ВІСНИК ЛЬВІВ. УН-ТУ VISNYK LVIV UNIV.
Серія мех.-мат. 2005. Вип.63. С.224-232 Ser. Mech.-Math. 2005. Vol.63. P.224-232

УДК 512.46, 513.83

ON ISOMORPHISMS OF THE FREE PARATOPOLOGICAL GROUPS AND FREE HOMOGENEOUS SPACES I

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In the paper we consider an isomorphic classification of the free (abelian) paratopological groups and free homogeneous spaces. We give methods for constructing examples of nonhomeomorphic spaces with topologically isomorphic free (abelian) paratopological groups and free homogeneous spaces. We propose methods for constructing examples of M_p-equivalent mappings.

Key words: free paratopological group, free homogeneous space, isomorphism of paratopological groups, isomorphism of homogeneous spaces.

1. Preliminaries. Under a *paratopological group* we understand a pair (G, τ) consisting of a group G and a topology τ on G making the group operation $\cdot\colon G \times G \to G$ on G continuous. If, in addition, the operation $(\cdot)^{-1} : G \to G$ of taking the inverse is continuous with respect to the topology τ, then (G, τ) is a *topological group*.

In the paper the word "space" means "topological space".

Definition 1. *Let X be a subspace of a paratopological group G with the identity e such that $e \in X$. Suppose that*

1. *The set X generates G algebraically, that is $\langle X \rangle = G$,*

2. *Every continuous mapping $f\colon X \to H$ from X to an arbitrary paratopological group H satisfying $f(e) = e_H$, where e_H is the unit of the group H, extends to a continuous homomorphism $f^*\colon G \to H$.*

Then G is called the Graev free paratopological group on (X, e) and is denoted by $FG_p(X, e)$.

If we replace the word "group" by the words "abelian group" in the above definition we obtain the definition of *Graev free abelian paratopological group* on (X, e), which we denote by $AG_p(X, e)$.

Definition 2. *Let X be a subspace of a paratopological group G. Suppose that*

1. The set X generates G algebraically, that is $\langle X \rangle = G$, *2. Every continuous mapping $f\colon X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $f^*\colon G \to H$.*

Then G is called Markov free paratopological group *on X and is denoted by $F_p(X)$.*

If we replace the word "group" by the words "abelian group" in the above definition we obtain the definition of *Markov free abelian paratopological group* on X which we denote by $A_p(X)$.

Proposition 1 *([14]). Let X be a space.*

1. Let e be an arbitrary point of the space X. Then free paratopological groups $FP(X, e)$ and $AP(X, e)$ exist.

2. Let e_1 and e_2 be arbitrary points of the space X. Then the free paratopological groups $FP(X, e_1)$ and $FP(X, e_2)$ are topologically isomorphic. The free paratopological groups $AP(X, e_1)$ and $AP(X, e_2)$ are topologically isomorphic as well.

Let X be a space. Similarly to the case of free topological groups we can prove that the group $F_p(X)$ is topologically isomorphic to the group $FG_p(X^+)$ and the group $A_p(X)$ is topologically isomorphic to the group $AG_p(X^+)$, where X^+ is the space obtained from X by adding one isolated point.

Proposition 2 *([12]). For each space X the following claims hold.*

1. The free paratopological groups $F_p(X)$ and $A_p(X)$ exist.

2. Let G_1, G_2 be arbitrary Markov free paratopological groups on X. Then there exists a topological isomorphism $i\colon G_1 \to G_2$ such that $i(x) = x$ for each point $x \in X$.

3. Let G_1, G_2 be arbitrary Markov free abelian paratopological groups on X. Then there exists a topological isomorphism $i\colon G_1 \to G_2$ such that $i(x) = x$ for each point $x \in X$.

In [3] V.K. Bel'nov have defined the category of homogeneous spaces and their morphisms.

