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ON SPACE OF OPEN MAPS OF THE CIRCLE

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The set $\Psi(S^1)$ of the equivalence classes of open onto maps of the unit circle is endowed with the topology generated by the Vietoris topology. The main results of this note consists in description of the connectedness components of the space $\Psi(S^1)$.

Key words: hyperspace, open map, circle.

1. Introduction. Every open map $f\colon X \longrightarrow Y$ of compact Hausdorff spaces generates a decomposition of X into the fibers of f. We say that two maps defined on X are equivalent if the corresponding decompositions coincide. One can define a topology on the set $\Psi(X)$ of open maps on X by using the embedding $f \mapsto \{f^{-1}(f(x)) \mid x \in X\}$ into the double hyperspace $\exp^2(X) = \exp(\exp X)$ of X and considering the Vietoris topology on $\exp^2 X$ (see, e.g. [4]).

In the previous papers the cases $X = \omega + 1$ (convergent sequence) and $X = I$ (the segment) are considered.

In this note we deal with the unit circle S^1. The main result is a description of the components of the space $\Psi(S^1)$.

2. Preliminaries. A map $f\colon X \longrightarrow Y$ is called a *local homeomorphism* if for any point $x \in X$ there exists a neighborhood $U \subset X$ such that $f(U)$ is an open subset in Y and $f|_U\colon U \longrightarrow f(U)$ is a homeomorphism.

The symbol Bd denotes the boundary.

Let $\mathfrak{n}$ be a natural number. Let $p \in X$. The space X has the *order* $\leqslant \mathfrak{n}$ *at the point* p, we write $\mathrm{ord}_p X \leqslant \mathfrak{n}$, if for every $\varepsilon > 0$ there exists an open set U such that

$$p \in U, \qquad \mathrm{diam}\, U < \varepsilon \qquad \text{i} \qquad |\mathrm{Fr}(U)| \leqslant \mathfrak{n}.$$

Recall that $X^{[\mathfrak{n}]} = \{p \in X \mid \mathrm{ord}_p X \leqslant \mathfrak{n}\}$.

The points of order $< \omega$ are called *regular* (that are the points, for which the set $|\mathrm{Bd}(U)|$ is finite). The points of order 1 are called *endpoints.*

The space X is called *regular*, according to the order theory, if every point $x \in X$ is a regular point. The space X is called *rational* , according to the order theory, if every point $x \in X$ is rational point (see [6], §51, p. 274).

A map of topological spaces is called open if the image of every open set is open.

We will use the following result (see [7]).

Theorem 1. *The order of a point is never increased under an open mapping.*

Let X and Y be two topological spaces. Two maps $f, g\colon X \longrightarrow Y$ are called *homotopic* (notation $f \simeq g$) if there is a map $F\colon X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

Two spaces X and Y are called *homotopy equivalent* ($X \simeq Y$) if there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

The symmetric group S_n acts on the n-th power of a topological space X^n of a topological space X by permutation of coordinates. The *n-th symmetric product* or the *n-th symmetric power* of the space X is the orbit space $SP^n = X^n/S_n$.

By S^1 we denote the unit circle, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

The following result holds (see [8]):

Theorem 2. *For n odd, $SP^n(S^1) \cong S^1 \times I^{n-1}$, the trivial $(n-1)$-disc bundle over S^1. For n even, $SP^n(S^1) \cong S^1 \overline{\times} I^{n-1}$ non-orientable $(n-1)$-disc bundle over S^1.*

Recall that maps $f\colon S^1 \longrightarrow S^1$ and $g\colon S^1 \longrightarrow S^1$ are *topologically equivalent* (write $f \sim g$), if there exist homeomorphisms $h_1\colon S^1 \longrightarrow S^1$ and $h_2\colon S^1 \longrightarrow S^1$ such that the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\ f\ } & S^1 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ S^1 & \xrightarrow[\ g\ ]{} & S^1 \end{array}$$

is commutative.

A separable metrizable space is an l^2-manifold, if every its point has a neighborhood homeomorphic to the open subset of an separable Hilbert space l^2.

Theorem 3. *Let M and N be l^2-manifolds. The spaces M and N are homeomorphic if they are homotopy equivalent.*

See [10] for the proof.

