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INVERSE PROBLEM TO FRACTIONAL DIFFUSION EQUATION WITH UNKNOWN YOUNG COEFFICIENT

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We study the inverse problem for linear nonhomogeneous diffusion equation with regulating fractional derivative of order $\beta \in (0,2)$ with respect to time on a bounded cylindrical domain $\Omega_0 \times (0,T]$, the inverse problem on determination of a pair of functions: a classical solution u of the first boundary value problem to such an equation and unknown, depending on time variable, continuous coefficient in young term of the equation under the over-determination condition

$$\int\limits_{\Omega_0} u(x,t)\varphi(x)dx = F(t), \quad t \in [0,T]$$

with some given functions φ and F. The unique solvability of the problem is established. The Green function method is used.

Key words: fractional derivative, inverse boundary value problem, the Green vector function, operator equation.

1. Introduction. The conditions of classical solvability of the first boundary value problem to the equation

$$D_t^\beta u(x,t) - a^2 \Delta u(x,t) = F_0(x,t), \quad a^2 = \text{const} > 0$$

with regulating fractional derivative (see [1] – [3])

$$D_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)} \int\limits_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\beta} d\tau = \frac{1}{\Gamma(1-\beta)} \left[\frac{\partial}{\partial t} \int\limits_0^t \frac{u(x,\tau)}{(t-\tau)^\beta} d\tau - \frac{u(x,0)}{t^\beta} \right], \quad \beta \in (0,1),$$

$$D_t^\beta u(x,t) = \frac{1}{\Gamma(2-\beta)} \int\limits_0^t \frac{u_{\tau\tau}(x,\tau)}{(t-\tau)^{\beta-1}} d\tau = \frac{1}{\Gamma(2-\beta)} \left[\frac{\partial}{\partial t} \int\limits_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^{\beta-1}} d\tau - \frac{u_t(x,0)}{(t-\tau)^{\beta-1}} \right], \text{ for } \beta \in (1,2)$$

were obtained in [4], [5]. In [6]–[11] there were proved the existence and uniqueness theorems and also the representations, by means of the Green functions, of classical solutions of fractional Cauchy problems to equations of such kind. In [12]–[14] the solvability of fractional Cauchy problems in spaces of generalizes functions was proved.

In this note we prove the existence and uniqueness of the solution (u, b) of the inverse boundary value problem

$$D_t^\beta u - \Delta u - b(t)u = F_0(x,t), \quad (x,t) \in \Omega_0 \times (0,T], \tag{1}$$

$$u(x,t) = 0, \quad (x,t) \in \Omega_0 \times [0,T], \tag{2}$$

$$u(x,0) = F_1(x), \quad u_t(x,0) = F_2(x), \quad x \in \overline{\Omega}_0, \tag{3}$$

$$\int\limits_{\Omega_0} u(x,t)\varphi_0(x)dx = F(t), \quad t \in [0,T] \tag{4}$$

where $\beta \in (0,2)$, Ω_0 is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with the boundary $\Omega_1 = \partial\Omega$ of class C^{1+s}, $s \in (0,1)$, F_0, F_1, F_2, F, φ_0 are given functions. The second initial condition in (3) is absent in the case $\beta \in (0,1]$.

Note that for $\beta = 1$ (respectively, $\frac{\partial}{\partial t}$ in place of D_t^1) the inverse boundary value problems on determination of a pair of functions (u, b) were studied in [15] and other works, where existence and uniqueness theorems were proved. Some inverse boundary value problems to diffusion-wave equation with different unknown functions or parameters were investigated, for example, in [16]–[23].

We shall use the Green function method to prove the solvability of this problem.