Definition 3. *A triple (Y, G, h) is a homogeneous space, if Y is a topological space and G is a topological group which acts effectively and transitively on Y by the continuous mapping h.*

Definition 4. *A morphism of two homogeneous spaces $p\colon (Y_1, G_1, h_1) \to (Y_2, G_2, h_2)$ is a pair $p = (f, \psi)$, where $f\colon Y_1 \to Y_2$ is a continuous mapping, $\psi\colon G_1 \to G_2$ is a continuous homomorphism such that the diagram*

$$\begin{array}{ccc} G_1 \times Y_1 & \xrightarrow{h_1} & Y_1 \\ {\scriptstyle \psi\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G_2 \times Y_2 & \xrightarrow[h_2]{} & Y_2 \end{array}$$

is commutative.

One may naturally define the composition of morphisms and the identity morphism. A morphism $p\colon (Y_1, G_1, h_1) \to (Y_2, G_2, h_2)$ is called an isomorphism, if there exists a morphism $p'\colon (Y_2, G_2, h_2) \to (Y_1, G_1, h_1)$ such that $p \circ p'$ and $p' \circ p$ are the identity morphisms.

Definition 5. *Let (Y, G, h) be a homogeneous space. We say that a subset $Y_0 \subseteq Y$ generates (Y, G, h), if any morphism $p = (f, \psi)\colon (Y, G, h) \to (Y, G, h)$ is the identity morphism provided $f(Y_0) = Y_0$ and $f|_{Y_0}$ is the identity mapping.*

Definition 6. *A homogeneous space (Y, G, h) is called a free homogeneous space on the space X, if the following holds:*

1) X is a subspace of Y;

2) X generates (Y, G, h);

3) for any homogeneous space (Y_1, G_1, h_1) and any continuous mapping $f_0\colon X \to Y_1$ there exists a continuous morphism $p = (f, \psi)\colon (Y, G, h) \to (Y_1, G_1, h_1)$ such that $f|_X = f_0$.

In [3] it was proved that for every topological space X the free homogeneous space on X exists and is unique up to the natural isomorphism.

In [8] Megrelishvili constructed examples of the nonhomeomorphic spaces with isomorphic free homogeneous spaces. We will denote by $H(X)$ the free homogeneous space of a topological space X.

Definition 7. *Topological spaces X and Y are called B-equivalent ($X \overset{B}{\sim} Y$) if free homogeneous spaces $H(X)$ and $H(Y)$ are isomorphic.*

Definition 8. *Topological spaces X and Y are called M_p-equivalent ($X \overset{M_p}{\sim} Y$) if the Markov free paratopological groups $F_p(X)$ and $F_p(Y)$ are topologically isomorphic.*

Definition 9. *Let X_1, X_2, Y_1, Y_2 be topological spaces. A mapping $f\colon X_1 \to Y_1$ is called M_p-*equivalent *to a mapping $g\colon X_2 \to Y_2$ if there exist topological isomorphisms $i\colon F_p(X_1) \to F_p(X_2)$ and $j\colon F_p(Y_1) \to F_p(Y_2)$ such that $j \circ f^* = g^* \circ i$ where $f^*\colon F_p(X_1) \to \to F_p(Y_1)$ and $g^*\colon F_p(X_2) \to F_p(Y_2)$ are homomorphisms extending the mappings f and g respectively.*

If we replace the words “free paratopological group” by the words “free abelian paratopological group” in Definitions 1.10 and 1.11 we obtain the definitions of A_p-equivalent spaces and A_p-equivalent mappings.

Similarly to the case of free topological groups, the M_p-equivalence of two spaces implies the A_p-equivalence (see [12, Proposition 2.8]).

The problem of isomorphic classification of free topological groups has been studied by many authors. Important results in this direction were obtained by Baars [2], Okunev [9], [10] and Tkachuk [15]. In [12] Pyrch and Ravsky have considered the basic properties of free paratopological groups related mostly to the separation properties. In this paper the author continues the investigation of free paratopological groups, focusing on their isomorphic classification.

The second section contains the methods for constructing the examples of nonhomeomorphic spaces with topologically isomorphic free paratopological (abelian) groups and free homogeneous spaces.