3. Topological type of the space $\Psi(S^1)$. Let us consider the structure of the space

$$\Psi(S^1) = \{\langle f \rangle \mid f\colon S^1 \longrightarrow X\},$$

where f is a continuous open map of the circle onto a metric compact space X. By Theorem 1 we obtain the following cases:

(1) If for every $z \in S^1$ we have $\mathrm{ord}_{f(z)} X = 2$ then the image of the circle is a simple closed curve and a map f is a local homeomorphism. Let $\Psi_{\mathrm{lh}}(X)$ be the set of all local homeomorphisms of the space X.

(2) If there exist $a, b \in S^1$ such that $\mathrm{ord}_{f(a)} X = \mathrm{ord}_{f(b)} X = 1$ and for every $z \in S^1 \setminus \{a, b\}$ we have $\mathrm{ord}_{f(z)} X = 2$ then the image of the circle is an arc

(an interval) and f is called an oscillating map. Let $\Psi_{\mathrm{osc}}(X)$ be the set of all oscillating maps of the space X.

(3) If for every $z \in S^1$ we have $\mathrm{ord}_{f(z)}X = 0$ then $X = \{*\}$ is the one point space and f is a constant map. By $\langle c\rangle$ we denote the set of all constant maps.

Then $\Psi(X) = \Psi_{\mathrm{lh}}(X) \cup \Psi_{\mathrm{osc}}(X) \cup \langle c\rangle$.

Let us consider the set of local homeomorphisms of the circle. Let $f\colon S^1 \longrightarrow S^1$ be a local homeomorphism. It is well-known that f is topologically equivalent to the map $z^n\colon S^1 \longrightarrow S^1$, for some $n \in \mathbb{Z}$ (see [9]). We denote by $\Psi^n_{\mathrm{lh}}(S^1)$ the set of all elements $\langle f\rangle \in \Psi(S^1)$ such that $f \sim z^n$. Then from $\langle f\rangle \in \Psi^n_{\mathrm{lh}}(S^1)$ we will have $|f^{-1}(f(s))| = n$, for every $s \in S^1$.

Then the set $\Psi_{\mathrm{lh}}(S^1)$ of all local homeomorphisms is defined as the union $\bigcup\limits_{n=1}^{\infty} \Psi^n_{\mathrm{lh}}(S^1)$.

Proposition 1. *The set $\Psi^n_{\mathrm{lh}}(S^1)$ is an open and closed subset of the space $\Psi_{\mathrm{lh}}(S^1)$, for all $n \in \mathbb{N}$.*

Proof. Let us consider $\langle f\rangle \in \Psi_{\mathrm{lh}}(S^1) \setminus \Psi^n_{\mathrm{lh}}(S^1)$. Then there exists $m \in \mathbb{N}$ such that $\langle f\rangle \in \Psi^m_{lh}(S^1)$. Than the proof splits into two cases:

1. Let $n < m$. Let $s_1 \in S^1$ and consider $f^{-1}(f(s_1)) = \{s_1, s_2, \dots, s_m\}$. Let us choose open disjoint subsets $U_i \subset S^1$ such, that $s_i \in U_i$ for all $i = 1, 2, \dots m$. Then $\langle U_1, \dots, U_m\rangle$ is an open subset in $\exp S^1$, and $\langle f\rangle \cap \langle U_1, \dots, U_m\rangle \neq \emptyset$. From this it follows that $\langle f\rangle \in \langle \exp S^1, \langle U_1, \dots, U_m\rangle\rangle = O(\langle f\rangle)$. The set $O(\langle f\rangle)$ is an open subset of the space $\exp^2 S^1$, i.e., is a neighborhood of the element $\langle f\rangle \in \exp^2 S^1$.

Let $\langle g\rangle \in O(\langle f\rangle)$. Then $\langle g\rangle \cap \langle U_1, \dots, U_m\rangle \neq \emptyset$, and there exists $x \in S^1$ such that $g^{-1}(x) \in \langle U_1, \dots, U_m\rangle$. From this it follows that $g^{-1}(x) \cap U_i \neq \emptyset$, for every $i = 1, 2, \dots, m$. Since, as the sets $U_1, U_2 \dots, U_m$ are disjoint, we can draw a conclusion that $|g^{-1}(x)| \geqslant m > n$. Therefore $\langle g\rangle$ does not belong to the set $\Psi^n_{\mathrm{lh}}(S^1)$.