2. Main notations and auxiliary results. We shall use the following notations $Q_i = \Omega_i \times (0,T]$, $i = 0,1$, $Q_2 = \Omega_0$,

$\mathfrak{D}(R^m)$ is the space of indefinitely differentiable functions with compact supports in R^m, $m = 1,2,\ldots$, $\mathfrak{D}(\overline{Q}_0) = \{v \in C^\infty(\overline{Q}_0) : (\frac{\partial}{\partial t})^k v|_{t=T} = 0, \quad k = 0,1,\ldots\}$,

$\mathfrak{D}'(R^m)$ and $\mathfrak{D}'(\overline{Q}_0)$ are the spaces of linear continuous functionals (generalized functions [24], p. 13-15) on $\mathfrak{D}(R^m)$ and $\mathfrak{D}(\overline{Q}_0)$, respectively, (f,φ) is the value of $f \in \mathfrak{D}'(R^m)$ on the test function $\varphi \in \mathfrak{D}(R^m)$ and also the value of $f \in \mathfrak{D}'(\overline{Q}_0)$ onto $\varphi \in \mathfrak{D}(\overline{Q}_0)$.

We denote by $*$ the operation of convolution of generalized functions f and g, use the function

$$f_\lambda(t) = \tfrac{\theta(t)t^{\lambda-1}}{\Gamma(\lambda)} \text{ for } \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \text{ for } \lambda \leqslant 0,$$

where $\Gamma(z)$ is the Gamma-function, $\theta(t)$ the Heaviside function. The relation

$$f_\lambda * f_\mu = f_{\lambda+\mu} holds.$$

Note that Riemann-Lioville derivative $v_t^{(\beta)}(x,t)$ of order $\beta > 0$ is defined by

$$v_t^{(\beta)}(x,t) = f_{-\beta}(t) * v(x,t)$$

and $D_t^\beta v(x,t) = v_t^{(\beta)}(x,t) - f_{1-\beta}(t)v(x,0)$ for $\beta \in (0,1)$,
$D_t^\beta v(x,t) = v_t^{(\beta)}(x,t) - f_{1-\beta}(t)v(x,0) - f_{2-\beta}(t)v_t(x,0)$ for $\beta \in (1,2)$.

Let $C(Q_0)$, $C(\overline{Q}_0)$, $C[0,T]$ be the classes of continuous functions on Q_0, $\overline{Q}_0$ and $[0,T]$, respectively, $C^\gamma(\Omega_0)$ $(C^\gamma(\bar{\Omega}_0))$ be the class of bounded continuous functions on Ω_0 $(\bar{\Omega}_0)$ satisfying the Hölder condition, $C^\gamma(Q_i)$ $(C^\gamma(\bar{Q}_i))$ be the class of bounded continuous functions on Q_i $(\bar{Q}_i)$ which for all $t \in (0,T]$ satisfy the Hölder condition with respect to the space variables, $i = 0,1$, $C_{2,\beta}(Q_0) = \{v \in C(Q_0) \,|\, \Delta v, D_t^\beta v \in C(Q_0)\}$,

$C_{2,\beta}(\bar{Q}_0) = C_{2,\beta}(Q_0) \cap C(\bar{Q}_0)$ in the case $\beta \in (0,1]$, $C_{2,\beta}(\bar{Q}_0) = \{v \in C_{2,\beta}(Q_0) \mid v, v_t \in C(\bar{Q}_0)\}$ if $\beta \in (1,2)$.

Suppose that the following assumptions hold:

(F0) $F_0 \in C^{\gamma}(Q_0)$, $\gamma \in (0,1)$,

(F1) $F_1 \in C^{\gamma}(\bar{\Omega}_0)$, $F_1|_{\Omega_1} = 0$,

(F2) $F_2 \in C^{\gamma}(\bar{\Omega}_0)$,

(F) $F, D^{\beta}F \in C[0,T]$, it exists $f := \inf\limits_{t \in [0,T]} |F(t)| > 0$,

(Φ) $\varphi_0 \in C^2(\bar{\Omega}_0)$, $\varphi_0|_{\Omega_1} = 0$.

Definition 1. *A pair of functions*

$$(u, b) \in \mathfrak{M}_{\beta}(Q_0) = \mathfrak{M}_{\beta} := C_{2,\beta}(\bar{Q}_0) \times C[0,T]$$

satisfying equation (1) on Q_0 and conditions (2)–(4) is called the solution of problem (1)–(4).