The third section contains the method for constructing the examples of M_p-equivalent mappings.

Some results of the paper were announced in [1] and [11].

2. On the method for constructing examples of M_p-equivalent and B-equivalent spaces. Let X be space. Denote by $G(X)$ the subgroup of the abstract group

$F(X)$ (here $F(X)$ is an abstract free group with the set of generators X) generated by the set $\{xy^{-1} \in F(X) | x, y \in X\}$ and $H(X) = \{gx \in F(X) | g \in G(X), x \in X\}$. Taking on $G(X)$ the discrete topology, we can consider the natural mapping $P\colon G(X) \times X \to H(X)$ defined by $P(g, x) = gx$. The set $H(X)$ equipped with quotient topology generated by the mapping P is denoted by $H_B(X)$. The group $G(X)$ acts on $H_B(X)$ by the continuous mapping h, where $h(g, x) = gx$. The triple $(H_B(X), G(X), h)$ is a free homogeneous space on X (see [3]). Sometimes we shall write shortly that the set $H(X)$ is a free homogeneous space on X. Consider on $F(X)$ the topology of the free paratopological group $F_p(X)$. Since $H(X) \subset F(X)$, the set $H(X)$ equipped with the subspace topology of $F_p(X)$ is denoted by $H_p(X)$.

Retractions r_1 and r_2 of a topological space X are called parallel provided $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$. By $X \oplus Y$ we denote the disjoint sum of topological spaces X and Y.

Let X, Y be spaces, $f\colon X \to Y$ be a continuous mapping. Then we can consruct a morphism of homogeneous spaces $\overline{f}\colon H(X) \to H(Y)$ by the following way. The morphism $\overline{f}$ is a pair (f^*, ψ), where

$$f^*(x_1 x_2^{-1} ... x_{2n}^{-1} x_{2n+1}) = f(x_1) f(x_2)^{-1} ... f(x_{2n})^{-1} f(x_{2n+1}) \text{ and}$$
$$\psi(x_1 x_2^{-1} ... x_{2n}^{-1}) = f(x_1) f(x_2)^{-1} ... f(x_{2n})^{-1}, \text{ for all } x_1, x_2, ..., x_{2n+1} \in X.$$

Proposition 3. *Let $r_i\colon X \to K_i$, $i = 1, 2$, be parallel retractions of a topological space X onto its discrete subspaces K_1 and K_2 and $p_i\colon X \to X/K_i$ be the quotient mappings. Then the quotient spaces X/K_1 and X/K_2 are B-equivalent.*

Proof. Let $K_1 = \bigoplus_{s \in S} \{a_s\}$, then $K_2 = \bigoplus_{s \in S} \{b_s\}$, where $b_s = r_2(a_s)$. Put $X_s = r_1^{-1}(a_s) = r_2^{-1}(b_s)$. Then $X = \bigoplus_{s \in S} X_s$. Consider the mappings $f_1, f_2\colon X \to H(X)$ defined by $f_1(x) = b_s a_s^{-1} x$ and $f_2(x) = a_s b_s^{-1} x$ if $x \in X_s$. Obviously, these mappings are continuous on each X_s, thus they are continuous on X. Let $\overline{f}_i\colon H(X) \to H(X)$ be the morphisms constructed from the mappings f_i as described before the proposition. It was proved in [10] that (f_1^*, ψ_1) is inverse to (f_2^*, ψ_2), hence (f_1^*, ψ_1) is an isomorphism. Let $\overline{p_i}\colon H(X) \to H(X/K_i)$ be the morphisms constructed from the mappings p_i as described before the proposition. Similarly to [10] one can easily check that there exists an isomorphism $j\colon H(X/K_1) \to H(X/K_2)$ such that $j \circ p_1^* = p_2^* \circ f_1^*$. Thus $X/K_1 \overset{B}{\sim} X/K_2$.

