a more clear understanding of the reasoning we formulate some results, on which we base the presentation.

Proposition 1. ([8]) A continuous function $f(t) : \mathbb{R} \to Y$ is almost periodic, if and only if for any $\varepsilon > 0$ the set

$$U_\varepsilon = \{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \| f(t+\tau) - f(t) \| < \varepsilon\}$$

is relatively dense.

Proposition 1 makes it possible to introduce a topology on the group by the set $U_\varepsilon(f)$. On the other hand, any continuous function in this topology is an a.p. function.

Proposition 2. ([11]) Each a.a. function $f(t) : \mathbb{R} \to Y$ is continuous in the topology $\Im_f$, defined by the sets

$$B_{N,\varepsilon} = \{\tau \in \mathbb{R} : \max_{t\in N} \| f(t+\tau) - f(t) \| < \varepsilon\}$$

where N,a compact set of numbers, $\varepsilon > 0$.

Proposition 3. ([11]) Let it is given an a.a. function $f(t) : \mathbb{R} \to Y$ and with its help is introduced the topology $\Im_f$ on $\mathbb{R}$. Any compact function $g(t)$, defined on the group $\mathbb{R}$,which is continuous in the topology $\Im_f$, is an a.a. function.

Propositions 2 and 3 do the same work as a proposition 1, but only for the a.a. functions.

Proposition 4. ([9],[10]) Let $f(t) : \mathbb{R} \to Y$ be differentiable function in the natural topology $\Im_0$ on the axis $\mathbb{R}$ and $f(t)$ is continuous in a weaker topology $\Im$ ($\Im \prec \Im_0$). The derivative $f'(t)$ is continuous in the weak topology $\Im$ if and only if for any $\varepsilon > 0$ and $x \in (-\infty; \infty)$ there was a neigborhood U in the topology $\Im$ and interval $(-\delta; \delta)$ so that

$$\sup_{t \in x+U} \parallel f'(t+h) - f'(t) \parallel < \varepsilon, \forall h \in (-\delta; \delta).$$

3. Main parts

Criterion for almost uniform convergence of numerical functions is given by G. Sirvint ([6], lemma 1.2).For functions with values in a Banach space it can be found in ([3], theorem 5.8).This criterion holds for abstract functions with values in a Freshet space, namely:

Theorem 1. The sequence $\{f_n(t)\}_{n=1}^{\infty}$,$f_n(t) : \mathbb{R} \to Y$ converges almost uniformly to the function $f(t) : \mathbb{R} \to Y$ if and only if:

$$\lim_{n} \underline{\lim}_{m} \parallel f_n(x_m) - f(x_m) \parallel = 0$$

for any set of numbers $\{x_m\}_{m=1}^{\infty}, x_m \in \mathbb{R}$.

Theorem 2. Let the function $f(t) : \mathbb{R} \to Y$ and its derivative $f'(t)$ are continuous a.a. functions. Then each of the functions $f_a(t)$ is differentiable and

$$[f_a(t)]' = [f']_a(t).$$

Proof. The function $f(t)$ and its derivative $f'(t)$ are almost automorphic. Then the function $f(t)$ and its derivative $f'(t)$ are uniformly continuous in the natural topology $\Im_0$ from ([7], lemma 4.1.1).On the axis we introduce the topology $\Im$ using the neighborhoods of the form:

$$B_{N,\varepsilon} = \{\tau \in \mathbb{R} : \sup_{t \in N} \parallel f(t+\tau) - f(t) \parallel < \varepsilon\},$$

$$B'_{M,\varepsilon} = \{\tau \in \mathbb{R} : \sup_{t \in M} \parallel f'(t+\tau) - f'(t) \parallel < \varepsilon\},$$

where N and M are compact sets, $\varepsilon > 0$.The function $f(t)$ and its derivative $f'(t)$ are continuous in the topology $\Im$. This is possible because the two functions are almost automorphic. We introduce the functions

$$\varphi_n(t) = n[f(t + \frac{1}{n}) - f(t)], \varphi_{n,a}(t) = n[f_a(t + \frac{1}{n}) - f_a(t)], n = 1, 2, 3, ...$$