It follows from (3) and (4) the necessary concordance conditions

$$\int\limits_{\Omega_0} F_1(x)\varphi_0(x)dx = F(0), \quad \int\limits_{\Omega_0} F_2(x)\varphi_0(x)dx = F'(0). \tag{5}$$

We introduce the operators

$$L: \quad (Lv)(x,t) \equiv v_t^{(\beta)}(x,t) - \Delta v(x,t), \qquad (x,t) \in \overline{Q}_0, \qquad v \in \mathfrak{D}'(\overline{Q}_0),$$

$$L^{reg}: \quad (L^{reg}v)(x,t) \equiv D_t^{\beta} v(x,t) - \Delta v(x,t), \qquad (x,t) \in \overline{Q}_0, \quad v \in C_{2,\beta}(\bar{Q}_0).$$

Definition 2. *The vector-function $(G_0(x,t,y,\tau), G_1(x,t,y), G_2(x,t,y))$ such that under rather regular g_0, g_1, g_2 the function*

$$u(x,t) = \int\limits_0^t d\tau \int\limits_{\Omega_0} G_0(x,t,y,\tau) g_0(y,\tau) dy + \sum_{j=1}^{2} \int\limits_{\Omega_0} G_j(x,t,y) g_j(y) dy, \quad (x,t) \in \overline{Q}_0 \tag{6}$$

is the classical (from $C_{2,\beta}(\bar{Q}_0)$) solution of the problem

$$L^{reg} u(x,t) = g_0(x,t), \quad (x,t) \in Q_0, \tag{7}$$

$$u(x,t) = 0, \quad (x,t) \in \bar{Q}_1, \tag{8}$$

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \quad x \in \bar{\Omega}_0 \tag{9}$$

is called the Green vector-function of this problem.

From Definition 2 it follows that

$(LG_0)(x,t,y,\tau) = \delta(x-y, t-\tau)$, $(x,t), (y,\tau) \in Q_0$ where δ is the Dirac delta-function,

$(L^{reg}G_j)(x,t,y) = 0$, $(x,t) \in Q_0$, $y \in \Omega_0$, $j = 1,2$, $G_1(x,0,y) = \delta(x-y)$,

$\frac{\partial}{\partial t} G_1(x,0,y) = 0$, $G_2(x,0,y) = 0$, $\frac{\partial}{\partial t} G_2(x,0,y) = \delta(x-y)$, $x, y \in \Omega_0$.

Lemma 1. *The following relations hold:*

$$G_j(x,t,y) = \int_0^t f_{j-\beta}(\tau) G_0(x,t,y,\tau)d\tau, \quad (x,t) \in \bar{Q}_0,\ y \in \Omega_0,\ j{=}1{,}2.$$

The lemma can be proved by using the scheme of the proof of the corresponding result in [12].

Lemma [23]**.** *The Green vector-function of the first boundary value problem (7)–(9) exists.*

Remark 1. From the maximum principle [4] and Lemma 1 it follows that

$$G_0(x,t,y,\tau) > 0 \ \text{ for } \ (x,t),(y,\tau) \in Q_0, \quad G_i(x,t,y) > 0, \quad (x,t) \in Q_0, \quad y \in \Omega_0,$$

From the results of [6], [11] it follows that

$$G(x,t) = \frac{\pi^{N/2} t^{\beta-1}}{|x|^N} H^{2,0}_{1,2}\left(\frac{|x|^2}{4t^\beta}\middle| \begin{matrix}(\beta,\beta) & \\ (1,1) & (N/2,1)\end{matrix}\right), \tag{10}$$

is the fundamental function of the operator L where
$H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(a_1,\gamma_1) & \dots & (a_p,\gamma_p)\\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}\right)$ is the H-function of Fox ([25]).

Using the properties of the H-functions the following estimates are found (see, for example, [6], [14]):

$$|G(x,t)| \le \frac{C_0^*}{t|x|^{N-2}}, \quad |f_{j-\beta}(t) * G(x,t)| \le \frac{C_j^*}{t^{\beta-j+1}|x|^{N-2}}, \quad j = 1,2, \quad |x|^2 < 4t^\beta,$$

$$|G(x,t)| \le \frac{\hat{C}_0 t^{\beta-1}}{|x|^N} \cdot \left(\frac{|x|^2}{4t^\beta}\right)^{1+\frac{N}{2(2-\beta)}} e^{-c\left(\frac{|x|^2}{4a_0 t^\beta}\right)^{\frac{1}{2-\beta}}} \le \frac{C_0}{t^{1-\beta}|x|^N},$$