Lemma 1. *Let X be a topological space. Then the natural mapping $h\colon H_B(X) \to H_p(X)$ is continuous.*

Proof. Consider the mapping $P_1\colon G(X) \times X \to H_p(X)$, defined as $P_1(g, a) = ga$ and the quotient mapping $P\colon G(X) \times X \to H_B(X)$ defined as $P(g, a) = ga$ (see [8]). Since for each $g \in G(X)$ the restriction $P_1|_{g \times X}\colon g \times X \to H_p(X)$ is continuous and $G(X) \times X = \bigoplus_{g \in G(X)} g \times X$. Then we see that the mapping P_1 is continuous on $G(X) \times X$. The continuity of the mapping h follows from the fact that the mapping $P_1 = h \circ P$ and the quotience of the mapping P.

Theorem 1. *Let X, Y be topological spaces with isomorphic free homogeneous spaces. Then Markov free paratopological groups of the spaces X and Y are topologically isomorphic.*

Proof. Let $(i,\psi)\colon (H_B(X),G(X),h_1) \to (H_B(Y),G(Y),h_2)$ be an isomorphism of homogeneous spaces. Denote by $p_X\colon H_B(X) \to H_p(X)$ and $p_Y\colon H_B(Y) \to H_p(Y)$ the natural mappings. Consider the mapping $g\colon X \to H_p(Y)$ defined by $g = p_Y \circ i|_X$. Let us extend the mapping g to a continuous homomorphism $g^*\colon F_p(X) \to F_p(Y)$. Since $G(X)$ acts transitively on $H(X)$, we obtain $p_Y \circ i = g^* \circ p_X$. Similarly, consider the mapping $f = p_X \circ i^{-1}|_Y$. Let us extend f to a continuous homomorphism $f^*\colon F_p(Y) \to F_p(X)$ satisfying the property $p_X \circ i^{-1} = f^* \circ p_Y$. From the last two equalities it follows that $p_X \circ i^{-1} \circ i|_X = f^* \circ g \circ p_X$, therefore $f^* \circ g(x) = x$ for all $x \in X$. Thus $f^* \circ g^* = 1_{F_p(X)}$. Similarly, we can check that $g^* \circ f^* = 1_{F_p(Y)}$, therefore g^* is a topological isomorphism.

In [6] free (abelian) topological groups on functionally Hausdorff spaces were considered.

Definition 10. *[6] Markov free topological group on a space X is a pair consisting of a topological group $F_M(X)$ and a continuous function $\eta_X\colon X \to F_M(X)$ such that any continuous function from X to a topological group G "lifts" to a unique continuous group homomorphism $\overline{f}\colon F_M(X) \to G$ such that $\overline{f} \circ \eta_X = f$.*

In the classic definitions of the free objects the mapping η_X is an embedding, in the above definition η_X need not to be an embedding.

If X is a Tychonoff space then the mapping η_X is a closed embedding and $F_M(X)$ is free topological group of X in the sense of [7].

If we change the word "group" to the words "abelian group" in the above definition we obtain the definition of *Markov free abelian topological group* on X which we denote by $A_M(X)$.

The next propositions follows from [12, proposition 2.2].

Proposition 4. *Let X and Y be functionally Hausdorff spaces with topologically isomorphic free paratopological groups. Then the groups $F_M(X)$ and $F_M(Y)$ on spaces X and Y are topologically isomorphic.*

Proposition 5. *Let X and Y be functionally Hausdorff spaces with topologically isomorphic free abelian paratopological groups. Then the groups $A_M(X)$ and $A_M(Y)$ on spaces X and Y are topologically isomorphic.*

The following proposition provides a method for constructing examples of nonhomeomorphic spaces with topologically isomorphic free (abelian) paratopological groups.