The uniform continuity of the derivative $f'(t)$ implies that the sequence $\varphi_n(t)$, $n = 1, 2, 3, ...$ converges uniformly on the axis to the function $f'(t)$.This means that the sequence $\{\varphi_n(t)-f'(t)\}_{n=1}^{\infty}$ converges to zero almost uniformly. Functions $\varphi_n(t)$ and $f'(t)$ are continuous on the axis in the topology $\Im$ and uniformly continuous in the topology $\Im_0$ and relatively compact. Applying theorem 1 to the sequence $\{\varphi_n(t) - f'(t)\}_{n=1}^{\infty}$ to an arbitrary numerical sequence $a = \{t + a_n\}_{n=1}^{\infty}$ for which

$$\lim_n \lim_k \parallel \varphi_n(t + a_{m_k}) - f'(t + a_{m_k}) \parallel$$

$$= \lim_n \underline{\lim}_m \parallel \varphi_n(t + a_m) - f'(t + a_m) \parallel = 0.$$

Equality

$$\lim_k \lim_n \parallel \varphi_n(t + a_{m_k}) - f'(t + a_{m_k}) \parallel = 0$$

follows from the pointwise convergence of the sequence $\varphi_n(t)$ to the function $f'(t)$. Using the relative compactness of the set values of the derivative we can find a sequence $\{\bar{a_k}\}_{k=1}^{\infty} \in \{a_{n_k}\}_{k=1}^{\infty}$ for which the sequence $f'(t + \bar{a_k})\}_{k=1}^{\infty}$ converges pointwise. Its limit is denoted by $f'_a(t)$, $f'_a(t) = \lim_k f'(t + \bar{a_k})$, $\forall t \in \mathbb{R}$. Then

$$f'_a(t) = \lim_k f'(t + \bar{a_k}) = \lim_k \lim_n \varphi_n(t + \bar{a_k}) = \lim_n \varphi_{n,a}(t) = (f_a(t))'.$$

The theorem is proved.

Theorem 3. Let the function $f(t) : \mathbb{R} \to Y$ be a. a., $\forall a$ $f_a(t)$ is differentiable and the derivative is compact. If each function $f_a(t)$ satisfies

$$[f_a(t)]' = [f']_a(t)$$

then the derivative $f'(t)$ is almost automorphic function.

Proof. For the sequence $a = \{a_m\}_{m=1}^{\infty}$, using the relatively compact range of the function $f(t)$ and the derivative $f'(t)$, we found a subsequence $\{\bar{a_k}\}_{k=1}^{\infty} \subset \{a_m\}_{m=1}^{\infty}$ for which there are all limits:$\lim_n \varphi_{n,a}(t)$, $\lim_k f'(t + \bar{a_n})$. Then

$$\lim_k \lim_n \varphi_n(t + \bar{a_k}) = \lim_k f'(t + \bar{a_k}) = [f']_a(t) = [f_a(t)]' =$$

$$= \lim_n \varphi_{n,a}(t) = \lim_n \lim_k \varphi_n(t + \bar{a_k})$$

or

$$\lim_n \underline{\lim}_k \parallel \varphi_n(t + \bar{a_k}) - f'(t + \bar{a_k} \parallel \leq \lim_n \lim_k \parallel \varphi_n(t + \bar{a_k}) - f'(t + \bar{a_k} \parallel =$$

$$= \lim_n \parallel \varphi_{n,a}(t) - (f'(t))_a \parallel = \lim_n \parallel \varphi_{n,a}(t) - (f_a(t))' \parallel = 0.$$

According to theorem 1 the sequence of functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ converges to $f'(t)$ almost uniformly. All functions $\{\varphi_n(t)\}_{n=1}^{\infty}$ are almost automorphic. Then the limit function $f'(t)$ according to [12] is almost automorphic. □

Remark. If the function $f(t)$ is continuous almost automorphic then functions $\varphi_n(t)$ are continuous almost automorphic function also. According to [12]

(Corollary 2) it follows that the limit is a continuous almost automorphic function, that is, the derivative $f'(t)$ is a continuous almost automorphic function.

Theorem 4. Let the function $f(t) : \mathbb{R} \to Y$ is almost periodic and it has a relatively compact derivative $f'(t)$. The derivative $f'(t)$ is almost periodic if /and only if/any function $f_a(t)$ satisfy the conditions of theorem 2 /respectively theorem 3/.

