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Lower bound on the number of meet-irreducible elements in extremal lattices

B.O. Chornomaz

V.N. Karazin Kharkiv National University
4, Svobody sqr., Kharkiv, 61022, Ukraine
markyz.karabas@gmail.com

Extremal lattices are lattices maximal in size with respect to the number n of their join-irreducible elements with bounded Vapnik-Chervonekis dimension k. It is natural, however, to estimate the size of a lattice also with respect to the number of its meet-irreducible elements. Although this number may differ for nonequivalent $(n, k+1)$-extremal lattices, we show that each $(n, k+1)$-extremal lattice has k disjoint chains of meet-irreducible elements, each of length $n-k+2$.
Keywords: Extremal lattices, Vapnik-Chervonekis dimension, meet-irreducible elements.

Чорномаз Б. О. **Нижня границя на кількість конерозкладних елементів в екстремальних решітках.** Екстремальними називаються решітки, максимальні за розміром відносно кількості n своїх нерозкладних елементів, при обмеженні на розмірність Вапніка-Червонекіса k. Цілком природньо, з іншого боку, оцінювати розмір решітки також відносно кількості її конерозкладних елементів. Ми покажемо, що в кожній $(n, k+1)$-екстремальній решітці існує k неперетинаючихся ланцюгів конерозкладних елементів, кожний довжини $n-k+2$.
Ключові слова: Екстремальні решітки, розмірність Вапника-Червонекіса, конерозкладні елементи.

Черномаз Б. А. **Нижняя граница на количество конеразложимых элементов в экстремальных решётках.** Экстремальными называются решётки, имеющие максимальный размер относительно количества n своих неразложимых элементов, при ограничении на размерность Вапника-Червонекиса k. Естественно, с друой стороны, оценивать размер решетки также относительно количества её конеразложимых элементов. Мы покажем, что в каждой $(n, k+1)$-экстремальной решетке есть k непересекающихся цепей конеразложимых элементов, каждый длины $n-k+2$.
Ключевые слова: Экстремальные решетки, размерность Вапника-Червонекиса, конеразложимые элементы.

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1. Introduction

This paper deals with extremal problems for lattices with bounded *Vapnik-Chervonekis (VC) dimension.* For a finite lattice L, the VC dimension of L, denoted $vc(L)$, is the maximal k such that that L admit an order embedding of boolean lattice on k generators $\mathcal{B}(k)$. A lattice L is called (n, k)*-free* if it has at most n join-irreducible elements and its VC-dimension is at most $k-1$, that is, if it does not admit order embedding of $\mathcal{B}(k)$. Let us define $f(n, k)$ as

$$f(n,k) = \sum_{i=0}^{k-1} \binom{n}{i}.$$

It is known [2] that $f(n, k)$ is an upper bound on the size of (n, k)-free lattices, which is sharp for all n and k. Thus, we define (n, k)*-extremal lattices* to be (n, k)-free lattices that reach this bound. As a matter of convenience, we will mostly work with $(n, k+1)$-extremal lattices, as they can be considered maximal lattices of VC dimension k.

The idea of constraining VC dimension arises from the fact that, as it was shown in [2], the growth of VC dimension is the only reason for exponential growth of the lattice with respect to $|J(L)|$ or to $|M(L)|$, where $J(L)$ and $M(L)$ are the sets of join-irreducible and of meet-irreducible elements of L correspondingly. The bound $f(n, k)$, restricting the size of L, however, obviously depends only on $|J(L)|$, while it is rather natural to consider either $|J(L)| + |M(L)|$ or $|J(L)||M(L)|$, or other bounds symmetric in $|J(L)|$ and $|M(L)|$, as some natural measure of "complexity" of the lattice. For example, $|J(L)||M(L)|$ is the size of the *reduced formal context* describing L, see [6] for examples.

The first step towards building such symmetric bounds could be an estimation of the size of $M(L)$ for $(n, k+1)$-extremal lattices. As extremal lattices are not unique for $k \geq 2$, this number may vary. Here we are interested in producing a simple lower bound for this case, as we generally seek to maximize $|L|$ and minimize $|J(L)|$ and $|M(L)|$. Namely, we will prove that in L there are at least $k(n - k + 2)$ meet-irreducible elements arranged in k disjoint chains, descending from the top of the lattice almost to its bottom. We will also show that this bound is sharp for $k = 2$, that is, for $(n, 3)$-extremal lattices. It seems that for larger k, however, this is not so, and even the rate of growth of $|M(L)|$ is not clear. As for the upper bound on the size of $|M(L)|$, it can be rather big. We refer to [1] for the example of the family of $(n, k+1)$-extremal lattices, $k \leq n/2$, for which every k-th element is meet-irreducible.

The structure of the paper is as follows. In Section 2, for the sake of self-sufficiency, we recall some basic facts about extremal lattices, as well as about lattices and partial orders in general. In Section 3 we explore how extremal lattices can be decomposed, and how these decompositions can be stacked. Then, in Section 4, we introduce *extremal decompositions* of lattices, and prove, in Theorem 2, that there is a one-to-one correspondence between extremal lattices and extremal decompositions via *root decompositions*. In Section 5 we use this technique in order to provide, in Theorem 3, the desired lower bound on the number of meet-irreducible elements. Finally, in Section 6 we outline the possible directions for application or extension of obtained results.

2. Preliminary definitions

In this section we recall basic definitions from lattice theory, as well as some facts about extremal lattices. We refer to [6] for further details and for general background.

For a function f and a subset X of the domain of f, we write $f[X]$ to denote the image of X under f. From time to time we deal with unions of disjoint sets, in this case, in order to stress their disjointness, we write $A \bigsqcup B$ instead of $A \bigcup B$. We write $\mathbf{k}$ to denote the standard set of k elements $\{1, \ldots, k\}$.

All lattices and other objects considered in this paper are finite. Throughout the text we will be dealing with three types of embeddings of lattices, which we will explicitly differentiate: proper lattice embeddings, that is, $(\vee, \wedge)$-embeddings; $(1, \wedge)$-embeddings, which will be the most common case; and order embeddings, that is, embeddings of lattices as posets. Sometimes we will refer to $(1, \wedge)$-embeddings as simply embeddings, two other cases will always be indicated explicitly.

For a lattice L, an element $x \in L$ is called *join-irreducible* if it does not have proper join-decomposition, that is, if $x = u \vee v$ implies $x = u$ or $x = v$. Meet-irreducible elements are defined dually; the sets of the join-irreducible elements and of the meet-irreducible elements of L are denoted $J(L)$ and $M(L)$ correspondingly. It is a well-known fact that for a finite lattice each element can be represented via join-irreducibles, namely

$$x = \bigvee \{j \in J(L) \mid j \leq x\}.$$

We denote semi-intervals in L as

$$(x] := \{y \mid y \leq x\},$$
$$[x) := \{y \mid y \geq x\}.$$

For each $x \in L$ we introduce notation $J(x) = (x] \cap J(L)$ and $M(x) = [x) \cap M(L)$.

We say that a set $X \subseteq J(L)$ is a *representation of an element* x if $\bigvee X = x$; X is a *minimal representation* if no proper subset of X joins to x. Notice that, in general, minimal representation is not unique, the simplest counterexample provided by the lattice M_3, also called *diamond*. *Atoms* are elements of L that cover 0, the set of all atoms is denoted $A(L)$. Every atom is, obviously, join-irreducible, thus $A(L) \subseteq J(L)$. Lattice is called *atomistic* if each element in it can be represented as a join of atoms. For atomistic lattices it holds that $A(L) = J(L)$. The notion of *coatoms* is defined dually, and the set of coatoms is denoted $Co(L)$.

The notion of *maximal chain of a poset* is quite common in order theory. Here we find useful to introduce a slightly different notion of a *covering chain in a* *poset*. We say that C is a covering chain in P if C, with induced order, is a chain, and if from $x \prec_C y$ it follows that $x \prec_P y$. It is easy to see that for a finite P any covering chain is a subinterval of some maximal chain.

A lattice is called *graded*, if for each element x the lengths of all maximal chains in $(x]$ are equal, in which case this length is called *rank* of x, denoted $r(x)$. In graded lattice, the zero is the only element with rank 0. Note that graded lattice is a particular case of *graded poset*, which, however, would require a little more elaborated definition.