Proposition 6. *Let X_k, Y_k be topological spaces and $p_k\colon X_k \to Y_k$, $k = 1,2$, be quotient mappings and $p_k^*\colon F_p(X_k) \to F_p(Y_k)$ be the homomorphic extensions of the mappings p_k. If there exists a topological isomorphism $i\colon F_p(X_1) \to F_p(X_2)$ such that $i(\ker p_1^*) = \ker p_2^*$ then the mappings p_1 and p_2 are M_p-equivalent.*

Proof. Suppose that such the mapping i exists. Let us define a mapping $j\colon F_p(Y_1) \to$ $\to F_p(Y_2)$ by putting $j(a) = p_2^* \circ i(p_1^*)^{-1}(a)$ for each $a \in F_p(Y_1)$. One can easy check that j is well defined and is a topological isomorphism such that $j \circ p_1^* = p_2^* \circ i$. By [12, Proposition 2.10], the homomorphism p_1^* is open. The composition $p_2^* \circ i$ is continuous, thus the mapping j is continuous. The continuity of j^{-1} can be checked similarly.

Proposition 7. *Let $r_i\colon X \to K_i$ $i = 1, 2$, be parallel retractions of the topological space X onto its subspaces K_1 and K_2 such that $F_p(K_1)$ and $F_p(K_2)$ are topological groups and $p_i\colon X \to X/K_i$ be quotient mappings. Then $p_1 \overset{M_p}{\sim} p_2$. In particular, $X/K_1 \overset{M_p}{\sim} X/K_2$.*

Proof. Since $F_p(K_i)$ are topological groups, then the mappings $r_1(x)^{-1}\colon X \to F_p(K_1)$, $r_2(x)^{-1}\colon X \to F_p(K_2)$ are continuous. Since K_i are retracts of X, the embeddings $K_i \hookrightarrow X$ extend to embeddings $t_i\colon F_p(K_i) \hookrightarrow F_p(X)$ [12], so the mappings

$$t_1 \circ r_1(x)^{-1}\colon X \to F_p(X), \quad t_2 \circ r_2(x)^{-1}\colon X \to F_p(X)$$

are continuous. Let us define the mapping $j\colon X \to F_p(X)$ by putting $j(x) = r_1(x)^{-1}xr_2(x)^{-1}$. The mappings $x \longmapsto r_i(x)^{-1}$ are continuous because $F_p(K_i)$ are topological groups, thus the mapping j is continuous. Denote by $J\colon F_p(X) \to F_p(X)$ the homomorphic extension of the mapping j. It was proved in [10] that $J \circ J = 1_{F_p(X)}$. Denote by $p_i\colon X \to X/K_i$ the quotient mappings and by $p_i^*\colon F_p(X) \to F_p(X/K_i)$ their homomorphic extensions. It was proved in [10] that $J(K_1) = K_2$, therefore $J(\ker p_1^*) = \ker p_2^*$. And now the proof follows from Proposition 6.

Let us characterize the spaces for which their free (abelian) paratopological group is a topological group.

Proposition 8. *The following conditions are equivalent for a topological space X:*

i) the paratopological group $F_p(X)$ is the disjoint sum $\underset{s\in S}{\oplus} Z_s$ of its antidiscrete subspaces Z_s,

i') the paratopological group $A_p(X)$ is the disjoint sum $\underset{s\in S}{\oplus} A_s$ of its antidiscrete subspaces A_s,

ii) the paratopological group $F_p(X)$ is a topological group,

ii') the paratopological group $A_p(X)$ is a topological group,

iii) the topological space X is the disjoint sum $\underset{s\in S}{\oplus} X_s$ of its antidiscrete subspaces X_s.

Proof. The implications $(i \Longrightarrow ii)$ and $(i' \Longrightarrow ii')$ follows from the fact that each locally compact paratopological group is a topological group (see [13]).

