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ON AN INVERSE BOUNDARY VALUE PROBLEM FOR A SECOND ORDER ELLIPTIC EQUATION WITH INTEGRAL CONDITION

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In the paper, an inverse boundary value problem for a second order elliptic equation is investigated. At first the initial problem is reduced to the equivalent problem for which the existence and uniqueness theorem of the solution is proved. Further, using these facts, the existence and uniqueness of the classic solution of the initial problem is proved.

Key words: inverse boundary value problem, elliptic equation, Fourier method, classic solution.

1. Introduction. The inverse problems are favorably developing section of up-to-date mathematics. Recently, the inverse problems are widely applied in various fields of science.

Different inverse problems for various types of partial differential equations have been studied in many papers. First of all we note the papers of A.N. Tikhonov [1], M.M. Lavrentyev [2,3], A.M. Denisov [4], M.I. Ivanchov [5] and their followers.

The goal our paper to prove the uniqueness and existence of the solution of a boundary value problem for a second order elliptic equation with integral condition.

The inverse problems with an integral predetermination condition for parabolic equations were investigated in [6-10].

In the papers [11-15] the inverse boundary value problems were investigated for a second order elliptic equation in a rectangular domain.

2. Problem statement and its reduction to equivalent problem. Consider the equation

$$u_{tt}(x,t)+u_{xx}(x,t)=a(t)u(x,t)+f(x,t) \tag{1}$$

and substitute for it in the domain $D_T=\{(x,t)\,:\,0\le x\le 1,\ 0\le t\le T\}$ an inverse boundary value problem with boundary conditions

$$u(x,0)=\varphi(x),\quad u_t(x,T)=\psi(x)\quad (0\le x\le 1), \tag{2}$$

$$u_x(0,t)=0,\quad (0\le t\le T), \tag{3}$$

with integral condition

$$\int_0^1 u(x,t)dx = 0 \quad (0 \leqslant t \leqslant T) \tag{4}$$

and additional condition

$$u(0,t) = h(t) \quad (0 \le t \le T), \tag{5}$$

where $f(x,t)$, $\varphi(x)$, $\psi(x)$, $h(t)$ are the given functions, and $u(x,t)$, $a(t)$ are the desired functions.

Definition 1. *A classic solution of problem (1)-(5) is a pair $\{u(x,t),\ a(t)\}$ of functions $u(x,t)$ and $a(t)$ possessing the following properties:*

1) *the function $u(x,t)$ is continuous in D_T together with all its derivatives contained in equation (1);*
2) *the function $a(t)$ is continuous on $[0,T]$;*
3) *all the conditions (1)-(5) are satisfied in the ordinary sense.*

For investigating problem (1)-(5), at first consider the following problem:

$$y''(t) = a(t)y(t) \quad (0 \le t \le T), \tag{6}$$

$$y(0) = 0, \quad y'(T) = 0, \tag{7}$$

where $a(t) \in C[0,T]$ is a given function, $y = y(t)$ is a desired function, and under the solution of problem (6),(7) we understand a function $y(t) \in C^2[0,T]$ satisfying in $[0,T]$ equation (6) and conditions (7).

The following lemma is valid.

Lemma 1 ([14])**.** *Let a function $a(t) \in C[0,T]$ be such that*

$$\|a(t)\|_{C[0,T]} \le R = const.$$

Furthermore,

$$\frac{1}{2}T^2R < 1 \tag{8}$$

Then problem (6),(7) has only a trivial solution.

Alongside with inverse boundary value problem consider the following auxiliary inverse boundary value problem. It is required to determine a pair $\{u(x,t),\ a(t)\}$ of functions $u(x,t)$ and $a(t)$ possessing the properties 1) and 2) of the definition of the classic solution of problem (1)-(5) from relations (1)-(3), and

$$u_x(1,t) = 0 \quad (0 \le t \le T), \tag{9}$$

$$h''(t) + u_{xx}(0,t) = a(t)h(t) + f(0,t) \quad (0 \le t \le T). \tag{10}$$

The following lemma is valid.