We refer to [2] (or to its extended version [3]) for detailed discussion on extremal lattices and lattices with bounded VC dimension. In particular, we refer to these papers for the proofs of all statements in this section. Note also that there authors would call an (n, k)-free lattice by a more correct, but more cumbersome name a $\mathcal{B}(k)$-free lattice on n join-irreducible elements.

A convenient characterization of $\mathcal{B}(k)$-freeness can be given in terms of minimal generators. The following Proposition is an easy consequence of [3, Lemma 6].

Proposition 1 *Lattice L is $\mathcal{B}(k{+}1)$-free if the size of each minimal representation is at most k.*

The general bound which connects the lattice size with its VC dimension is as follows:

Theorem 1 (Vapnik-Chervonekis bound) *For a finite lattice L with $vc(L) \leq k$ and $|J(L)| \leq n$ it holds*

$$|L| \leq f(n, k+1). \tag{1}$$

This bound is sharp for all $n, k \geq 1$.

As was mentioned before, lattices reaching the bound (1) are called $(n, k+1)$-extremal. The following proposition states basic properties of extremal lattices, and describes their construction for several simple cases.

Proposition 2

1. *An $(n, 1)$-extremal lattice is a one-element lattice, for all $n \geq 1$;*
2. *an $(n, 2)$-extremal lattice is a chain of length n;*
3. *for $n \leq k$, an $(n, k+1)$-extremal lattice is $B(n)$;*
4. *for $n, k \geq 1$, an $(n, k+1)$-extremal lattice is a graded lattice of height n with $r(x) = |J(x)|$;*
5. *for $n \geq 1$ and $k \geq 2$, every $(n, k+1)$-extremal lattice is atomistic.*

Note. In Theorem 1, which establishes the upper bound on $|L|$, we demand that $vc(L) \leq k$ and $|J(L)| \leq n$, not $vc(L) = k$ and $|J(L)| = n$. This formulation is rather a technicality, as for extremal lattices these inequalities will always turn out to be equalities, for one exception: as stated, an $(n,1)$-extremal lattice is a one-element lattice, its VC dimension is 0, but it has no join-irreducible elements, that is, $|J(L)| = 0 < n$.

In contrast to the general case, in an extremal lattice each element x has a unique minimal representation, which we denote by $G(x)$. Note that the uniqueness of minimal representations can be considered as an alternative definition of the *meet-distributivity* of a lattice. Thus, stating the uniqueness of minimal representations in extremal lattices is equivalent to stating that all extremal lattices are meet-distributive.

As it turns out, all extremal lattices can be iteratively constructed through procedure called *doubling*. For a poset L and its subposet K, doubling of K in L, denoted $L[K]$, is a poset with elements $L \cup \overset{\bullet}{K}$, and order

$$\leq' = \leq \cup \left\{ (x, \overset{\bullet}{y}) \in L \times \overset{\bullet}{K} \mid x \leq y \right\} \cup \left\{ (\overset{\bullet}{x}, \overset{\bullet}{y}) \in \overset{\bullet}{K} \times \overset{\bullet}{K} \mid x \leq y \right\},$$

where $\overset{\bullet}{K}$ is a disjoint copy of K. Although doublings are defined for arbitrary posets, mostly we will be interested in doublings of lattices.

Proposition 3 *If L and K are lattices and K $(1, \wedge)$-embeds into L then $L[K]$ is a lattice.*

The procedure for construction of arbitrary extremal lattices by doublings is provided by the following lemma:

Lemma 1 *For an $(n, k+1)$-extremal lattice L, $n \geq 1$, $k \geq 2$, and an (n, k)-extremal lattice K, order-embedded into L, $L[K]$ is an $(n+1, k+1)$-extremal lattice.*

In the following two sections, which constitute the core of the paper, we will widely generalize the doubling procedure from Lemma 1 above, arriving at *root decompositions* of extremal lattices.

3. Decompositions of extremal lattices

In this section we show that doubling can be used to deconstruct extremal lattices, as well as to construct them. This and the following sections follow in general Section 3 and Section 4 in [4]. The methods that we develop here are, however, far more refined and close connections can only be made at the beginning.

Lemma 2 *Let L be an $(n, k+1)$-extremal lattice, $n \geq 1$, $k \geq 2$. Then there is a one-to-one correspondence between the coatoms of L and the elements of $G(1_L)$, established by:*

$$c \in Co(L) \ngeq a \in G(1_L). \tag{2}$$

Moreover, given such c and a, $K' = [a)$ is an $(n-1, k)$-extremal lattice, $L' = (c] = L - [a)$ is $(n-1, k+1)$-extremal, and the mapping $\delta\colon K' \to L'$,

$$\delta(x) = \bigvee \big(J(x) - \{a\}\big), \tag{3}$$

defines a $(1, \wedge)$*-embedding of* K' *into* L'*. Thus,* $L \cong L'[K']$.

Note. Sometimes, when referring to δ, we write δ_a in order to explicitly identify the element a used in (3).

Proof. We will only establish the correspondence between coatoms and elements of $G(1_L)$, the rest follows verbally [2, Theorem 4].

Let us take $a \in G(1_L)$, $X = \big(J(L) - \{a\}\big)$, and $c = \bigvee X$. As $G(1_L)$ is the only minimal decomposition of 1_L, and as $X \not\supseteq G(1_L)$ (because $a \notin X$), we get $x < 1_G$. It is also clear that x is covered by 1_L, because there is exactly one element (a) in $J(1_L) - J(x)$. Thus, c is a coatom, and it is, trivially, the only coatom not above a. Thus, (2) establishes an injection of $G(1_L)$ into $Co(L)$.

For the bijection, let us note that for any coatom d there is $a \in G(1_L) - J(d)$. Now, if we construct $c = c(a)$ using the procedure above, then c is a coatom, and $d \leq c$, implying $d = c$. ■

We write c_a, or, in a functional form, $c(a)$, to denote the unique coatom of L satisfying (2), for $a \in G(1_L)$. An easy corollary from [4, Proposition 2.3] is the following representation for c_a.

Proposition 4 *In the notation of Lemma 2,* $c_a = \bigvee\big(J(L) - \{a\}\big)$.

For an $(n, k+1)$-extremal lattice L and an element $a \in G(1_L)$, let us denote by L_a a lattice $L - [a) = (c_a]$, and by L^a a $(1, \wedge)$-embedding of semi-interval $[a)$ into L_a by δ from (3). We also use notation $\dot{L}_a$ and $\dot{L}^a$ to denote semi-intervals $(c_a]$ and $[a)$ correspondingly, the notation paralleled with that in the doubling construction. Notice that $L \cong L_a[L^a]$ and $L = L_a \sqcup \dot{L}^a = \dot{L}_a \sqcup \dot{L}^a$. As $L_a = \dot{L}_a$, introducing the latter may seem excessive. We, however, will find it useful further on, when we will be constructing families L^B_A and $\dot{L}^B_A$, for which the mentioned lattices would serve as building blocks. In general, different elements of $G(1_L)$ yield nonequivalent decompositions of L in a sense that $L_a \ncong L_b$ for different $a, b \in G(1_L)$.

Now we are going to prepare a method for stacking decompositions. First of all, we examine how join-irreducible elements behave under decomposition.

Proposition 5 *For an* $(n, k+1)$*-extremal lattice* L *and an element* $a \in G(1_L)$*, holds:*

1. *if* $k \geq 3$ *then* $J(L_a) = J(L^a) = J(L) - \{a\}$*;*
2. *if* $k = 2$ *then* $J(L_a) = J(L) - \{a\}$ *and* $J(L^a)$ *is an* n*-element chain,* $J(L^a) = L^a - \{0_{L^a}\}$*, and there is a natural correspondence between* $J(L_a)$ *and* $J(L^a)$*, established as follows:*
    - *for* $x \in J(L_a)$ *we define* $x' \in J(L^a)$ *as* $\delta(x \vee a)$*,*
    - *for* $y \in J(L^a)$ *we define* $y' \in J(L_a)$ *as a unique* y' *for which* $\delta(y' \vee a) = y$*;*

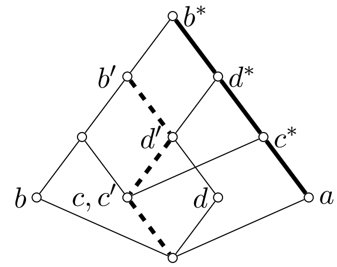

Fig. 1: Correspondence between join-irreducible elements of an $(n,2)$-extremal lattice embedded into an $(n,3)$-extremal lattice.