$(ii \Longrightarrow iii)$,$(ii' \Longrightarrow iii)$ Let U be an open subset of the space X. If the assumption (ii) holds then [12, theorem 2.4] implies that U is a closed subset of X. If the assumption (ii') holds then similarly to [12, theorem 2.4] we can prove that U is a closed subset of X. Therefore, each open subset of X is clopen. Define the relation "$\sim$" on X by the following. Let $x, y \in X$. We put $x \sim y$ if and only if there is no open subset of the space X containing exactly one of the points x and y. Let T_0X be the quotient space of X determined by the relation "$\sim$" and $q\colon X \to T_0X$ be the quotient mapping. Let U be a closed subspace of T_0X. Then $q^{-1}(U)$ is closed and hence open subset in X. By the quotience of q we see that the set U is open in T_0X. Let $a, b \in T_0X$ be an arbitrary distinct points. Then there exists an open subset V of T_0X containing exactly one of these points. The set $X \setminus V$ is also open and contains the other point. Thus the topological space T_0X is a T_2 space. So every point is closed, and hence is open in T_0X, that is the topological space T_0X is discrete. Hence X is a disjoint sum of its antidiscrete subspaces.

$(iii \Longrightarrow i)$,$(iii \Longrightarrow i')$ Consider the quotient mapping $q\colon X \to Y$, where the image of each X_s is a singleton. The topological space Y is discrete, therefore $\ker q^*$ is a clopen antidiscrete subgroup of the paratopological group $F_p(X)$, that is $F_p(X)$ is the disjoint sum $\underset{s\in S}{\oplus} Z_s$ of its antidiscrete subspaces Z_s.

Example 1. [5] Let $X = C\times\mathbb{N}$, where C is a convergent sequence with the limit point c_0, $\mathbb{N}$ is a countable discrete space, $a \in C\setminus\{c_0\}$. Then the sets $K_1 = \{c_0\}\times\mathbb{N}$, $K_2 = \{a\}\times\mathbb{N}$ are discrete parallel retracts of the topological space X. The quotient space X/K_1 is homeomorphic to the Fréchet fan, and the quotient space X/K_2 is homeomorphic to X. Thus, local compactness, metrizability, the first and second axioms of countability, Čech completeness are not preserved by the relation of M_p-equivalence. The quotient mapping $p_1\colon X \to X/K_1$ is not open, and has no right inverse, while the quotient mapping $p_2\colon X \to X/K_2$ is open and has a right inverse.

Let X, Y be topological spaces. A mapping $f\colon X \to Y$ is called a *local homeomorphism* if for each $x \in X$ there exists an open neighborhood $U(x)$ such that the restriction $f|_{U(x)}$ is a homeomorphism from $U(x)$ onto an open subspace of Y.

Example 2. Let $C_i = \{x_i, y_i\}$, $\tau_i = \{\{\emptyset\}, \{x_i\}, \{x_i, y_i\}\}$ for $i = 1, 2$. Denote by X the disjoint sum of the topological spaces (C_1, τ_1) and (C_2, τ_2). The subsets $K_1 = \{x_1, x_2\}$ and $K_2 = \{y_1, y_2\}$ are discrete parallel retracts of X, so the quotient mappings $p_1\colon X \to X/K_1$ and $p_2\colon X \to X/K_2$ are M_p-equivalent. The mapping p_1 is open not closed, local homeomorphism. The mapping p_2 is closed not open and it is not a local homeomorphism.

A topological space X is called *resolvable* (respectively *ω-resolvable)* if X can be partitioned into two (respectively countably many) dense subsets.

Example 3. Let $X = [0, 1] \cup [2, 3] \cup \{4\}$ be a subspace of the reals with the natural topology. Then the subspaces $K_1 = \{1, 4\}$ and $K_2 = \{1, 2\}$ are discrete parallel retracts of X. Then $X/K_1 = I \oplus I$, $X/K_2 = I \oplus \{x\}$, where I is the closed unit interval. Thus $I \oplus I \overset{M_p}{\sim} I \oplus \{x\}$. The space $I \oplus I$ is ω-resolvable, while $I \oplus \{x\}$ is not resolvable. Hence resolvability and ω-resolvability are not preserved by the relation of M_p-equivalence.

3. On the method for constructing examples of M_p-equivalent mappings.

Proposition 9. *Let X be a Tychonoff space and $r_i\colon X \to K$, $i = 1, 2$, its retractions onto the same retract K such that $F_p(K)$ is a topological group. Then $r_1 \overset{M_p}{\sim} r_2$.*

Proof. Obviously $r_1 \circ r_2 = r_2$ and $r_2 \circ r_1 = r_1$.