$$|f_{j-\beta}(t) * G(x,t)| \le \frac{\hat{C}_j t^{j-1}}{|x|^N} \cdot \left(\frac{|x|^2}{4t^\beta}\right)^{\frac{N}{2(2-\beta)}} e^{-c\left(\frac{|x|^2}{4t^\beta}\right)^{\frac{1}{2-\beta}}} \le \frac{C_j t^{j-1}}{|x|^N}, \quad j = 1,2, \quad |x|^2 > 4t^\beta$$

where $c, C_i^*, C_i, \hat{C}_i \quad (i = 0,1,2)$ are some positive constants.

Remark 2. According to Levy method we obtain the same kind of estimates for the functions $G_0(x,t,y,\tau)$, $G_j(x,t,y)$ as for $G(x-y,t-\tau)$, $\int_0^t f_{j-\beta}(\tau)G(x-y,t-\tau)d\tau$, $j = 1,2$, respectively.

From the results of [7], [8] it follows that the estimates

$$|G_i(x+\Delta x, t+\Delta t, y,\tau) - G_i(x,t,y,\tau)| \le A_i(x,t,y,\tau)[|\Delta x| + |\Delta t|^{\beta/2}]^\gamma \tag{11}$$
$$\forall (x,t),(x+\Delta x,t+\Delta t) \in \bar{Q}_0, (y,\tau) \in \bar{Q}_i, \quad i = 0,1,2$$

hold with some $0 < \gamma < 1$ where non-negative functions $A_i(x,t,y,\tau)$ have the same kind of estimates as for $G_i(x,t,y,\tau)$, $i = 0,1,2$, respectively, and $G_i(x,t,y,\tau) = G_i(x,t,y)$, $A_i(x,t,y,\tau) = A_i(x,t,y)$ for $i = 1,2$. Note that for general parabolic boundary value problem the Hölder conditions for all components of the Green vector-function were obtained in [26], [27].

Theorem 1. *If $g_0 \in C^\gamma(Q_0)$, $\gamma \in (0,1)$, $g_j \in C^\gamma(\bar\Omega_0)$, $j = 1,2$, $g_1|_{\Omega_1} = 0$ then there exists the unique solution $u \in C_{2,\beta}(\bar Q_0)$ of the problem (7), (8), (9), it is defined by*

$$u(x,t) = \big(\mathfrak{G}_0 g_0\big)(x,t) + \big(\mathfrak{G}_1 g_1\big)(x,t) + \big(\mathfrak{G}_2 g_2\big)(x,t), \quad (x,t) \in \bar Q_0 \qquad (12)$$

where $(\mathfrak{G}_0 g_0)(x,t) = \int\limits_0^t d\tau \int\limits_{\Omega_0} G_0(x,t,y,\tau) g_0(y,\tau)dy$, $(\mathfrak{G}_j g_j)(x,t) = \int\limits_{\Omega_0} G_j(x,t,y) g_j(y)dy$, $j = 1,2$.

Proof. Taking into account Lemma 1 and Remark 2, as in [8], [13], [26], we show that function (12) belongs to $C_{2,\beta}(\bar Q_0)$ and satisfies problem (1)–(3). Namely, we have