Lemma 2. *Let $\varphi(x)$, $\psi(x) \in C[0,1]$, $h(t) \in C^2[0,T]$, $h(t) \neq 0$ $(0 \le t \le T)$, $f(x,t) \in C(D_T)$, $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, and the following consistency conditions be fulfilled:*

$$\int_0^1 \varphi(x)dx = 0, \qquad \int_0^1 \psi(x)dx = 0, \tag{11}$$

$$\varphi(0) = h(0), \quad \psi(0) = h'(T). \tag{12}$$

Then the following statements are true:

1) *each classic solution $u(x,t)$, $a(t)$ of problem (1)-(5) is the solution of problem (1)-(3), (9), (10) as well;*
2) *each solution $u(x,t)$, $a(t)$ of problem (1)-(3), (9), (10) is such that*

$$\frac{1}{2}T^2 \|a(t)\|_{C[0,T]} < 1, \tag{13}$$

is the classic solution of problem (1)-(5).

Proof. Let $u(x,t)$, $a(t)$ be a solution of problem (1)-(5). Integrating equation (1) with respect to x from 0 to 1, we have:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx + u_x(1,t) - u_x(0,t) = a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx \quad (0 \le t \le T). \tag{14}$$

Hence, by means of $\int_0^1 f(x,t)dx = 0 \ \ (0 \le t \le T)$ and (3) we obtain (9).

Substituting $x = 0$ in equation (1), we find:

$$u_{tt}(0,t) + u_{xx}(0,t) = a(t)u(0,t) + f(0,t) \quad (0 \le t \le T). \tag{15}$$

Further assuming that $h(t) \in C^2[0,T]$, and differentiating (5) twice, we have:

$$u_{tt}(0,t) = h''(t) \quad (0 \le t \le T).$$

Taking into account the last relation and condition (5) in (15) we obtain (10).

Now suppose that $u(x,t)$, $a(t)$ is a solution of problem (1)-(3), (9), (10), moreover, (13) is fulfilled. Then from (14), allowing for (3) and (9), we find:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - a(t)\int_0^1 u(x,t)dx = 0 \quad (0 \le t \le T). \tag{16}$$

By (2), and (11), it is obvious that

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0, \qquad \int_0^1 u_t(x,T)dx = \int_0^1 \psi(x)dx = 0. \tag{17}$$

Since by Lemma 1, problem (16), (17) has only a trivial solution, then $\int_0^1 u(x,t)dx = 0$ $(0 \le t \le T)$ i.e. condition (4) is satisfied.

Further, from (10) and (15) we get:

$$\frac{d^2}{dt^2}(u(0,t) - h(t)) = a(t)(u(0,t) - h(t)) \quad (0 \le t \le T). \tag{18}$$

By (2) and agreement conditions (12) we have:

$$u(0,0) - h(0) = \varphi(0) - h(0) = 0, \quad u_t(0,T) - h'(T) = \psi(0) - h'(T) = 0. \tag{19}$$

From (18) and (19), by Lemma 2 we conclude that condition (5) is satisfied. The lemma is proved.

3. Investigation of the existence and uniqueness of the classic solution of the inverse boundary value problem.

Let us look for the first component $u(x,t)$ of the solution $u(x,t),\ a(t)$ of problem (1)-(3), (9), (10) in the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x \quad (\lambda_k=k\pi), \tag{20}$$

where

$$u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k x dx \quad (k=0,1,2,...),$$

moreover,

$$m_k=\begin{cases}1, & k=0\\ 2, & k=1,2,....\end{cases}$$

Then applying the formal scheme of the Fourier method, from (1), (2) we get

$$u_k''(t)-\lambda_k^2 u_k(t)=F_k(t;u,a) \quad (k=0,1,2,...;\quad 0\le t\le T), \tag{21}$$

$$u_k(0)=\varphi_k,\quad u_k'(T)=\psi_k \quad (k=0,1,2,...), \tag{22}$$

where

$$F_k(t;u,a)=f_k(t)+a(t)u_k(t),\quad f_k(t)=m_k\int_0^1 f(x,t)\cos\lambda_k x dx$$

$$\varphi_k=m_k\int_0^1\varphi(x)\cos\lambda_k x dx,\quad \psi_k=m_k\int_0^1\psi(x)\cos\lambda_k x dx \quad (k=0,1,2,...)$$