3. *if $k = 1$ then L and L_a are $n+1$ and n-element chains correspondingly, $a = 1_L$, $J(L_a) = J(L) - \{a\}$, and L^a is a one-element lattice, $L^a = 1_{L_a}$, $J(L^a) = \emptyset$.*

Proof. All statements, except for the explicit correspondence in (2), follow from the fact that L_a and L^a are $(n-1,k)$ and $(n-1,k-1)$-extremal correspondingly, and from structural properties of extremal lattices given in Proposition 2. In particular, atomicity implies that L_a and L^a not only have the same number of join-irreducible elements, but that these elements are exactly atoms, and thus coincide.

So we only need to prove the explicit correspondence between join-irreducible elements in an $(n-1,3)$-extremal lattice L_a and an $(n-1,2)$-extremal lattice L^a. First of all, as L^a is a chain, $J(L^a) = L^a - \{0_{L^a}\}$. Now, as L is $(n,3)$-extremal, each subset of $J(L)$ of size at most 2 is a unique minimal representation for some $x \in L$. Thus, the set $A = \{x \vee a \mid x \in J(L)\}$ contains exactly n elements, all lying above a. But, as $[a) = \overset{\bullet}{L}{}^a$, we get $|[a)| = |A| = n$. Thus, the mapping $x \mapsto x^* = x \vee a$ establishes a one-to-one correspondence between $J(L)$ and $\overset{\bullet}{L}{}^a$. Moreover, as $a^* = a \vee a = a = 0_{\overset{\bullet}{L}{}^a}$, this is also a correspondence between $J(L_a) = J(L) - \{a\}$ and $J(\overset{\bullet}{L}{}^a)$. The application of δ to the right-hand side establishes the desired correspondence.

See Figure 1 for the illustration of the argument. ■

From Proposition 5 also easily follows the correspondence between $J(L^a)$ and $J(\overset{\bullet}{L}{}^a)$:

Corollary 1 *In terms of Proposition 5,*

$$J(\overset{\bullet}{L}_a) = J(L_a);$$
$$J(\overset{\bullet}{L}{}^a) = a \vee J(L^a) = a \vee J(L_a),$$

where $a \vee J = \{a \vee j \mid j \in J\}$.

Proposition 6 *For an $(n,k+1)$-extremal lattice L and an element $a \in G(1_L)$, holds:*

1. $G(1_{L^a}) = \begin{cases} G(1_L) - \{a\}, & k \geq 3, \\ \big(G(1_L) - \{a\}\big)', & k = 2, \\ \emptyset, & k = 1; \end{cases}$

2. $G(1_{L_a}) = \begin{cases} G(1_L) - \{a\}, & n \leq k, \\ G(1_L) - \{a\} + \{b\} \text{ for some } b \in J(L) - G(1_L), & n > k. \end{cases}$

Proof.

Item (2) is exactly [4, Lemma 3.4], and cases $k = 1, 2$ in (1) are trivial, so we need only to prove (1) for $k \geq 3$. Let us observe that the lattice L^a is an $(n-1, k)$-extremal with $J(L) = J(L^a)$ and $|G(1_{L^a})| = k - 1 = |G(1_L) - \{a\}|$. On the other hand, notice that

$$\bigvee_L G(1_L) = a \vee \bigvee_L \big(G(1_L) - \{a\}\big) = \bigvee_L \big\{x \vee a \mid x \in G(1_L) - \{a\}\big\}$$
$$= \bigvee_{L^a} \big(G(1_L) - \{a\}\big).$$

Thus, $G(1_L) - \{a\}$ is a minimal representation of 1_{L^a} in L^a. ■

With Proposition 6 we now can take two (or more) elements $a, b \in G(1_L)$ and construct lattices $L_{ab} = \big(L_a\big)_b$ and $L_{ba} = \big(L_b\big)_a$. Fortunately, these lattices are equal, as we will soon prove. First of all, however, we need to introduce some intermediary terminology, which is a technical, but necessary step.

Note. The terminology developed below, until Lemma 3, will only be used in formulation and the proof of the lemma, which then would enable us to drop it and introduce a more concise formulations.

For an $(n, k+1)$-extremal lattice L, let A and B be disjoint, and possibly empty, subsets of $G(1_L)$, $|A|+|B| = p \leq k$. Let $X = x_1, \ldots, x_p$ be an enumeration of $A \bigsqcup B$. We denote by $L_X^{A,B}$ the lattice, embedded into L and obtained as the result of the following process: $L_0 = L$, $L_{i+1} = \big(L_i\big)_{x_i}$ if $x_i \in A$ and $L_{i+1} = \big(L_i\big)^{x_i}$ if $x_i \in B$, $L_X^{A,B} = L_n$. We write simply L_X, if A and B are clear from the context. When X is an enumeration of a set $\{a, b\}$ with only two elements, we use instead a simplified notation L_{ab}, ${L_a}^b$, ${L^b}_a$ and L^{ab} to denote four possible ways of decomposition.

Similarly, we define $\overset{\bullet}{L}_X$ by putting $L_{i+1} = (\overset{\bullet}{L}_i)^{x_i}$ instead of $\big(L_i\big)^{x_i}$, and $L_{i+1} = (\overset{\bullet}{L}_i)_{x_i}$ instead of $\big(L_i\big)_{x_i}$, in the iterative definition above. Note that $\overset{\bullet}{L}{}_X^{A,B}$ is embedded into L itself, for all A, B and X.

Proposition 7 *For an $(n, k+1)$-extremal lattice L and $a, b \in G(1_L)$, holds:*

$$c_{L_a}(b) = c_{L^a}(b) = c_a \wedge c_b,$$
$$c_{\overset{\bullet}{L}_a}(b) = c_a \wedge c_b,$$
$$c_{\overset{\bullet}{L}{}^a}(b^*) = c_b,$$

where $b^* = b \vee a \in J(\overset{\bullet}{L}{}^a)$.

Proof. By Proposition 4, $c_a = \bigvee\big(J(L) - \{a\}\big)$, thus

$$\begin{aligned} c_{L_a}(b) = c_{(c_a]}(b) &= \bigvee\big(J(c_a) - \{b\}\big) = \bigvee\big(J(L) - \{a, b\}\big) \\ &= \bigvee\big(J(c_a) \cap J(c_b)\big) = c_a \wedge c_b = c_{L_b}(a). \end{aligned}$$

As $L_a = \overset{\bullet}{L}_a$, the second equation is obvious, for the third one we use the representation $J(\overset{\bullet}{L}{}^a) = a \vee J(L_a)$ from Corollary 1, to get:

$$\begin{aligned} c_{\overset{\bullet}{L}_a}(b^*) &= \bigvee\big(J(\overset{\bullet}{L}_a) - \{b^*\}\big) = \bigvee\big(a \vee J(L_a) - \{a \vee b\}\big) \\ &= a \vee \bigvee\big(J(L_a) - \{b\}\big) = a \vee \bigvee\big(J(L) - \{a, b\}\big) \\ &= \bigvee\big(J(L) - \{b\}\big) = c_b. \end{aligned}$$

■

Lemma 3 $L_X^{A,B}$ *is independent of enumeration* X*, for an* $(n, k+1)$*-extremal lattice* L *and disjoint* $A, B \subseteq G(1_L)$*. That is,* $L_X = L_Y$*, for all enumerations* X, Y *of* $A \sqcup B$*.*

Thus, L_X *depends only on* A *and* B*, and we denote it by* L_A^B*. Moreover,* L_A^B *is an* $(n - |A| - |B|, k + 1 - |B|)$*-extremal lattice.*

Proof. Trivially, all we have to do is to prove three cases, namely that $L_{ab} = L_{ba}$, ${L_a}^b = {L^b}_a$ and $L^{ab} = L^{ba}$, for an $(n, k+1)$-extremal L with $n \geq 2$ and $k \geq 1$, 2 and 3 correspondingly. Figure 2 below depicts this equivalence.