Proof of necessity. If the derivative is almost periodic, it is relatively compact and uniformly continuous. Any slight shift $f'_a(t)$ of the derivative $f'(t)$ is almost periodic, and therefore it is almost automorphic. The function $f_a(t)$ satisfies the conditions of theorem 2. From theorem 2 it follows $[f_a(t)]' = [f']_a(t)$. $\square$

Proof of sufficiency. Applying theorem 3 we receive that $f'_a(t)$ is an almost automorphic function. Applying Veech theorem [7] to the numerical almost automorphic functions $< y^*, f'_a(t) >$, $y^* \in Y^*$, we find that the function $f'(t)$ is weakly almost periodic. Using relatively compact range of the derivate $f'(t)$, we see that it was strongly almost periodic. $\square$

Theorem 5. Let $f(t) : [0, \infty) \to Y$, $f(t) = g(t) + w(t)$ is an asymptotically almost automorphic function, where $g(t)$ is a continuous almost automorphic function on the line, $w(t)$ - a function that has a limit $\lim_{t\to\infty} w(t) = 0$. Let the derivative $w'(t)$ is uniformly continuous on $[0, \infty)$.The derivative function $f'(t)$ is asymptotically almost automorphic function if and only if for any $a = \{a_n\}_{n=1}^{\infty}$ it satisfies the conditions $[f_a(t)]' = [f']_a(t)$, where all derivatives exist.

Proof of necessity. If the derivative $f'(t) = g'(t) + w'(t)$ is asymptotically almost automorphic function, the continuity $w'(t)$ and the limit 0 at infinity ensure its uniform continuity. Therefore $[w_a(t)]' = [w']_a(t)$. Then, according to theorem 2 follows that the function $g(t) = f(t) - w(t)$ satisfies the condition $[g_a(t)]' = [g']_a(t)$. Consequently

$$[f_a(t)]' = [g_a(t)]' + [w_a(t)]' = [g']_a(t) + [w']_a(t) = [f']_a(t).$$

The conditions for the function $f(t)$ are necessary. $\square$

Proof of sufficiency. Let us consider the sequence $\varphi_{\Delta t}(t)$, $\Delta t \to 0$

$$\varphi_{\Delta t}(t) = \frac{w(t + \Delta t) - w(t)}{\Delta t} = \frac{1}{\Delta t} \int_t^{t+\Delta t} w'(\tau) d\tau.$$

It consists of uniformly continuous functions $\varphi_{\Delta t}(t)$ with the limit of zero to infinity, and it converges uniformly to $w'(t)$. The limit $w'(t)$ is a uniformly continuous function on $[0, \infty)$, and has a limit $\lim_{t\to\infty} w(t) = 0$. Therefore

$$[w_a(t)]' = [w']_a(t).$$

Then the almost automorphic function $g(t) = f(t) - w(t)$ satisfies the condition

$$[g_a(t)]' = [g']_a(t).$$

According to theorem 3 $g'(t)$ is almost automorphic function. Thus, the derivative $f'(t) = g'(t) + w'(t)$ is asymptotically almost automorphic function. $\square$

Theorem 6. Let the function $f(t):\mathbb{R}\to Y$ be almost periodic and has a continuous derivative. Its derivative $f'(t)$ is almost periodic if and only if for any $\varepsilon>0$ and for any point $x\in(-\infty;\infty)$ there exists a neighborhood of zero U_α

$$U_\alpha=\{\tau:\sup_{t\in\mathbb{R}}\parallel f(t+\tau)-f(t)\parallel<\alpha\},\alpha>0$$

interval $(-\delta;+\delta)$ such that:

$$\sup_{t\in x+U_\alpha}\parallel f'(t+h)-f'(t)\parallel<\varepsilon,\forall h\in(-\delta;+\delta).$$

Proof of necessity. On the axis is entered topology $\Im_U$ with the help of the neighborhoods

$$U_\alpha=\{\tau:\max\{\sup_{t\in R}\parallel f(t+\tau)-f(t)\parallel,\sup_{t\in R}\parallel f'(t+\tau)-f'(t)\parallel\}<\alpha\},$$

where $\alpha>0$.