Consider the mappings $h(x), g(x)\colon X \to F_p(X)$ defined by the formulas
$h(x) = xr_1(x)^{-1}r_2(x)$, $g(x) = xr_2(x)^{-1}r_1(x)$.

Since $F_p(K)$ is topological group, the mappings $r_1(x)^{-1}\colon X \to F_p(K)$, $r_2(x)^{-1}\colon X \to F_p(K)$ are continuous. Since K is a retract of X, the embedding $K \hookrightarrow X$ extends to an embedding $t\colon F_p(K) \hookrightarrow F_p(X)$ [12], so the mappings $t \circ r_1(x)^{-1}\colon X \to F_p(X)$, $t\circ r_2(x)^{-1}\colon X \to F_p(X)$ are continuous. Thus the mappings h and g are also continuous. Let $h^*, g^*\colon F_p(X) \to F_p(X)$ be a homomorphic extensions of the mappings h and g. If $x \in X$ then

$$h^* \circ g(x) = [xr_2(x)^{-1}r_1(x)] \times r_1[xr_2(x)^{-1}r_1(x)]^{-1} \times r_2[xr_2(x)^{-1}r_1(x)] =$$
$$= x \times r_2(x)^{-1} \times r_1(x) \times r_1(r_1(x))^{-1} \times r_1(r_2(x)) \times r_1(x)^{-1} \times r_2(x)\times$$
$$\times r_2(r_2(x))^{-1} \times r_2(r_1(x)) = x$$

So, $h^* \circ g^* = 1_{F_p(X)}$. Similarly one can check that $g^* \circ h^* = 1_{F_p(X)}$. Hence, h^* is a topological isomorphism. Moreover,
$r_2(g(x)) = r_2(x) \times r_2(r_2(x))^{-1} \times r_2(r_1(x)) = r_2(x) \times r_2(x)^{-1} \times r_1(x) = r_1(x)$
$r_1(h(x)) = r_1(x) \times r_1(r_1(x))^{-1} \times r_1(r_2(x)) = r_1(x) \times r_1(x)^{-1} \times r_2(x) = r_2(x)$

From these facts we conclude that $r_1 \overset{M_p}{\sim} r_2$.

Let X and Y be spaces. A map $f\colon X \to Y$ is called monotone (respectively easy, zero-dimensional) if any $f^{-1}(y)$ is connected (respectively hereditary disconnected, zero-dimensional) for each point $y \in Y$ [4, p. 526, 538].

Example 4. Let X be the space of reals. Consider on X the following topology: the set $(-\infty, 0]$ is equipped with the standart topology and $[0, +\infty)$ is an antidiscrete subset of X. Then the mappings f, g defined as $f(x) = |x|$ and $g(x) = x^+ = (x + f(x))/2$ for $x \in \mathbb{R}$ are retractions from $\mathbb{R}$ onto $\mathbb{R}^+ = [0, \infty)$. So, by Proposition 9 we obtain that $f \overset{M_p}{\sim} g$. The mapping f is not monotone, easy, zero-dimensional, the mapping g is monotone, not easy, not zero-dimensional.

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ІЗОМОРФІЗМИ ВІЛЬНИХ ПАРАТОПОЛОГІЧНИХ ГРУП ТА ВІЛЬНИХ ОДНОРІДНИХ ПРОСТОРІВ І

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Розглянуто ізоморфну класифікацію вільних (абелевих) паратопологічних груп і вільних однорідних просторів. Подано методи для побудови негомеоморфних просторів з топологічно ізоморфними вільними (абелевими) паратопологічними групами та вільними однорідними просторами. Подано також методи побудови M_p - еквівалентних відображень.

Ключові слова: вільна паратопологічна група, вільний однорідний простір, ізоморфізм паратопологічних груп, ізоморфізм однорідних просторів.

Стаття надійшла до редколегії 03.05.2006

Прийнята до друку 24.10.2007