From (21), (22) we obtain:

$$u_0(t)=\varphi_0+\psi_0 t+\int_0^T G_0(t,\tau)F_0(\tau;u,a)d\tau, \tag{23}$$

$$u_k(t)=\frac{ch(\lambda_k(T-t))}{ch(\lambda_k T)}\varphi_k+\frac{sh(\lambda_k t)}{\lambda_k ch(\lambda_k T)}\psi_k+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau \quad (k=1,2,\ldots), \tag{24}$$

where

$$G_0(t,\tau)=\begin{cases}-t, & t\in[0,\tau],\\ -\tau, & t\in[\tau,T],\end{cases} \tag{25}$$

$$G_k(t,\tau)=\begin{cases}\frac{-1}{2\lambda_k ch(\lambda_k T)}[sh(\lambda_k(T+t-\tau))-sh(\lambda_k(T-(t+\tau)))], & t\in[0,\tau],\\ \frac{1}{2\lambda_k ch(\lambda_k T)}[sh(\lambda_k(T-(t+\tau)))-sh(\lambda_k(T-(t-\tau)))], & t\in[\tau,T].\end{cases} \tag{26}$$

After substituting the expressions from (23), (24) into (20), for determining the component of the solution of problem (1)-(3), (9), (10) we get

$$u(x,t)=\varphi_0+t\psi_0+\int_0^T G_0(t,\tau)F_0(\tau;u,a)d\tau+$$

$$+\sum_{k=1}^{\infty}\left\{\frac{ch(\lambda_k(T-t))}{ch(\lambda_k T)}\varphi_k+\frac{sh(\lambda_k t)}{\lambda_k ch(\lambda_k T)}\psi_k+\int_0^T G_k(t,\tau)F_k(\tau;u,a)d\tau\right\}\cos\lambda_k x, \tag{27}$$

where

$$F_k(t;u,a) = f_k(t) + a(t)u_k(t) = m_k \int\limits_0^1 (f(x,t) + a(t)u(x,t)) \cos\lambda_k x dx \quad (k = 0, 1, 2, ...).$$

Now, from (10), allowing for (20) we have:

$$a(t) = h^{-1}(t) \left\{ h''(t) - f(0,t) - \sum_{k=1}^{\infty} \lambda_k^2 u_k(t) \right\}. \tag{28}$$

For obtaining an equation for the second component $a(t)$ of the solution $\{u(x,t)$, $a(t)\}$ of problem (1)-(3), (9) , (10), substitute expression (24) into (28):

$$a(t) = h^{-1}(t) \left\{ h''(t) - f(0,t) - \sum_{k=1}^{\infty} \lambda_k^2 \left[\frac{ch(\lambda_k(T-t))}{ch(\lambda_k T)} \varphi_k + \right.\right.$$

$$\left.\left. + \frac{sh(\lambda_k t)}{\lambda_k ch(\lambda_k T)} \psi_k + \int\limits_0^T G_k(t,\tau) F_k(\tau; u, a) d\tau \right] \right\}. \tag{29}$$

where

$$F_k(t;u,a) = f_k(t) + a(t)u_k(t) = 2 \int\limits_0^1 (f(x,t) + a(t)u(x,t)) \cos\lambda_k x dx \quad (k = 1, 2, ...).$$

Thus, problem (1)-(3), (9), (10) was reduced to system (27), (29) with respect to the unknown functions $u(x,t)$ and a(t).

The following lemma is important for studying the uniqueness of the solution of problem (1)-(3), (9), (10).

Lemma 3. *If* $\{u(x,t),\ a(t)\}$ *is any solution of problem (1)-(3), (9), (10), then the functions*

$$u_k(t) = m_k \int\limits_0^1 u(x,t) \cos\lambda_k x dx \quad (k = 0, 1, 2, ...)$$

satisfy system (23), (24) in $[0,T]$.