The proof itself, however, is a straightforward application of Proposition 7:

1. $L_{ab} = (c_{L_a} b] = (c_a \wedge c_b] = L_{ba}$;

2. $$\begin{aligned} {L_a}^b = \delta_b[b)_{L_a} = \delta_b[b, c_a] &= \{\delta_b(x) \mid b \leq x, x \leq c_a\} \\ &= \{\delta_b(x) \mid b \leq x, \delta_b(x) \leq \delta_b(c_a)\} \\ &= \delta_b[b) \cap (\delta_b(c_a)] = \delta_b[b) \cap (c_a \wedge c_b] = {L^b}_a; \end{aligned}$$

3. $$\begin{aligned} L^{ab} &= \delta_b[b)_{L^a} = \delta_b\big([b) \cap \delta_a[a)\big) \\ &= \delta_b \circ \delta_a\big([a \vee b)\big) = \delta_a \circ \delta_b\big([a \vee b)\big) = L^{ba}. \end{aligned}$$

■

Lemma 3 then easily extends to $\overset{\bullet}{L}{}_A^B$.

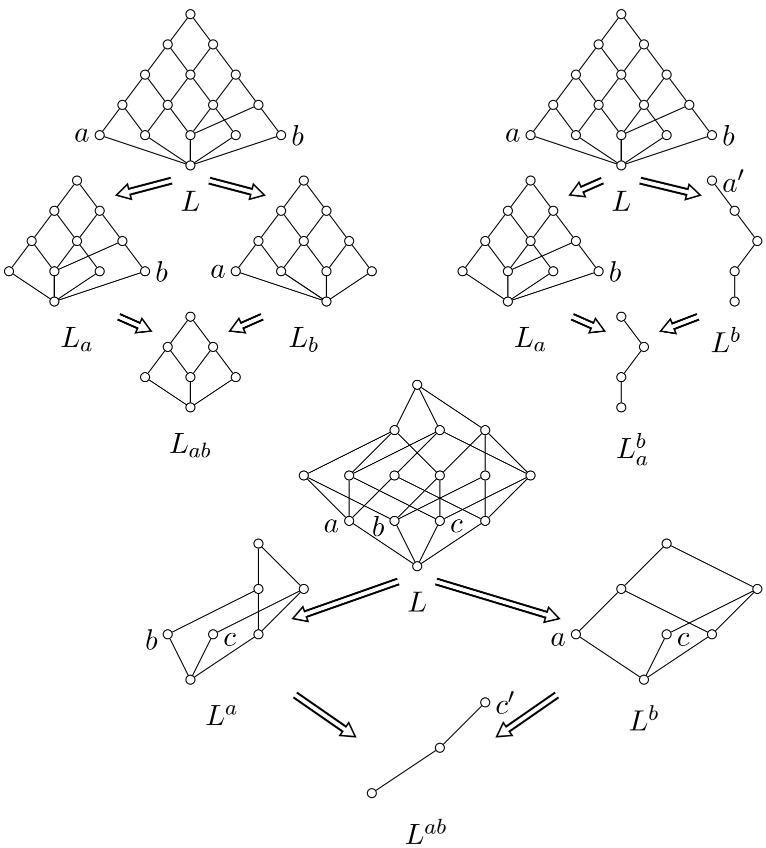

Fig. 2: Equivalence of enumerations.

Proposition 8 *In terms of Lemma 3,* $\overset{\bullet}{L}{}_X^{A,B}$ *is independent of enumeration, thus justifying the notation* $\overset{\bullet}{L}{}_A^B$*. Moreover,*

$$\overset{\bullet}{L}{}_A^B = \left[\bigvee B, \bigwedge c[A]\right].$$

Mapping $\delta_B \colon \left[\bigvee B\right) \to \left(\bigwedge c[B]\right]$*, defined as*

$$\delta_B(x) = \bigvee\bigl(J(x) - B\bigr),$$

establishes an isomorphism between $\overset{\bullet}{L}{}_A^B$ *and* L_A^B*, and*

$$\delta_B^{-1}(x) = \bigvee\bigl(J(x) + B\bigr).$$

Proof. The independence proof follows the one from Lemma 3, and is only simplified by the fact we do not use δ. Namely, we get $\overset{\bullet}{L}_{ab} = (c_a \wedge c_b] = \overset{\bullet}{L}_{ba}$, $\overset{\bullet}{L}{}_a{}^b = [b, c_a] = \overset{\bullet}{L}{}^b{}_a$ and $\overset{\bullet}{L}_{ab} = [a \vee b) = \overset{\bullet}{L}_{ba}$. Same argument also yields two other statements of the proposition. ■

Corollary 2 *In terms of Lemma 3,* L_A^B $(1, \wedge)$*-embeds into* $L_{A \sqcup B}$*. Moreover, for any disjoint* A' *and* B'*, such that* $A' \sqcup B' = A \sqcup B$*, a lattice* L_A^B $(1, \wedge)$*-embeds into* $L_{A'}^{B'}$*, whenever* $A \subseteq A'$*.*

Note. At this point, as mentioned above, we no longer need any notation involving enumerations, like $L_X^{A,B}$. Further on we will write simply L_A^B.

4. Root decompositions

Notions of *root* and *root decomposition* were introduced in [4] in order to count isomorphism classes of $(n, 3)$-extremal lattices. There it was shown that for $k \leq 3$, but not for larger ones, isomorphism of decompositions is equivalent to isomorphism of lattices. Here we generalize root decompositions in order to obtain similar equivalence for larger k, thus, our definition of root decomposition will be different. The definition of root, however, stays the same.

Definition 1 (Extremal decomposition) *An* $(n, k+1)$-extremal decomposition *is a family* $\{L_X^*\}$ *of extremal lattices, parametrized by a set* $X \subseteq \mathbf{k}$*, together with a family* $\{\phi_{X,Y} \colon L_X^* \to L_Y^*\}_{Y \subseteq X \subseteq \mathbf{k}}$ *of embeddings, such that:*

1. L_X^* *is* $(n, k - |X| + 1)$*-extremal, for all* X*;*

2. $\phi_{X,Y}$ *is a* $(1, \wedge)$*-embedding of* L_X^* *into* L_Y^**, for all* $Y \subseteq X$*;*

3. $\phi_{X,X} = id$ *and* $\phi_{Y,Z} \circ \phi_{X,Y} = \phi_{X,Z}$*, for all* $Z \subseteq Y \subseteq Z \subseteq \mathbf{k}$*; that is, all embeddings are compatible.*

We typically denote an $(n, k+1)$-extremal decomposition by $\mathcal{L} = (L^*_X, \phi_{X,Y})$ and often omit the word extremal, whenever the context is clear. The lattice $L^*_\emptyset$, or simply L^*, is called a *root* of the decomposition, all other lattices L^*_X are embedded into it. With abuse of notation we denote embedding of L^*_X into L^* as ϕ_X. We will write simply ϕ if the image and the domain are clear from the context. At times, instead of $\mathbf{k}$ we use custom fixed set of size k, which we call a *base set* of $\mathcal{L}$.

Note. For most embeddings in an $(n, k+1)$-decomposition, the unit preservation is automatically satisfied, as, by Proposition 2, for $k \geq 1$ all $(n, k+1)$-extremal lattices are graded of height n. The only place where it matters is when we considering embeddings of $L^*_{\mathbf{k}}$, as it is a one-element $(n, 1)$-extremal lattice, and the condition ensures that this element is always mapped to the unit element.

Our goal is to show that every $(n+k, k+1)$-extremal lattice can be in a unique way put to correspondence with an $(n, k+1)$-decomposition. The road for such correspondence is already paved by Lemma 3.

Definition 2 (Root decomposition) *For an $(n+k, k+1)$-extremal lattice L we define its* root decomposition $\mathcal{L}(L)$ *as an $(n, k+1)$-extremal decomposition $\mathcal{L}(L) = (L^*_X, \phi_{X,Y})$, where $G = G(1_L)$, $L^*_X = L^X_{G-X}$, and $\phi_{X,Y} \colon L^*_Y \to L^*_X$ is a natural embedding of $L^*_Y = L^Y_{G-Y}$ into $L^*_X = L^X_{G-X}$, for $X \subseteq Y$.*

It follows from Corollary 2 that all ϕ are $(1, \wedge)$-embeddings and the compatibility is straightforward. We also define a *root element* of an extremal lattice L as $x^* = \bigwedge G(1_L)$ and a *root* L^* of L as $L^* = [0, x^*]_L$. Note, that L^* will also be the root of $\mathcal{L}(L)$, and further we will not distinguish these definitions.

For further justification of putting $\mathcal{L}(L)$ to correspondence with L, we make the following digression.

Proposition 9 *For an extremal lattice L it holds*

$$L = \bigsqcup_{X \subseteq G} \dot{L}^X_{G-X},$$

where $G = G(1)$.