2. Let $n > m$ and $\langle f\rangle \in \Psi^m_{\mathrm{lh}}(S^1)$. Let $x, y \in f(S^1)$, $x \neq y$ and

$$\varepsilon = \min\{d(a, b) \mid a, b \in f^{-1}(\{x, y\}), a \neq b\}.$$

Let $\langle g\rangle \in \Psi^n_{lh}(S^1)$ be such that $d_{HH}(\langle f\rangle, \langle g\rangle) < \varepsilon/3$. There exists $\xi \in g(S^1)$ such that $d_H(g^{-1}(\xi), f^{-1}(x)) < \varepsilon/3$. According to the pigeon-hole principle there exists $a \in f^{-1}(x)$ such that $O_{\varepsilon/3}(a)$ contains at least two different points of the set $g^{-1}(\xi)$. The map g is a local homeomorphism so it is homeomorphic to the map $z \mapsto z^n$, where $n \in \mathbb{N}$. Whence, for every $\eta \in g(S^1)$ the set $g^{-1}(\eta)$ intersects the interval between two points of the set $g^{-1}(\xi)$. Therefore $g^{-1}(\eta) \cap O_{\varepsilon/3}(a) \neq \emptyset$, for every $\eta \in g(S^1)$. Then there exists $\widetilde{\eta} \in g(S^1)$ such that $d_H(g^{-1}(\widetilde{\eta}), f^{-1}(y)) < \varepsilon/3$ and $g^{-1}(\widetilde{\eta}) \cap O_{\varepsilon/3}(a) \neq \emptyset$.

Let $O_{\varepsilon/3}(f^{-1}(y))$ be a neighborhood of the set $f^{-1}(y)$. According to the choice of ε, $O_{\varepsilon/3}(a) \cap O_{\varepsilon/3}(f^{-1}(y)) = \emptyset$. However, $g^{-1}(\widetilde{\eta}) \cap O_{\varepsilon/3}(a) \neq \emptyset$ and we have a contradiction.

Let $\langle f\rangle \in \Psi_{lh}(S^1)$ then there exist $n \in \mathbb{Z}\setminus\{0\}$ such that $f \sim z^n$. Every $\langle f\rangle \in \Psi^n_{lh}(S^1)$ can be identified with some element of the space $(0,1)^{n-1} \times (C([0,1],[0,1]))^{n-1}$.

Let $z_0 = 1$ and $f^{-1}(f(1)) = \{z_0, z_1, \dots z_{n-1}\}$.

We consider the points $z_0, z_1, \dots z_{n-1}$ to be situated in the order counterclockwise on the circle. There exists $t_1 \in (0,1)$ such that the ratio of the lengths of the arcs (z_0, z_1) and (z_1, z_0) is $t_1 : (1 - t_1)$. Next, we define $t_i \in (0,1)$ such that the ratio of the lengths of the arcs (z_{i-1}, z_i) and (z_i, z_0) is $t_i : (1-t_i)$. In this way, we obtain a one-to-one continuous map $\{z_0, z_1, \dots z_{n-1}\} \mapsto (t_1, t_2, \dots t_{n-1})$.

Let us consider the following maps:

$$h_0 \colon [0,1] \to [z_0, z_1] \text{ such that } h_0(t) = e^{ct}, \text{ where } c = \mathrm{Ln}(z_1),$$

$$h_k \colon [0,1] \to [z_k, z_{k+1}] \text{ such that } h_k(t) = e^{c_k t + a_k}$$

$$\text{where } c_k = \mathrm{Ln}\left(\frac{z_{k+1}}{z_k}\right), a_k = \mathrm{Ln}(z_k),$$

$$\psi_k \colon [z_0, z_1] \to [z_k, z_{k+1}] \text{ such that } \psi_k(z) = f^{-1}(f(z)) \cap (z_k, z_{k+1}),$$

$$\text{where } 1 \leqslant k \leqslant n-1.$$

Let us consider the map $\varphi_k = h_k^{-1} \circ \psi_k \circ h_0$, then $\varphi_k \colon [0,1] \to [0,1]$ such that $\varphi_k(0) = 0$ i $\varphi_k(1) = 1$, for all $k \in \{1, 2, \dots, n-1\}$.