Proof. Let $\{u(x,t),\ a(t)\}$ be any solution of problem (1)-(3), (9), (10). Then, having multiplied the both sides of equation (1) by the function $m_k \cos\lambda_k x$ $(k = 0, 1, 2, ...)$, integrating the obtained equality with respect to x from 0 to 1, and using the relations

$$m_k \int\limits_0^1 u_{tt}(x,t) \cos\lambda_k x dx = \frac{d^2}{dt^2} \left(m_k \int\limits_0^1 u(x,t) \cos\lambda_k x dx \right) = u_k''(t) \quad (k = 0, 1, 2, ...),$$

$$m_k \int\limits_0^1 u_{xx}(x,t) \cos\lambda_k x dx = -\lambda_k^2 \left(m_k \int\limits_0^1 u(x,t) \cos\lambda_k x dx \right) = -\lambda_k^2 u_k(t) \quad (k = 0, 1, 2, ...),$$

we get that equation (21) is satisfied.

Similarly, from (2) we get that condition (22) is fulfilled.

Thus, $u_k(t)\ \ (k = 0, 1, 2, ...)$ is a solution of problem (21), (22). Hence it directly follows that the functions $u_k(t)\ \ (k = 0, 1, 2, ...)$ satisfy on $[0, T]$ system (23), (24). The lemma is proved.

Remark 1. From Lemma 3 it follows that for proving the uniqueness of the solution of problem (1)-(3),(9),(10), it suffices to prove the uniqueness of the solution of system (27), (29).

In order to investigate problem (1)-(3), (9), (10), consider the following spaces:
Denote by $B^3_{2,T}$ the set of all the functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = \pi k)$$

considered in D_T, where each of the functions $u_k(t)\ \ (k = 0, 1, 2, ...)$ is continuous on $[0, T]$, and

$$J_T(u) \equiv \|u_0(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} < \infty.$$

In this set, we determine the operation of addition and multiplication by the number (real) in the usual way: under the zero element of this set we understand the function $u(x,t) \equiv 0$ on D_T, and determine the norm in this set by the formula

$$\|u(x,t)\|_{B^3_{2,T}} = J_T(u).$$

Prove that all these spaces are Banach spaces. Indeed, the validity of the first two axioms of the norms is obvious, and validity of the third axiom of the norm is easily established by means of the summator inequality of Minkowsky; consequently, $B^3_{2,T}$ is a linear normalized space. Now prove its completeness. Let

$$u_n(x,t) = \sum_{k=0}^{\infty} u_{k,n}(t) \cos \lambda_k x \quad (n = 1, 2, ...)$$

be any sequence which is fundamental in $B^3_{2,T}$. Then for any $\varepsilon > 0$ there exists a number n_ε such that

$$\|u_n(x,t) - u_m(x,t)\|_{B^3_{2,T}} = \|u_{0,n}(t) - u_{0,m}(t)\|_{C[0,T]} +$$

$$+ \left(\sum_{k=1}^{\infty} \left(\lambda_k^3 \|u_{k,n}(t) - u_{k,m}(t)\|_{C[0,T]} \right)^2 \right)^{\frac{1}{2}} < \varepsilon \qquad (30)$$

$\forall n, m \geq n_\varepsilon$

Consequently, for any fixed $k\ \ (k = 1, 2, ...)$:

$$\|u_{0,n}(t) - u_{0,m}(t)\|_{C[0,T]} < \varepsilon,$$

$$\|u_{k,n}(t) - u_{k,m}(t)\|_{C[0,T]} < \varepsilon \quad \forall n, m \geq n_\varepsilon \qquad (31)$$

This means that the sequences $\{u_{0,n}(t)\}_{n-1}^{\infty}$ and for any fixed k $(k=1,2,...)$: the sequences $\{u_{0,n}(t)\}_{n-1}^{\infty}$ are fundamental in $C[0,T]$ and consequently by the completeness of $C[0,T]$ they converge in the space $C[0,T]$:

$$u_{0,n}(t) \overset{C[0,T]}{\to} u_{0,0}(t) \in C[0,T] \ \ as \ \ n\to\infty,$$
$$u_{k,n}(t) \overset{C[0,T]}{\to} u_{k,0}(t) \in C[0,T] \ \ as \ \ n\to\infty. \qquad (32)$$

Further, by (30), for any fixed number N:

$$\|u_{0,n}(t)-u_{0,m}(t)\|_{C[0,T]} + \left(\sum_{k=1}^{N}\left(\lambda_k^3\left\|u_{k,n}(t)-u_{k,m}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} < \varepsilon \ \ \forall n,m\geq n_\varepsilon. \qquad (33)$$