Proof. We recall from Proposition 8 that $\dot{L}^X_{G-X} = \left[\bigvee X, \bigwedge c[G-X]\right]$. For $x \in L$, let us introduce $H(x) = G \cap J(x)$ (we recall that the suggestive notation $G(x)$ is already used to denote the minimal representation of x). Then $H(x) \subseteq G$ and $x \geq \bigvee H(x)$. Moreover, Lemma 2 implies that $j \leq x \Leftrightarrow x \not\leq c_j$, for $h \in J(L)$. Thus, $x \leq c_h$ for all $h \in G - H(x)$, and consequently $x \leq \bigwedge c[G - H(x)]$, that is, $x \in \dot{L}^{H(x)}_{G-H(x)}$.

On the other hand, if $x \in \dot{L}^H_{G-H}$, for some $H \subseteq G$, then $x \geq h$ for all $h \in H$, and $x \leq c_j \Leftrightarrow j \not\leq x$ for all $j \in G - H$, from which it follows that $H = H(x)$.

All in all, the family $\dot{L}^X_{G-X}$ is nonintersecting and covers entire L, so the statement of the proposition follows. ■

While root decomposition gives us a transition from extremal lattices to extremal decompositions, the following definition enables us to pass from decompositions back to lattices.

Definition 3 (Canonical lattice) Canonical lattice $L(\mathcal{L})$ *of an* $(n, k+1)$*-decomposition* $\mathcal{L}$ *is a poset*

$$\{(X, x) \mid X \subseteq \mathbf{k}, x \in L_X\},$$

with an order defined by

$$(X, x) \leq (Y, y) \Leftrightarrow X \subseteq Y \text{ and } x \leq \phi y.$$

The fact that $L(\mathcal{L})$ is a lattice, let alone an extremal one, is not that trivial and we pose it as a separate lemma.

Lemma 4 *The canonical lattice of an* $(n, k+1)$*-extremal decomposition is* $(n+k, k+1)$*-extremal.*

Proof. For $k = 1$ the statement is trivial, so we consider $k > 1$, fix an $(n, k+1)$-extremal decomposition $\mathcal{L} = (L^*_X, \phi_{X,Y})$, and denote its canonical lattice by L. The lattice $L^*_{\mathbf{k}}$ is $(n, 1)$-extremal and thus, by Proposition 2, is a one-element lattice. We denote this element by u and note that $\phi_{\mathbf{k},X}$ always maps u to $1_{L^*_X}$. Thus, $(\mathbf{k}, u)$ is the largest element, that is, a unit, of L.

Now, for $x \in L^*_X$ and $y \in L^*_Y$, let us consider the element $z = (X \cap Y, \phi(x) \wedge \phi(y))$, which is trivially a lower bound of (X, x) and (Y, y). On the other hand, if some (W, w) is another lower bound, then $W \subseteq X, Y$ and thus $W \subseteq X \cap Y$. Finally, $w \leq \phi_{X,W}(x)$ and $w \leq \phi_{Y,W}(y)$. Thus

$$\begin{aligned} w \leq \phi_{X,W} x \wedge \phi_{Y,W}(y) &= \phi_{X \cap Y, W} \circ \phi_{X, X\cap Y}(x) \wedge \phi_{X\cap Y, W} \circ \phi_{Y, X \cap Y}(y) \\ &= \phi_{X\cap Y, W} \circ (\phi_{X, X\cap Y}(x) \wedge \phi_{Y, X\cap Y}(y)), \end{aligned}$$

and $(W, w) \leq z$, that is, z is a meet of (X, x) and (Y, y). All in all, L is a $(1, \wedge)$-semilattice and, consequently, a lattice. Still, let us describe join in L explicitly. We claim that

$$\bigvee_i (A_i, a_i) = (A, a),$$

where $A = \bigcup_i A_i$, $a = \bigvee_i \psi_{A_i, A}(a_i)$ and $\psi_{X,Y}(x) = \bigwedge\{y \in Y \mid x \leq \phi_{Y,X}(y)\}$ for $X \subseteq Y$. As it is with ϕ, we write simply ψ when the image and the domain are clear. Trivially, $\psi_{X,Y} \circ \phi_{Y,X}(x) = x$, for all $X \subseteq Y$.

The proof of explicit construction of joins is almost immediate. Indeed, $(A_i, a_i) \leq (A, a)$ for all i. Now, if we take (W, w) such that $(A_i, a_i) \leq (W, w)$ for all i, then $A = \bigcup_i A_i \subseteq W$ and $a_i \leq \phi_{W, A_i}(w)$, for all i. But then

$$\begin{aligned} \psi_{A_i, A}(a_i) \leq \psi_{A_i, A} \circ \phi_{W, A_i}(w) &= \psi_{A_i, A} \circ \phi_{A, A_i} \circ \phi_{W, A}(w) \\ &= \phi_{W, A}(w), \end{aligned}$$

and $a = \bigvee_i \psi a_i \leq \phi w$, implying $(A, a) \leq (W, w)$.

Now we consider join-irreducible elements of L. We claim that there are exactly two kinds of them: n elements $(\emptyset, j)$ for $j \in J(L^*)$, and k elements $(\{i\}, 0)$ for $i \in \mathbf{k}$. Indeed, all these $n+k$ elements cover $(\emptyset, 0)$, that is, the zero of L, and so they are trivially join irreducible. Let us show that no other join irreducible element exists.

First of all, for $X \subseteq \mathbf{k}$, if $|X| \geq 2$ then $(X, x) = (X - a, \phi x) \vee (X - b, \phi x)$, where a and b are any two distinct elements of X, and thus (X, x) is not join irreducible. If $|X| = 1$ and $x > 0$ then $(X, x) = (X, 0) \vee (\emptyset, \phi x)$, and (X, x) is again not join irreducible. Finally, if $x \in L^*$ and $x = y \vee z$ is a proper join decomposition of x then $(\emptyset, x) = (\emptyset, y) \vee (\emptyset, z)$ is a proper join decomposition of $(\emptyset, x)$, which finishes our claim about the structure of $J(L)$.

Simple manipulation with binomial coefficients show that L has

$$\sum_{X \subseteq \mathbf{k}, i \leq k - |X|} \binom{n}{i} = \sum_{l \leq k} \binom{k-l}{l} \sum_{i \leq k-l} \binom{n}{i} = \sum_{l \leq k} \binom{k}{l} \sum_{i \leq l} \binom{n}{i}$$
$$= \sum_{l \leq k} \sum_{i \leq l} \binom{k}{l} \binom{n}{i} = \sum_{j \leq n+k} \binom{n+k}{j}$$

elements, so in order to finish the proof we only need to show that L is $\mathcal{B}(k+1)$-free.

To show this, we employ Proposition 1 and argue that each minimal join representation has at most k elements. Let us fix an element (X, x) of L, and let $H \subseteq J(L)$ be a minimal representation of (X, x). Again, we may take $X \subsetneq \mathbf{k}$, for otherwise the statement holds trivially, and denote $l = |X|$. Recalling the structure of $J(L)$, we split H into $H' = H \cap \{(\emptyset, j) \mid J \in J(L^*)\}$ and $H'' = H \cap \{(\{i\}, 0) \mid i \in \mathbf{k}\}$. Then

$$\begin{aligned}
(X, x) &= \bigvee H' \vee \bigvee H'' \\
&= \bigvee \{(\emptyset, j) \mid (\emptyset, j) \ \in H'\} \vee \bigvee \{(\{i\}, 0) \mid (\{i\}, 0) \in H''\} \\
&= \Big(\emptyset, \bigvee \{j, \mid (\emptyset, j) \ \in H'\}\Big) \vee \Big(\{i \mid (\{i\}, 0) \in H''\}, 0\Big) \\
&= (Y, y)\,,
\end{aligned}$$

where $Y = \{i \mid (\{i\}, 0) \in H''\}$ and $y = \bigvee \{\psi_{\emptyset, Y}(j) \mid (\emptyset, j) \ \in H'\}$. We may conclude that $X = Y$ and $x = y$. From the first equation we get $X = \{i \mid (\{i\}, 0) \in H''\}$ and thus $H'' = \{(\{i\}, 0) \mid i \in X\}$. In particular, $|H''| = |X| = l$. Now, let us notice that for $j \in J(L^*)$, $\psi_{\emptyset, X}(j)$ lies in $J(L_X^*)$: if $|X| \leq k - 2$ then $J(L_X^*) = \psi\,[J(L_*)]$, and if $|X| = k - 1$ then L_X^* is a chain and all its nonzero elements are join irreducible. As H is minimal, then the representation $y = \bigvee \{\psi_{\emptyset, X}(j) \mid (\emptyset, j) \ \in H'\}$ is also minimal, for otherwise we could exclude some elements from H'' without changing the join of H. However, this representation is in L_X^*, which is $(n, k-l+1)$-extremal. Thus, $|H''| \leq k - l$, and $|H| = |H'| + |H''| \leq l + k - l = k$. ∎