As usual, $C(S^1)$ denotes the set of all continuous real functions on S^1, endowed with the uniform convergence topology. We denote by $C_0(I)$ the set of all $f \in C(I)$ such that f preserves the endpoints of the segment. Then the map

$$\langle f \rangle \longmapsto (t_1, t_2, \dots t_{n-1}; \varphi_1, \varphi_2, \dots \varphi_{n-1}) \in I^{n-1} \times (C_0(I))^{n-1}$$

is an embedding.

The image of this embedding is the set

$$\mathcal{K} = \{(t_1, t_2, \dots, t_n) \in I^{n-1} \mid 0 < t_1 < \dots < t_{n-1} < 1\} \times (C_0(I))^{n-1}.$$

The set K is an open convex subset of $I^{n-1} \times (C_0(I))^{n-1}$ and therefore is homeomorphic to l^2. It is well known that every $n \in \mathbb{N}$ the space $I^{n-1} \times (C_0(I))^{n-1}$ is homeomorphic to the Hilbert space l^2 (see [10]).

Theorem 4. $\Psi_{\mathrm{lh}}(S^1) \cong \bigoplus\limits_{n=1}^{\infty} (l^2)_n$.

Let us consider the set of oscillating maps of the circle. Let $\langle g \rangle \in \Psi_{\mathrm{osc}}(X)$, then $g \colon S^1 \longrightarrow I$.

Proposition 2. *The sets $Z = g^{-1}(0)$ and $W = g^{-1}(1)$ are finite.*

Proof. Let us consider the set $Z = g^{-1}(0) = \{z_1, z_2, \dots\} \subset S^1$. Since the map $g \colon S^1 \longrightarrow I$ is open $\mathrm{Int}(Z) = \emptyset$. Suppose that Z is an infinite set. Then for every $\delta > 0$ there exist elements $z_i, z_{i+1} \in Z$ such that $\rho(z_i, z_{i+1}) < \delta$.

Since $g \colon S^1 \longrightarrow I$ is uniformly continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in S^1$ with $\rho(x, y) < \delta$, we have $|g(x) - g(y)| < \varepsilon$.

Let $\varepsilon = 1/3$, then there exists $\delta_0 > 0$ such that from $\rho(x, y) < \delta_0$ it follows that $|g(x) - g(y)| < 1/3$. In the set Z one can find elements z_i and z_{i+1} such that $\rho(z_i, z_{i+1}) < \delta_0$. Let us consider the arc $\omega_i = (z_i, z_{i+1}) \subset S^1$. For any elements $x, y \in \omega_i$ we have $|g(x) - g(y)| < 1/3$, then $g(\omega_i) = [0, a] \subset [0, 1/3]$, and this contradicts the openness of the map $g \colon S^1 \longrightarrow I$. Thus Z is a finite subset in S^1. Similarly, we can prove that W is a finite set.

Thus, $|Z| = |W| = n$, where $n \in \mathbb{N}$. Let $z_1 \in Z$. Then we will move along the circle in the direction counterclockwise. Denote by z_2 the element $z \in Z \setminus \{z_1\}$ which is the closest to the element z_1. Denote by z_3 the element $z \in Z \setminus \{z_1, z_2\}$ which is the closest to the element z_2. So on, unless we number all the elements from the set Z.

Then there exists $w \in W$ such that $z_1 < w < z_2$, denote it by w_1. Let us choose $w_2 \in W \setminus \{w_1\}$ such that $z_2 < w_2 < z_3$. We proceed similarly, unless we number all the elements from the set W. In such a way all the elements from the sets Z and W are situated on the circle $(z_1, w_1, z_2, w_2, \dots z_n, w_n) \subset S^1$ in the order counterclockwise. Therefore, $2n$ points are located on the circle and they divide the circle into $2n$ arcs. Denote by $J_k = [z_k, w_k]$ and $J_{k'} = [w_k, z_{k+1}]$ where $k \in \{1, 2, \dots n\}$, $k' \in \{1, 2, \dots n\}$ and $n \in \mathbb{N}$. Then the restriction of $g\colon S^1 \longrightarrow I$ to J_k or $J_{k'}$ is a homeomorphism, for all $k, k' \in \{1, 2, \dots n\}$.