Using relations (32) and passing to limit as $m\to\infty$ in (33), we obtain

$$\|u_{0,n}(t)-u_{0,0}(t)\|_{C[0,T]} + \left(\sum_{k=1}^{N}\left(\lambda_k^3\left\|u_{k,n}(t)-u_{k,m}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} < \varepsilon \ \ \forall n\geq n_\varepsilon. \quad (34)$$

Hence, by arbitrariness of N (or equivalently, passing to limit as $N\to\infty$), we obtain

$$\|u_{0,n}(t)-u_{0,0}(t)\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_{k,n}(t)-u_{k,0}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} < \varepsilon \ \ \forall n\geq n_\varepsilon. \quad (35)$$

Accept the denotation

$$u_0(x,t)=\sum_{k=0}^{\infty}u_{k,0}(t)\cos\lambda_k x. \qquad (36)$$

Since $u_0(x,t)=[u_0(x,t)-u_{n_\varepsilon}(x,t)]+u_{n_\varepsilon}(x,t)$, and by (35) $u_0(x,t)-u_{n_\varepsilon}(x,t)\in B_{2,T}^3$, and also $u_{n_\varepsilon}(x,t)\in B_{2,T}^3$, we get that

$$u_0(x,t)\in B_{2,T}^3.$$

Then, by (35) for any $\varepsilon>0$ there exists a number n_ε such that

$$\|u_n(x,t)-u_0(x,t)\|_{B_{2,T}^3}\leq\varepsilon \ \ \forall n\geq n_\varepsilon.$$

And this means that the sequence $u_n(x,t)$ converges in $B_{2,T}^3$ to the element $u_0(x,t)\in B_{2,T}^3$. This proves the completeness and consequently the Banach property of the space $B_{2,T}^3$.

Denote by E_T^3 the space $B_{2,T}^3\times C[0,T]$ of the vector-functions $z(x,t)=\{u(x,t),a(t)\}$ with the norm

$$\|z(x,t)\|_{E_T^3}=\|u(x,t)\|_{B_{2,T}^3}+\|a(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^3$ and E_T^3 are Banach spaces.

Now, in the space E_T^3 consider the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\},$$

where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,$$

$$\Phi_2(u,a)=\tilde{a}(t),$$

$\tilde{u}_0(t),\tilde{u}_k(t)$ $(k=1,2,...)$ and $\tilde{a}(t)$ equal the right sides of (23), (24) and (29), respectively.

It is easy to see that

$$\frac{\operatorname{sh}(\lambda_k t)}{\operatorname{ch}(\lambda_k T)}<1 \quad (0\le t\le T), \qquad \frac{\operatorname{ch}(\lambda_k (T-t))}{\operatorname{ch}(\lambda_k T)}\le 1 \quad (0\le t\le T),$$

$$\frac{\operatorname{sh}(\lambda_k (T+t-\tau))}{\operatorname{ch}(\lambda_k T)}\le 1 \quad (0\le t\le \tau\le T), \qquad \frac{\operatorname{sh}(\lambda_k (T-(t+\tau)))}{\operatorname{ch}(\lambda_k T)}\le 1 \quad (0\le t\le \tau\le T),$$

$$\frac{\operatorname{sh}(\lambda_k (T-(t+\tau)))}{\operatorname{ch}(\lambda_k T)}\le 1 \quad (0\le \tau\le t\le T), \qquad \frac{\operatorname{sh}(\lambda_k (T-(t-\tau)))}{\operatorname{ch}(\lambda_k T)}\le 1 \quad (0\le \tau\le t\le T).$$