Corollary 3 *For an* $(n, k+1)$*-extremal decomposition* $\mathcal{L}$ *there are* $n+k$ *elements in* $J = J(L(\mathcal{L}))$*, which have form* $J = J' \sqcup J''$*, where*

$$J' = \{(\emptyset, j) \mid J \in J(L^*)\},$$
$$J'' = \{(\{i\}, 0) \mid i \in \mathbf{k}\},$$

$|J'| = n$ *and* $|J''| = k$*.*

Corollary 4 *For an* $(n, k+1)$*-extremal decomposition* $\mathcal{L}$*, joins and meets in* $L(\mathcal{L})$ *are defined as*

$$\bigvee_i (A_i, a_i) = (A, a)\,,$$

where $A = \bigcup_i A_i$*,* $a = \bigvee_i \psi_{A_i,A}(a_i)$*,* $\psi_{X,Y}(x) = \bigwedge\{y \in Y \mid x \leq \phi_{Y,X}(y)\}$*, for* $X \subseteq Y$*. And*

$$\bigwedge_i (B_i, b_i) = (B, b)\,,$$

where $B = \bigcap_i B_i$ *and* $b = \bigwedge_i \phi_{B_i,B}(b_i)$*.*

To establish a correspondence between extremal lattices and decompositions we now clarify which decompositions we consider isomorphic.

Definition 4 Isomorphism of $(n, k+1)$ decompositions $\mathcal{L} = (L^*_X, \phi_{X,Y})$ and $\mathcal{K} = (K^*_X, \varphi_{X,Y})$ *is a pair* (σ, ε) *where* σ *is a permutation of* $\mathbf{k}$*, and* $\varepsilon\colon K^* \to L^*$ *is an isomorphism from* K *to* L*, such that* $\phi^{-1}_{\sigma(X)} \circ \varepsilon \circ \varphi_X$ *is an isomorphism of* K^*_X *into* L^*_X*, for all* $X \subseteq \mathbf{k}$*. Decomposition* $\mathcal{K}$ *is* isomorphic *to* $\mathcal{L}$ *if there is an isomorphism between them.*

It is trivial to check that, thus defined, isomorphism is an equivalence relation, and that canonical lattices and root decompositions are preserved under isomorphisms.

Proposition 10 *For* $(n+k, k+1)$*-extremal lattices* L *and* L'*, and* $(n, k+1)$*-extremal decompositions* $\mathcal{L}$ *and* $\mathcal{L}'$ *holds:*

- $\mathcal{L}(L) \cong \mathcal{L}(L')$*, whenever* $L \cong L'$*;*
- $L(\mathcal{L}) \cong L(\mathcal{L}')$*, whenever* $\mathcal{L} \cong \mathcal{L}'$*.*

Finally, the following Lemma shows that the operations of constructing canonical lattice and root decomposition are inverse up to isomorphism, which establishes the correspondence between extremal lattices and extremal decompositions.

Lemma 5 *For an $(n+k, k+1)$-extremal lattice L and an $(n, k+1)$-extremal decomposition $\mathcal{L}$ holds:*

$$L(\mathcal{L}(L)) \cong L,$$
$$\mathcal{L}(L(\mathcal{L})) \cong \mathcal{L}.$$

Proof. $L(\mathcal{L}(L)) \cong L$. Let us denote $G = G(1_L)$ and

$$L' = L(\mathcal{L}(L)) = \left\{(X, x) \mid X \subseteq G, x \in L^X_{G-X}\right\},$$

and let us recall that by Proposition 9

$$L = \bigsqcup_{X \subseteq G} \overset{\bullet}{L}{}^X_{G-X},$$

and that by Proposition 8, the mapping $\delta_X \colon \overset{\bullet}{L}{}^X_{G-X} \to L^X_{G-X}$,

$$\delta_X(x) = \bigvee\nolimits_{L^*} \big(J_L(x) - X\big),$$
$$\delta_X^{-1}(x) = \bigvee\nolimits_{L} \big(J_{L^*}(x) \sqcup X\big),$$

is an isomorphism between $\overset{\bullet}{L}{}^X_{G-X}$ and L^X_{G-X}.

Thus, the mapping $\alpha \colon L' \to L$, defined by $\alpha(X, x) = \delta_X^{-1}(x)$, provides a bijection between L' and L with $\alpha^{-1}(x) = (H(x), \delta_{H(x)}(x))$, where $H(x) = J(x) \cap G$. Now we recall, that $\phi_{X,Y}$ for $\mathcal{L}(L)$ is provided by the natural embedding of L^Y_{G-Y} into L^X_{G-X}, which means that $(X, x) \le (Y, y)$ if and only if $X \subseteq Y$ and $x \le_{L^*} y$.

Note that for $x \in \overset{\bullet}{L}{}^X_{G-X}$ holds $J(\delta_X(x)) = J(x) - X$, and thus for $x \in L^X_{G-X}$ holds $J_L(\delta_X^{-1}(x)) = J_{L^*}(x) \sqcup X$. Thus

$$\begin{aligned}
(X, x) \le (Y, y) &\Leftrightarrow X \subseteq Y, x \le_{L^*} y \\
&\Leftrightarrow X \subseteq Y, J_{L^*}(x) \subseteq J_{L^*}(y) \\
&\Leftrightarrow X \sqcup J_{L^*}(x) \subseteq Y \sqcup J_{L^*}(y) \\
&\Leftrightarrow J\big(\delta_X^{-1}(x)\big) \subseteq J\big(\delta_Y^{-1}(y)\big) \\
&\Leftrightarrow \alpha(X, x) \le_L \alpha(Y, y),
\end{aligned}$$

and the isomorphism of L and L' follows.

$\mathcal{L}(L(\mathcal{L})) \cong \mathcal{L}$. Let $\mathcal{L} = (L_X, \phi_{X,Y})$, $L = L(\mathcal{L})$ and $\mathcal{L}' = \mathcal{L}(L) = (L'_X, \phi'_{X,Y})$. By Corollary 3, L is $(n+k, k+1)$-extremal and $J(L) = J' \sqcup J''$, where

$$J' = \{(\emptyset, j) \mid J \in J(L^*)\},$$
$$J'' = \{(\{i\}, 0) \mid i \in \mathbf{k}\}.$$

Trivially, $\bigvee J'' = (X, 1) = 1_L$. Thus, $G = G(1_L) = J''$, and

$$L'^{*} = L'_G = \Big[0_L, \bigvee J(L) - G(L)\Big]_L$$
$$= \Big[0_L, \bigvee J'\Big]_L = \Big[(\emptyset, 0_{L^*}), \Big(\emptyset, \bigvee J(L^*)\Big)\Big]_L$$
$$= [(\emptyset, 0_{L^*}), (\emptyset, 1_{L^*})]_L \cong L^*.$$

We define $\sigma\colon \mathbf{k} \to J''$ and $\varepsilon\colon L^* \to L'^{*}$ as $\sigma(i) = (\{i\}, 0)$ and $\varepsilon(x) = (\emptyset, x)$. We claim that (σ, ε) is an isomorphism of $\mathcal{L}$ and $\mathcal{L}'$. Indeed, let $X' \subseteq J'' = \big\{(\{i\}, 0) \mid i \in X\big\} = \sigma[X]$ for $X \subseteq \mathbf{k}$. Then

$$L'^{*}_{X'} = L'^{X'}_{G-X'} = \Big[\bigvee X', \bigwedge c[G - X']\Big]_L$$
$$= \Big[\bigvee X', \bigvee J - (G - X')\Big]_L = \Big[\bigvee X', \bigvee J' \vee \bigvee X'\Big]_L$$
$$= [(X, 0_{L^*}), (\emptyset, 1_{L^*}) \vee (X, 0_{L^*})]_L$$
$$= [(X, 0_{L^*}), (X, 1_{L^*})]_L \cong L^*_X,$$

and the isomorphism between L^*_X and $L'^{*}_{X'}$ is established by the mapping $\alpha\colon L^*_X \to L'^{*}_{X'}$, $\alpha(x) = (X', x) = \phi^{-1}_{X'}\big(0, \phi_X(x)\big) = \phi^{-1}_{\sigma[X]} \circ \varepsilon \circ \phi(x)$. ■

As an easy consequence, we now obtain the most important structural result of the paper, which establishes a correspondence between extremal lattices and decompositions.