$$\Big|\int\limits_0^t d\tau \int\limits_{\Omega_0} G_0(x,t,y,\tau) g_0(y,\tau)dy\Big|$$
$$\le \int\limits_0^t d\tau \Big[\int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|<2(t-\tau)^{\beta/2}\}} G_0(x,t,y,\tau)|g_0(y,\tau)|dy$$
$$+ \int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|>2(t-\tau)^{\beta/2}\}} G_0(x,t,y,\tau)|g_0(y,\tau)|dy\Big]$$
$$\le \int\limits_0^t d\tau \Big[\int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|<2(t-\tau)^{\beta/2}\}} \tfrac{C^* dy}{(t-\tau)|y-x|^{N-2}}|g_0(y,\tau)|dy$$
$$+ \int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|>2(t-\tau)^{\beta/2}\}} \tfrac{C^* dy}{(t-\tau)^{1-\beta}|y-x|^{N}}|g_0(y,\tau)|dy\Big]$$
$$\le C\int\limits_0^t \Big[\tfrac{1}{(t-\tau)} \int\limits_0^{2(t-\tau)^{\beta/2}} r dr + \tfrac{1}{(t-\tau)^{1-\beta}} \int\limits_{2(t-\tau)^{\beta/2}}^{diam\Omega_0} \tfrac{dr}{r}\Big] d\tau \cdot \|g_0\|_{C(\bar Q_0)}$$
$$\le \hat C \int\limits_0^t \Big[(t-\tau)^{\beta-1} + (t-\tau)^{\beta-1} ln \tfrac{diam\Omega_0}{(t-\tau)^{\beta/2}}\Big] d\tau \cdot \|g_0\|_{C(\bar Q_0)}$$
$$\le k_0 t^{\beta_1} \cdot \|g_0\|_{C(\bar Q_0)} \quad \forall (x,t) \in \bar Q_0,$$

where $C^*, C, \hat C, \hat k, k_0$ are some positive constants, $\beta_1 = \beta - \varrho$, ϱ is an arbitrary number from $(0,1)$, $||g_0||_{C(\bar Q_0)} := \sup\limits_{(x,t)\in Q_0} |g_0(x,t)|$.

Similarly, we obtain the uniform convergence, and therefore continuity on $\bar Q_0$, of the other term in the right-hand side of (12):

$$\Big|\int\limits_{\Omega_0} G_j(x,t,y) g_j(y)dy\Big| \le \Big[\int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|<2t^{\beta/2}\}} G_j(x,t,y)dy +$$
$$\int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|>2t^{\beta/2}\}} G_j(x,t,y)dy\Big] \cdot ||g_j||_{C(\bar\Omega_0)}$$
$$\le c_0 \Big[\int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|<2t^{\beta/2}\}} \tfrac{t^{j-1-\beta}}{|x|^{N-2}} dy$$
$$+ \int\limits_{\{(y,\tau)\in\Omega_0 : |y-x|>2t^{\beta/2}\}} \tfrac{t^{j-1}}{|y-x|^N} \cdot \big(\tfrac{|y-x|^2}{4t^\beta}\big)^{\frac{N}{2(2-\beta)}} e^{-c\left(\frac{|y-x|^2}{4t^\beta}\right)^{\frac{1}{2-\beta}}} dy\Big] \cdot ||g_j||_{C(\bar\Omega_0)}$$
$$\le \hat k_j t^{j-1} \Big[1 + \int\limits_{2t^{\beta/2}}^{diam\Omega_0} r^{\frac{N}{2-\beta}-1} t^{-\frac{\beta N}{2(2-\beta)}} e^{-\hat c\left(\frac{r^2}{t^\beta}\right)^{\frac{1}{2-\beta}}} dr\Big] \cdot ||g_1||_{C(\bar\Omega_0)}$$
$$\le k_j t^{j-1} ||g_j||_{C(\bar\Omega_0)}, \quad (x,t) \in \bar Q_0, \quad j = 1,2$$

where $c_0, \hat c, \hat k_j, k_j$ $(j = 1,2)$ are positive constants.

As in [8], [27] we show that under conditions of the theorem function (12) belongs to the required space and satisfies the problem. The uniqueness of the solution follows from the maximum principle [4].

3. The existence and uniqueness theorems for inverse problem. We pass to proof of the existence of the solution of inverse problem (1)–(4).