It is weaker than the natural topology $\Im_0$. The function $f(t)$ and its derivative $f'(t)$ are continuous in this topology $\Im_U$, and even uniformly continuous. Let it is given $\varepsilon>0$. From the uniform continuity of the derivative in the natural topology $\Im_0$ there exist $\delta>0$ such that

$$\sup_{t\in\mathbb{R}}\parallel f'(t+h)-f'(t)\parallel<\varepsilon,\forall h<\delta.$$

Then $\forall h\in(-\delta;+\delta)$, $\forall x\in\mathbb{R}$

$$\sup_{t\in x+U_\alpha}\parallel f'(t+h)-f'(t)\parallel<\sup_{t\in\mathbb{R}}\parallel f'(t+h)-f'(t)\parallel<\varepsilon.$$

□

Proof of sufficiency. On the axis is entered the topology $\Im_U$ using the neighborhoods for $\alpha>0$

$$U_\alpha=\{\tau:\sup_{t\in\mathbb{R}}\parallel f(t+\tau)-f(t)\parallel<\alpha\}.$$

In this topology, the function $f(t)$ is continuous. Applying proposition 4, we see that the derivative $f'(t)$ is continuous in the topology $\Im_U$. For the almost periodicity of the derivative according to proposition 1 is sufficient to prove that the set

$$V_\beta=\{\tau:\sup_{t\in\mathbb{R}}\parallel f'(t+\tau)-f'(t)\parallel<\beta\},\forall\beta>0$$

is relatively dense. Assume the contrary, that for some $\varepsilon_0>0$ the set V_{ε_0} is not relatively dense. Using that the set is not relatively dense, we construct a numerical sequence $\{z_n\}_{n=1}^{\infty}$

$$z_1\in\mathbb{R},z_2\notin z_1+V_{\varepsilon_0},z_3\notin(z_1+V_{\varepsilon_0})\cup(z_2+V_{\varepsilon_0}),...$$

or

$$z_n - z_k \notin V_{\varepsilon_0}, \forall k < n.$$

Applying definition 3 (Bochner criterion) to almost periodic function $f(t)$ we select the subsequence , which is uniformly convergent $\{f(t + y_n)\}_{n=1}^{\infty}$,i.e.

$$\lim_{n>m, m\to\infty} \sup_t \parallel f(t + y_n) - f(t + y_m) \parallel = 0$$

or

$$\lim_{n>m, m\to\infty} \sup_x \parallel f(x + y_n - y_m) - f(x) \parallel = 0.$$

Hence the sequence $\{y_n - y_m\}_{n>m,m=1}^{\infty}$ converges to zero in the topology $\Im_U$. Since the function $f'(t)$ is continuous in the topology $\Im_U$ a number N can be found, so that

$$\sup_{t\in\mathbb{R}} \parallel f'(t + y_n - y_m) - f'(t) \parallel < \varepsilon_0, m < n, m > N$$

i.e. $y_n - y_m \in V_{\varepsilon_0}$. This contradicts the choice of the sequence $\{y_n\}_{n=1}^{\infty}$. Hence the set V_ε is relatively dense and according to the proposition 1 derivative $f'(t)$ is almost periodic.□

Theorem 7. Let it is given an asymptotically almost periodic function $f(t) : [0; \infty) \to Y$, $f(t) = g(t) + w(t)$ where $g(t)$ is almost periodic function on the line, $w(t)$ - a function with a limit $\lim_{t\to\infty} w(t) = 0$. Let there exist continuous derivatives $g'(t)$, $w'(t)$ and the derivative $w'(t)$ is uniformly continuous on $[0; \infty)$. The derivative $f'(t)$ is asymptotically almost periodic function if and only if for any $\varepsilon > 0$ and for any point $x \in (-\infty; +\infty)$ there exists a neighborhood of zero U_α:

$$U_\alpha = \{\tau : \sup_{t\in\mathbb{R}} \parallel g(t + \tau) - g(t) \parallel < \alpha\}$$

interval $(-\delta; +\delta)$ such that

$$\sup_{t\in x+U_\alpha} \parallel f'(t + h) - f'(t) \parallel < \varepsilon, \forall h \in (-\delta; +\delta).$$

Proof of necessity. Since $f'(t) = g'(t) + w'(t)$ is an asymptotically almost periodic function, $w'(t)$ has a limit of 0 to infinity, and is an uniformly continuous function on $[0; \infty)$. The function $g'(t)$ is an almost periodic function. It is easy to see that the conditions of the theorem follow from theorem 5 and the uniform continuity of the function $w'(t)$. □

Proof of sufficiency. Given a point $x \in (-\infty; +\infty)$ and a number $\varepsilon > 0$, as in the proof of theorem 5 we can show that the derivative $w'(t)$ is uniformly continuous on the axis and has a limit of zero to infinity. Then there exists a number $\delta_1 > 0$ corresponding to $\frac{\varepsilon}{2}$ such that

$$\sup_{t\in R} \parallel w'(t + h) - w'(t) \parallel < \frac{\varepsilon}{2}, h < \delta_1.$$