Theorem 2 *For an $(n+k, k+1)$-extremal lattice L and an $(n, k+1)$-extremal decomposition $\mathcal{L}$, $L \cong L(\mathcal{L})$ if and only if $\mathcal{L} \cong \mathcal{L}(L)$.*

Although technical details in this section were rather involved, the basic fact of the correspondence between extremal lattices and decompositions is quite transparent. Apart from providing structural information about extremal lattices, decompositions can be quite handy in depicting them, as illustrated by Figures 3 and 4.

5. Meet-irreducible elements in extremal decompositions

One possible application of structural insight we gain from extremal decompositions is the estimation of the number of meet-irreducible elements. We start by characterizing meet-irreducible elements of canonical lattices of extremal decompositions. We then apply this characterization to get a simple lower bound on the number of meet-irreducible elements of extremal lattices.

Proposition 11 *For an $(n, k+1)$-extremal decomposition $\mathcal{L}$, its canonical lattice $L = L(\mathcal{L})$, and elements (X, x) and $(Y, y) \in L$, (X, x) is covered by (Y, y) if and only if either $x = y$ and $X \prec Y$, or if $X = Y$ and $x \prec_{L^*_X} y$.*

Proof. It is trivial that under given conditions, $(X, x) \prec (Y, y)$, so we show that these conditions are also necessary. Indeed, if $(X, x) \prec (Y, y)$ then $(X, x) < (Y, y)$,

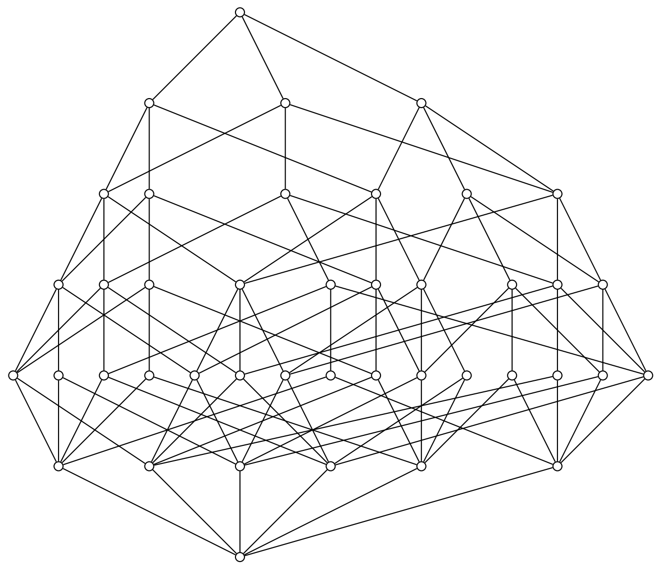

Fig. 3: $(6, 4)$-extremal lattice.

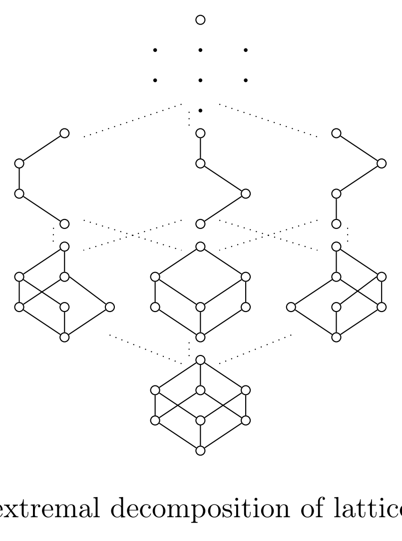

Fig. 4: $(3, 4)$-extremal decomposition of lattice on Figure 3.

meaning that $X \subseteq Y$ and $x \subseteq y$, and that at least one inequality is strict. If $X \not\preccurlyeq Y$ then there is Z such that $X \subsetneq Z \subsetneq Y$ and thus $(X,x) < (Z,x) < (Y,y)$, contradicting the covering. Thus, $X \preccurlyeq Y$, and similarly $x \preccurlyeq y$. Finally, if in both cases there is a proper covering, that is, if $X \prec Y$ and $x \prec y$, then $(X,x) < (X,y) < (Y,y)$, again contradicting the covering. ■

Lemma 6 *For an $(n,k+1)$-extremal decomposition $\mathcal{L}$ and its canonical lattice $L = L(\mathcal{L})$, an element $(X,x) \in L$ is meet-irreducible if and only if one of two mutually exclusive conditions hold:*

- *either x is meet-irreducible in L^*_X and $x \notin \phi_{Y,X}[L^*_Y]$, for all $Y \supsetneq X$;*
- *or x is a unit in L^*_X and $X = \mathbf{k} - a$, for some $a \in \mathbf{k}$.*

Note. By this lemma, all elements of the form $(\mathbf{k}-a, x)$ are meet-irreducible.
Proof. We use the property that an element is meet-irreducible if and only if it is covered by exactly one element. The statement follows from step by step classification of the elements of L:

- $\overline{x} = (\mathbf{k}, 1)$. $\overline{x}$ is a unit in L and thus is not meet-irreducible. Neither it satisfies any of two given conditions;
- $\overline{x} = (\mathbf{k}-a, 1)$, *for some* $a \in \mathbf{k}$. The only element covering $\overline{x}$ is $(\mathbf{k},1)$, thus it is meet-irreducible, while it also satisfies the second condition, and does not satisfy the first, because the unit element is not meet-irreducible;
- $\overline{x} = (\mathbf{k}-a, x)$, *for some* $a \in \mathbf{k}$ *and* $x < 1$. As $L^*_{\mathbf{k}-a}$ is a chain, x is meet-irreducible in $L^*_{\mathbf{k}-a}$, and thus $\overline{x}$ satisfies the first condition. The element $\overline{x} = (\mathbf{k}-a, x)$ is covered by $(\mathbf{k}-a, x')$, for a unique $x' \in L^*_{\mathbf{k}-a}$, covering x. In the same time $L^*_{\mathbf{k}}$ contains only unit, and $(\mathbf{k}, x)$ is not a cover of $\overline{x}$. Thus, $\overline{x}$ has a unique cover and thus it is meet-irreducible;
- $\overline{x} = (X, 1)$, *for* $X \subseteq \mathbf{k}$, $|X| \leq k-2$. $\overline{x}$ does not satisfy neither of two conditions. In the same time for $a, b \in \mathbf{k} - X$, $a \neq b$, elements $(X \sqcup a, 1)$ and $(X \sqcup b, 1)$ cover $\overline{x}$, thus $\overline{x}$ is not meet-irreducible;
- $\overline{x} = (X, x)$, *for* $X \subseteq \mathbf{k}$, $|X| \leq k-2$, $x < 1$. If x is meet-irreducible in L^*_X and $x \notin \phi_{Y,X}[L^*_Y]$ for all $Y \supsetneq X$, then $\overline{x}$ satisfies the first condition, does not satisfy the second, and (X, x') for a unique cover $x' \in L^*_X$ of x is a unique cover of $\overline{x}$ in L. Otherwise both conditions are not satisfied and there is a proper meet-decomposition of $\overline{x}$. If x is not meet-irreducible in L^*_X then this decomposition is given by $(X,x) = (X,x') \wedge (X,x'')$ for proper meet decomposition $x = x' \wedge x''$. Or, if $x \in L^*_Y$ for some $X \supsetneq Y$ then the decomposition is given by $(X,x) = (X,1) \wedge (Y,x)$.

■

Theorem 3 *Any $(n+k, k+1)$-extremal lattice L has at least $k(n+1)$ meet-irreducible elements, arranged in k disjoint covering chains of length n each. Each of these chains contains exactly one element of rank i, for $i \in k-1, \dots n+k-1$.*

Any covering chain of meet-irreducible elements in L of length n is one of those chains.