From Theorem 1 it follows that under assumptions (F0), (F1), (F2) and known $b \in C[0,T]$ the solution $u \in C_{2,\beta}(\bar{Q}_0)$ of the first boundary value problem (1)–(3) satisfies the integral equation

$$u(x,t) = \big(\mathfrak{G}_0\big(bu + F_0\big)\big)(x,t) + \big(\mathfrak{G}_1 F_1\big)(x,t) + \big(\mathfrak{G}_2 F_2\big)(x,t), \quad (x,t) \in \bar{Q}_0,$$

that is

$$u(x,t) = \int_0^t d\tau \int_{\Omega_0} \mathfrak{G}_0(x,t,y,\tau) b(\tau) u(y,\tau) dy + h(x,t), \quad (x,t) \in \bar{Q}_0 \tag{13}$$

where

$$h(x,t) = \int_0^t d\tau \int_{\Omega_0} G_0(x,t,y,\tau) F_0(y,\tau) dy + \sum_{j=1}^{2} \int_{\Omega_0} G_j(x,t,y) F_j(y) dy, \quad (x,t) \in \bar{Q}_0. \tag{14}$$

Conversely, any solution $u \in C^\gamma(Q_0)$ of integral equation (13) (with known $b \in C[0,T]$) belongs to the space $C_{2,\beta}(\bar{Q}_0)$ and is a solution of problem (1)–(3).

From equation (1) and conditions (2), (Φ) it follows that

$$\int_{\Omega_0} D_t^\beta u(x,t) \varphi_0(x) dx = \int_{\Omega_0} u(x,t) \Delta\varphi_0(x) dx + b(t) \int_{\Omega_0} u(x,t) \varphi_0(x) dx + \int_{\Omega_0} F_0(x,t) \varphi_0(x) dx,$$

$t \in (0,T]$. Using condition (4) we obtain

$$D_t^\beta F(t) = \int_{\Omega_0} u(x,t) \Delta\varphi_0(x) dx + b(t) F(t) + \int_{\Omega_0} F_0(x,t) \varphi_0(x) dx$$

and, using (F),

$$b(t) = \big[D_t^\beta F(t) - \int_{\Omega_0} F_0(x,t) \varphi_0(x) dx - \int_{\Omega_0} u(x,t) \Delta\varphi_0(x) dx\big] [F(t)]^{-1}, \quad t \in (0,T]. \tag{15}$$

Note that, according to (15), $b \in C[0,T]$ for $u \in C^\gamma(Q_0)$. By substituting the right-hand side of (15) into (13) in place of b we obtain the following nonlinear integral equation for unknown $u \in C^\gamma(\Omega_0)$

$$u(x,t) = \int_0^t [F(\tau)]^{-1} d\tau \int_{\Omega_0} G_0(x,t,y,\tau) \big[D_t^\beta F(\tau) - \int_{\Omega_0} F_0(z,\tau) \varphi_0(z) dz - \tag{16}$$

$$- \int_{\Omega_0} u(z,\tau) \Delta\varphi_0(z) dz\big] u(y,\tau) dy + h(x,t), \quad (x,t) \in \bar{Q}_0.$$

We have reduced problem (1)–(4) to system (15), (16). The converse conclusion holds and we obtain the following result.

Theorem 2. *Under assumptions (F0), (F1), (F2), (F), (Φ) and concordance condition (5) a pair of functions $(u,b)\in\mathfrak{M}_\beta(Q_0)$ is the solution of problem (1)–(4) if and only if the function $u\in C^\gamma(Q_0)$ is a solution of integral equation (16), $b\in C[0,T]$ is defined by (15).*

Theorem 3. *Under assumptions (F0), (F1), (F2), (F), (Φ) and concordance condition (5) there exists some $T^*\in(0,T]$ (respectively, $Q_0^*=\Omega_0\times(0,T^*]$) and the solution $(u,b)\in\mathfrak{M}_\beta(Q_0^*)=C_{2,\beta}(\bar{Q}_0^*)\times[0,T^*]$ of problem (1)–(4): the function u is a solution of integral equation (16), b is defined by (15).*

Proof. Granting Theorem 2 it is enough to prove the solvability of equation (16) in the class $C^\gamma(Q_0)$. Let

$$||b||_{C[0,T]}=\max_{t\in[0,T]}|b(t)|,$$

$$||v||_{C^\gamma(Q_0)}=\max\{\sup_{(x,t)\in Q_0}|v(x,t)|,\ \sup_{(x,t)\in Q_0,|\Delta x|<1}\tfrac{|v(x+\Delta x,t)-v(x,t)|}{|\Delta x|^\gamma}\},$$

$M_R=M_R(Q_0)=\{v\in C^\gamma(Q_0)\mid \ ||v||_{C^\gamma(Q_0)}\le R\}$ for some $R=const>0$.