Taking into account these relations, by means of simple transformations we find

$$|\tilde{u}_0(t)|\le|\varphi_0|+T\,|\psi_0|+2T\int_0^T|f_0(\tau)|\,d\tau+2T\int_0^T|a(\tau)|\,|u_0(\tau)|\,d\tau,$$

$$|\tilde{u}_k(t)|\le|\varphi_k|+\tfrac{1}{\lambda_k}\,|\psi_k|+\tfrac{1}{\lambda_k}\sqrt{T}\left(\int_0^T|f_k(\tau)|^2d\tau\right)^{\frac12}+\tfrac{1}{\lambda_k}T\,\|a(t)\|_{C[0,T]}\,\|u_k(t)\|_{C[0,T]}.$$

$$|\tilde{a}(t)|\le\left|h^{-1}(t)\right|\{|h''(t)-f(0,t)|$$
$$+\left(\sum_{k=1}^{\infty}\tfrac{1}{\lambda_k^2}\right)^{\frac12}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\,|\varphi_k|\right)^2\right)^{\frac12}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\,|\psi_k|\right)^2\right)^{\frac12}+\right.$$
$$\left.\left.+\tfrac{1}{\sqrt6}\sqrt{T}\left(\int_0^T(\lambda_k^2\,|f_k(\tau)|)^2d\tau\right)^{\frac12}+\tfrac{1}{\sqrt6}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac12}\right]\right\},$$

or

$$\|\tilde{u}_0(t)\|_{C[0,T]}\le$$
$$\le|\varphi_0|+T\,|\psi_0|+2T\sqrt{T}\left(\int_0^t|f_0(\tau)|^2\,d\tau\right)^{\frac12}+2T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,, \tag{37}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac12}\le 2\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac12}+2\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_k|)^2\right)^{\frac12}+$$
$$+2\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_k(\tau)|)^2d\tau\right)^{\frac12}+2T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac12}, \tag{38}$$

$$\left\|\breve{a}(t)\right\|_{C[0,T]}\le\left\|h^{-1}(t)\right\|_{C[0,T]}\left\{\|h''(t)-f(0,t)\|_{C[0,T]}+\tfrac{1}{\sqrt6}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac12}+\right.$$
$$+\tfrac{1}{\sqrt6}\left(\sum_{k=1}^{\infty}(\lambda_k^2\,|\psi_k|)^2\right)^{\frac12}++\tfrac{1}{\sqrt6}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2\,|f_k(\tau)|)^2d\tau\right)^{\frac12}+$$
$$\left.\left.+\frac{1}{\sqrt6}\;T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac12}\right]\right\}. \tag{39}$$

Suppose that the data of problem (1)-(3), (9), (10) satisfy the following conditions:

1) $\varphi(x) \in C^2[0,1]$, $\varphi'''(x) \in L_2(0,1)$ and $\varphi'(0) = \varphi'(1) = 0$;
2) $\psi(x) \in C^1[0,1]$, $\psi''(x) \in L_2(0,1)$ and $\psi'(0) = \psi'(1) = 0$;
3) $f(x,t)$, $f_x(x,t) \in C(D_T)$, $f_{xx}(x,t) \in L_2(D_T)$ and $f_x(0,t) = f_x(1,t) = 0$ $(0 \le t \le T)$;
4) $h(t) \in C^2[0,T]$, $h(t) \ne 0$ $(0 \le t \le T)$.

Then from (37)-(39), we obtain:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \le A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{40}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{41}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + 2T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} +$$
$$+2\,\|\varphi'''(x)\|_{L_2(0,1)} + 2\,\|\psi''(x)\|_{L_2(0,1)} + 2\sqrt{T}\,\|f_{xx}(x,t)\|_{L_2(D_T)},$$
$$B_1(T) = 2T(T+1);$$
$$A_2(T) = \left\|h^{-1}(t)\right\|_{C[0,T]}\left\{\|h''(t) - f(0,t)\|_{C[0,T]} + \frac{1}{\sqrt{6}}\,\|\varphi'''(x)\|_{L_2(0,1)} + \right.$$
$$\left. + \frac{1}{\sqrt{6}}\,\|\psi''(x)\|_{L_2(D_T)} + \frac{1}{\sqrt{6}}\sqrt{T}\,\|f_{xx}(x,t)\|_{L_2(D_T)}\right\};$$
$$B_2(T) = \left\|h^{-1}(t)\right\|_{C[0,T]} \cdot \frac{1}{\sqrt{6}} \cdot T \quad .$$

From inequalities (40), (41) we obtain:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}, \tag{42}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

So, we can prove the following theorem.

Theorem 1. *Let conditions 1-4 be satisfied, and*

$$(A(T)+2)^2 B(T) < 1. \tag{43}$$

Then problem (1)-(3), (9), (10) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \le R = A(T)+2)$ *of the space* E^3_T.