Using the number $\frac{\varepsilon}{2}$ the point $x \in (-\infty; +\infty)$ and the condition of the theorem we find a neighborhood U_α and the interval $(-\delta, \delta)$ so that

$$\sup_{t \in x+U_\alpha} \| f'(t+h) - f'(t) \| < \frac{\varepsilon}{2}, h < \delta_2.$$

Then for the number $\varepsilon > 0$, the point $x \in (-\infty; +\infty)$ there is a neighborhood U_α and the interval $(-\delta; +\delta)$, $\delta = \min\{\delta_1, \delta_2\}$ so that

$$\sup_{t \in x+U_\alpha} \| g'(t+h) - g'(t) \| \leq \sup_{t \in x+U_\alpha} \| f'(t+h) - f'(t) \| +$$

$$+ \sup_{t \in R} \| w'(t+h) - w'(t) \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall h \in (-\delta; +\delta).$$

According to theorem 6 derivative $g'(t)$ is an almost periodic function, i.e., $f'(t) = g'(t) + w'(t)$ is an asymptotically almost periodic function.□

4. Conclusions

Earlier to prove almost periodicity (almost automorphy) of an almost periodic (almost automorphic) function, an obligatory condition on uniform continuity of the derivative was to be exploited. In this paper a variety of conditions providing the preservation of type of a function for differentiating are presented. In this regard the results of the theorems 6 and 7 are especially important. The theorems show that the uniform continuity of derivative of an almost periodic (asymptotically almost periodic) function in only the relatively dense neighborhood of zero (each point has its own neighborhood) results in almost periodicity (asymptotic almost periodicity) and, moreover, it results in the uniform continuity of the derivative in the whole axis as well. For almost automorphic (asymptotically almost automorphic) functions instead of the uniform continuity of the derivative their compactness and fulfilling the equality $[f_a(t)]' = [f']_a(t)$ is required. In these conditions the derivative is an almost automorphic (asymptotically almost automorphic) function. It should be noted that the derived results are both sufficient and necessary.

The results obtained in the paper are new even in the case of Banach space.

REFERENCES

1. Arzela C. Intorno alla continuita della somma di 'infinit di funzioni continue / Rend. R. Accad. Sci. Istit. Bologna, 1883-1884. – P. 79–84.

2. Arzela C. Sulle serie di funzioni / Mem. R. Accad. Sci. Ist. Bologna, serie 5, 1899-1900. – (**8**). – P. 131–186 and 701–744.

3. Brace J. W. The topology of almost uniform convergence / Pacific J. Math., 1959. – v.9, No. **3** July. – P. 643–652.

4. N'Guerekata G.M. Almost Automorphic and Almost Periodic Functions in Abstract Spaces / Kluwer Academic, Plenum Publishers, New York, 2001.

5. Levitan B.M., Zhikov V.V. Almost periodic functions and differential equations / Cambridge University Press, New York, 1983. – 211 p.

6. Sirvint G. Weak compactness in Banach spaces / Studia Mathematica, 1957. – **6**. – P. 71–94.

7. Veech W.A. Almost automorphic functions on groups / Amer. J. Math., 1965. – **87**(3). – P. 719–751.

8. Dimitrova-Burlayenko S.D. On continuity properties of almost-periodic functions / Euromech 498 Colloquium Book of Abstracts, P. 1–4, 2008, Lublin University of Technology.

9. Dimitrova-Burlayenko S.D. The continuity of the derivative in a weaker topology. / Contemporary problems of mathematics, mechanics and computing sciences. Book of Abstracts. – P. 249–250, – Kharkov (2011) (in Russian).

10. Dimitrova-Burlayenko S.D. The conditions for saving continuity for differentiating functions. / Contemporary problems of mathematics, mechanics and computing sciences. N.N. Kizilova, G.N. Zholtkevych (eds). Kharkov (2011). – P. 332–338 (in Russian).

11. Dimitrova-Burlayenko S.D. Almost automorphic functions as compact continuous functions on the group / Bulletin of National Technical University 'KhPI'. Series: Mathematical modeling in engineering and technologies, 2012. – **27**. – P. 82–85 (in Russian).

12. Dimitrova-Burlayenko S.D. Quasiuniform limit Levitan's almost periodic functions / Bulletin of National Technical University 'KhPI', Series: Mathematical modeling in engineering and technologies 54'2013, Kharkiv. – P. 111–117 (in Russian).

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