Proof. Trivially, if we denote the initial lattice by L and denote $G = G(1_L)$ and $\mathcal{L} = \mathcal{L}(L)$, then $L \cong L(\mathcal{L})$ and by Lemma 6 set $\{(G-a, x) \mid x \in L^*_{G-a}\} \cong [\bigvee G - a, 1)$ gives such chain.

For the second statement, let us take a covering chain C of meet-irreducible elements in L of length n. Again, from Lemma 6 it easily follows that all these elements should lie in L^*_X for some fixed X. Otherwise there are elements $\overline{x} = (X, x)$ and $\overline{y} = (Y, y)$ in C such that $\overline{x} \prec \overline{y}$ and $X \subsetneq Y$, which is impossible. Now, let us note that height of L^*_X is n, and thus in order to fit in such chain, the unit of L^*_X should also be meet-irreducible, which is only the case for $X = G - a$. ■

We call meet-irreducible elements from Theorem 3 *canonical meet-irreducible elements*, and corresponding chains *canonical chains*. Note also that in an $(n+k, k+1)$-extremal lattice there are at least k meet-irreducible elements of rank $k-1$. However, all elements of rank lower than $k-1$ are situated trivially, as was shown in Lemma 3.2 in [4], which we repeat below as a Proposition 12. Its easy corollary is that there are no meet-irreducible elements of smaller rank.

Proposition 12 *For an element x of an $(n, k+1)$-extremal lattice, $G(x) = J(x)$, whenever $G(x) \leq k-1$.*

Corollary 5 *The smallest rank of a meet-irreducible element in an $(n+k, k+1)$-extremal lattice is $k-1$.*

Proof. Theorem 3 states that there are k meet-irreducible elements of rank $k-1$. On the other hand, let us take x such that $r(x) = |J(x)| \leq k-2$, and let us take two distinct join-irreducible elements $a, b \in J(L) - J(X)$. Then there are two elements x_a and x_b such that $G(x_a) = J(x) + a$ and $G(x_b) = J(x) + b$. By Proposition 12, $J(x_a) = G(x_a) = J(x) + a$ and $J(x_b) = J(x) + b$. Consequently, $J(x_a \wedge x_b) = J(x_a) \cap J(x_b) = J(x)$, and thus $x_a \wedge x_b = x$, which gives a proper meet-decomposition of x. ■

The easiest example of canonical chains in extremal lattice can be given by an *interval lattice on $n+2$ elements*, which is $(n+2, 3)$-extremal. This lattice is the lattice of all intervals of $[1, \dots, n+2]$, including the empty one, ordered by set inclusion. The meet-irreducible elements are $\{[1, i] \mid i = 1, \dots n+1\}$ and $\{[i, n+2] \mid i = 2, \dots n+2\}$ and there are $2(n+1)$ of them. Figure 5 provides an illustration of these lattices.

As it turns out, interval lattice has no meet-irreducible elements, other than those, provided by Theorem 3. Moreover, it is, in essence, the only extremal lattice with that property. All other $(n, 3)$-extremal lattices will have some additional elements, and for $k > 2$, construction of an $(n+k, k+1)$-extremal lattice with $k(n+1)$ elements is impossible for large n.

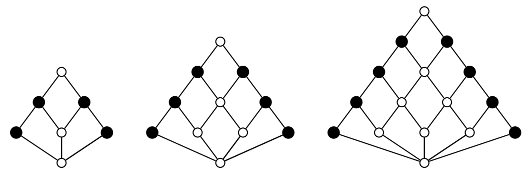

Fig. 5: Interval lattices. Black dots indicate meet-irreducible elements.

Lemma 7 *An interval lattice on $n+2$ atoms is $(n+2,3)$-extremal with $2(n+1)$ meet-irreducible elements, that is, it reaches the lower bound on the number of meet-irreducible elements from Theorem 3.*

Moreover, if an $(n+2,3)$-extremal lattice has $2(n+1)$ meet-irreducible elements, then it is isomorphic to the interval lattice on $n+2$ elements.

Proof. The fact that the interval lattice is extremal an reaches the lower bound is trivial, so we only need to prove the second statement. Let L be an $(n+2,3)$-extremal lattice with $J(L) = \mathbf{n+2}$. By Theorem 3, in L there are two disjoint chains $\{m_1, \ldots, m_{n+1}\}$ and $\{l_1, \ldots, l_{n+1}\}$ of meet-irreducible elements, such that $r(m_i) = r(l_i) = i$, for $i = 1, \ldots, n+1$. By proposition of the lemma, those are the only meet-irreducible elements of L. Let us additionally put $m_{n+2} = l_{n+2} = 1_L$. This way, chains $\{m_i\}$ and $\{l_i\}$ are still covering, but we can now state that each $x \in L$ can be represented, not necessarily in a unique way, as $x = m_i \wedge l_j$, for some i and j.

Without losing generality, we suppose that $J(m_i) = \mathbf{i}$, for all i: this can always be achieved by reordering of $J = J(L)$. Note that for each $x = m_i \wedge l_j$ we have $J(x) = J(m_i) \cap J(l_j)$.

We claim that $J(l_j) = [n+3-j, n+2]$. In this case the elements of L will be x such that $J(x) = \emptyset$, or such that $J(x) = [a,b]$ for $1 \leq a \leq b \leq n+2$. This structure will correspond exactly to the interval lattice and that would finish the proof of the lemma.

Let us suppose the contrary, and fix the largest j such that $J(l_j) \neq [n+3-j, n+2]$, notice that $j < n+2$ as $J(l_{n+2}) = J(1) = [1, n+2]$. As $J(l_{j+1}) = [n+2-j, n+2]$, we get $J(l_j) = J(l_{j+1}) - a = [n+2-j, n+2] - a$, for some $a \in [n+3-j, n+2]$.

Let us recall that $a = m_{i_0} \wedge l_{j_0}$, where i_0 and j_0 are smallest such that $a \leq m_{i_0}$ and $a \leq l_{j_0}$. Thus, $a = m_a \wedge l_{j+1}$ and

$$\begin{aligned} 1 = |J(a)| &= \big|[1,a] \cap [n+2-j, n+2]\big| \\ &= \big|[n+2-j, a]\big| \geq \big|[n+2-j, n+3-j]\big| = 2, \end{aligned}$$

a contradiction. ■

6. Discussion and open problems

As was mentioned in Introduction, the ultimate goal of our exploration is to arrive at bounds on the size of lattices with bounded VC dimension, symmetric with respect to $|J(L)|$ and $M(L)$. However far we may be from this goal, several improvements certainly can be made.

First, it seems that for $k \geq 3$ our lower bound may be significantly improved, although this would require a certain elaboration. As a first step, we are interested in the possibility of constructing extremal lattices with only canonical meet-irreducible elements:

Question 1 *Is there an $(n+k, k+1)$-extremal lattice with only canonical meet-irreducible elements, for $k \geq 3$ and for sufficiently large n*

The tentative answer to Question 1 is: no. What we can realistically expect is some limit on the rate of growth on the number of meet-irreducible elements, which we anticipate to be linear.

Question 2 *For given k, is there a constant $C = C(k)$ such that for all n there exists an $(n+k, k+1)$-extremal lattice L with $|M(L)| \leq C \cdot n$?*

An obvious way of providing an upper bound on the minimal number of meet-irreducible elements is to try and construct a family of lattices obtaining such bound. Obviously, root decompositions can be a handy tool for this. As a step in this direction, one can ask for an algorithm that, starting from given $(n, k+1)$-extremal lattice L, would go through all its $(n+k, k+1)$-*extensions*, possibly with repetitions. Here we use term *extensions* to denote extremal lattices, which share a given root, at least up to isomorphism.

Devising such algorithm may, on the other hand, be nontrivial, as it involves generating (n, l)-extremal lattices, embedded into intersection of several $(n, l+1)$-extremal lattices. This, apart from problems of practical realization, may require further theoretical elaboration.

Problem 1 *Device an effective algorithm for enumerating, possibly with repetitions, all extensions of an $(n, k+1)$-extremal lattice.*

As a curiosity, which, on the other hand, can help in shaping the theory, let us recall that the root decomposition in this paper appear as a generalization of a more simple construction from [4], which was used to count all possible non-isomorphic $(n, 3)$-extremal lattices. Now, we may ask the similar question about the number of non-isomorphic lattices for lager k, and see if our advanced decomposition can help in finding them.

Question 3 *How many non-isomorphic $(n+k, k+1)$-extremal lattices exist?*

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