Let us consider the following maps:

$$h_1\colon [0,1] \to J_1 \text{ such that } h_1(t) = e^{c_1 t + a_1} \text{ where } c_1 = \operatorname{Ln}\left(\frac{w_1}{z_1}\right), a_1 = \operatorname{Ln}(z_1),$$

$$h_k\colon [0,1] \to J_k \text{ such that } h_k(t) = e^{c_k t + a_k} \text{ where } c_k = \operatorname{Ln}\left(\frac{w_k}{z_k}\right), a_k = \operatorname{Ln}(z_k),$$

$$h_{k'}\colon [0,1] \to J_k \text{ such that } h_{k'}(t) = e^{c_k t + a_k} \text{ where } c_k = \operatorname{Ln}\left(\frac{z_{k+1}}{w_k}\right), a_k = \operatorname{Ln}(w_k),$$

$$\psi_k\colon J_1 \to J_k \text{ is defined by the formula } \psi_k(z) = g^{-1}(g(z)) \cap J_k,$$

$$\psi_{k'}\colon J_1 \to J_{k'} \text{ is defined by the formula } \psi_{k'}(z) = g^{-1}(g(z)) \cap J_{k'},$$

where $k \in \{1, 2, \dots n\}$, $k' \in \{1, 2, \dots n\}$ i $n \in \mathbb{N}$.

Let us consider the map $\varphi_k = h_k^{-1} \circ \psi_k \circ h_1$, then $\varphi_k\colon [0,1] \to [0,1]$ is such that $\varphi_k(0) = 0$ and $\varphi_k(1) = 1$, for all $k \in \{1, 2, \dots, n\}$. Similarly, we obtain a map $\varphi_{k'} = h_{k'}^{-1} \circ \psi_{k'} \circ h_1$ such that $\varphi_{k'}(0) = 1$ i $\varphi_{k'}(1) = 0$, for all $k' \in \{1, 2, \dots, n\}$.

Therefore, every $\langle g \rangle \in \Psi^{2n}_{\mathrm{osc}}(X)$ can be identified with the element

$$(z_1, w_1, z_2, w_2, \dots z_n, w_n, \varphi_1, \varphi_{1'}, \dots, \varphi_n, \varphi_{n'}) \in SP^2(SP^{2n}(S^1)) \times (C_0(I))^{2n}.$$

Since $SP^{2n}(S^1) \cong S^1 \overline{\times} I^{2n-1} \simeq S^1$ (see Theorem 3) and $(C_0(I))^{2n} \cong l^2$, we see that

$$SP^2(SP^{2n}(S^1)) \times (C_0(I))^{2n} \cong S^1 \times l^2.$$

Therefore $\Psi^{2n}_{\mathrm{osc}}(X) \cong S^1 \times l^2$, which gives the following result:

Theorem 5. $\Psi_{\mathrm{osc}}(X) \cong \bigoplus\limits_{n=1}^{\infty} (S^1 \times l^2)_n$.

Theorems 1 and 2 give a description of the topology of the connectedness components of the space of open quotient maps of the circle.

Remark. Similarly as in [5] one can show that the space $\Psi(S^1)$ is not the one-point compactification of the space of its components.

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ПРО ПРОСТІР ВІДКРИТИХ ВІДОБРАЖЕНЬ ОДИНИЧНОГО КОЛА

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Розглянуто множину класів еквівалентності неперервних відкритих відображень кола, наділених топологією Вієторіса. Описано топології компонент зв'язності цього простору.

Ключові слова: гіперпростір, відкрите відображення, коло.

О ПРОСТРАНСТВЕ ОТКРЫТЫХ ОТОБРАЖЕНИЙ ЕДИНИЧНОЙ ОКРУЖНОСТИ

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Рассмотрено множество классов эквивалентности открытых отображений окружности наделенных топологией Виеториса. Описано топологии компонент связности этого пространства.

Ключевые слова: гиперпространство, открытое отображение, окружность.