Proof. In the space E^3_T consider the equation

$$z = \Phi z, \tag{44}$$

where the components $\Phi_i(u,a)$ $(i = 1,2)$ of the operator $\Phi(u,a)$ are defined from the right sides of equations (27) and (29).

Consider the operator $\Phi(u,a)$ in the ball $K = K_R$ from E^3_T. Similar to (42), we get that for any $z, z_1, z_2 \in K_R$ the following estimates are valid:

$$\|\Phi z\|_{E^3_T} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} \le$$
$$\le A(T) + B(T)R^2 \le A(T) + B(T)\left(A(T)+2\right)^2, \tag{45}$$

$$\|\Phi z_1-\Phi z_2\|_{E^3_T}\le B(T)R\left(\|a_1(t)-a_2(t)\|_{C[0,T]}+\|u_1(x,t)-u_2(x,t)\|_{B^3_{2,T}}\right)\leqslant$$
$$\le B\left(T\right)\left(A\left(T\right)+2\right)\|z_1-z_2\|_{E^3_T}. \qquad (46)$$

Then allowing for (43), from estimations (45) and (46) it follows that the operator Φ acts in the ball $K=K_R$ and is contractive. Therefore, in the ball $K=K_R$ the operator Φ has a unique fixed point $\{u,a\}$ that is a unique solution of equation (44) in the ball $K=K_R$, i.e. it is a unique solution of system (27), (29) in the ball $K=K_R$.

The function $u(x,t)$ as an element of the space $B^3_{2,T}$ is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t)$ in D_T.

From (21) it in easy to see that

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\|u''_k(t)\|_{c[0,T]}\right)^2\right)^{\frac12}\le\sqrt2\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{c[0,T]}\right)^2\right)^{\frac12}+$$
$$+\sqrt2\left\|\|a(t)u_x(x,t)+f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}.$$

Hence it follows that $u_{tt}(x,t)$ is continuous in D_T.

It is easy to verify that equation (1) and conditions (2), (3), (9), (10) are satisfied in the ordinary sense.

Consequently, $u(x,t),\ a(t)$ is a solution of problem (1)-(3), (9), (10), and by Lemma 3 it is unique in the ball $K=K_R$. The theorem is proved.

By means of Lemma 2, a unique solvability of initial problem (1)-(5) follows from the last theorem.

Theorem 2. *Let all the conditions of Theorem 1 be satisfied*

$$\int_0^1 f(x,t)dx=0\quad(0\le t\le T),\qquad\int_0^1\varphi(x)dx=0,\qquad\int_0^1\psi(x)dx=0,$$
$$\varphi(0)=h(0),\quad\psi(0)=h'(T),\quad\frac12(A(T)+2)T^2<1.$$

Then problem (1)-(5) has a unique classic solution in the ball $K=K_R(\|z\|_{E^3_T}\le R=$ $=A(T)+2)$ *of the space* E^3_T.

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ПРО ОБЕРНЕНУ КРАЙОВУ ЗАДАЧУ ДЛЯ ЕЛІПТИЧНОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ З ІНТЕГРАЛЬНОЮ УМОВОЮ

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Розглянуто обернену крайову задачу для еліптичного рівняння другого порядку. Спочатку вихідна задача зводиться до еквівалентної задачі, для якої доведено теореми існування та єдиності її розв'язку. Використовуючи цей факт, доведено існування та єдиність розв'язку вихідної задачі.

Ключові слова: обернена крайова задача, еліптичне рівняння, метод Фур'є, класичний розв'язок.

ОБ ОБРАТНОЙ ЗАДАЧЕ ДЛЯ ЕЛЛИПТИЧЕСКОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ИНТЕГРАЛЬНЫМ УСЛОВИЕМ

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Рассмотрено обратную граничную задачу для еллиптического уравнения второго порядка. Сначала исходная задача сводится к эквивалентной задаче, для которой доказано теоремы существования и единственности её решения. Используя этот факт, доказано существование и единственность решения исходной задачи.

Ключевые слова: обратная граничная задача, еллиптическое уравнение, метод Фурьє, классическое решение.