We shall use the Schauder principle. On M_R we consider the operator

$$(Pv)(x,t):=\int_0^t[F(\tau)]^{-1}d\tau\int_{\Omega_0}G_0(x,t,y,\tau)\big[D_\tau^\beta F(\tau)-\int_{\Omega_0}F_0(z,\tau)\varphi_0(z)dz$$
$$-\int_{\Omega_0}u(z,\tau)\Delta\varphi_0(z)dz\big]u(y,\tau)dy+h(x,t),\quad (x,t)\in\Omega_0,\quad v\in M_R$$

where the function $h(x,t)$ is defined by (14).

At the beginning we show the existence of $R>0$, $T^*\in(0,T]$, and therefore $M_R^*=M_R(Q_0^*)$ such that $P:M_R^*\to M_R^*$.

Using the proof of Theorem 1, for $v\in M_R$, $(x,t)\in\bar{Q}_0$, we obtain

$$|(Pv)(x,t)|\le\frac{k_0}{f}t^{\beta_1}(c_1R+c_2R^2)+|h(x,t)|,$$

$$|h(x,t)|\le k_0t^{\beta_1}\|F_0\|_{C(Q_0)}+k_1\|F_1\|_{C(\bar\Omega_0)}+k_2t\|F_2\|_{C(\bar\Omega_0)},\quad (x,t)\in\bar{Q}_0.$$

Thus, $\quad |(Pv)(x,t)|\le\frac{k_0}{f}t^{\beta_1}(c_1R+c_2R^2)+B_0(t),\quad (x,t)\in\bar{Q}_0$

where $\quad c_1=\|D^\beta F-\int_{\Omega_0}F_0(z,\cdot)\varphi_0(z)dz\|_{C[0,T]},\quad c_2=\int_{\Omega_0}|\Delta\varphi_0(z)|dz,$

$B_0(t)=k_0t^{\beta_1}\|F_0\|_{C(Q_0)}+k_1\|F_1\|_{C(\bar\Omega_0)}+k_2t\|F_2\|_{C(\bar\Omega_0)}$.

Also, we have

$$\frac{|(Pv)(x+\Delta x,t)-(Pv)(x,t)|}{|\Delta x|^\gamma}\le$$

$$\le\frac{c_1R+c_2R^2}{f}\int_0^t d\tau\int_{\Omega_0}\frac{|G_0(x+\Delta x,t,y,\tau)-G_0(x,t,y,\tau)|}{|\Delta x|^\gamma}dy+\frac{|h(x+\Delta x,t)-h(x,t)|}{|\Delta x|^\gamma}$$

$$\le\frac{c_1R+c_2R^2}{f}\int_0^t d\tau\int_{\Omega_0}A_0(x,t,y,\tau)dy+\frac{|h(x+\Delta x,t)-h(x,t)|}{|\Delta x|^\gamma},\quad (x,t)\in Q_0.$$

We used Remark 2. In just the same way we estimate the second term. As in the proof of Theorem 1 for $v \in C^{\gamma}(Q_0)$, $(x,t) \in Q_0$, $|\Delta x| < 1$, we obtain

$$\frac{|(Pv)(x+\Delta x,t)-(Pv)(x,t)|}{|\Delta x|^{\gamma}} \le c_3 t^{\beta_1}\frac{c_1 R + c_2 R^2}{f} + B_1(t), \quad (x,t) \in Q_0$$

where

$$B_1(t) = c_3 t^{\beta_1}\|F_0\|_{C(Q_0)} + c_4\|F_1\|_{C(\bar\Omega_0)} + c_5\|F_2\|_{C(\bar\Omega_0)},$$

$$c_3 = \sup_{(x,t)\in Q_0} t^{-\beta_1}\int_0^t d\tau \int_{\Omega_0} A_0(x,t,y,\tau)dy,$$

$$c_4 = \sup_{x\in\Omega_0}\int_{\Omega_0} A_1(x,t,y)dy, \qquad c_5 = \sup_{x\in\Omega_0} t\int_{\Omega_0} A_2(x,t,y)dy.$$

As a result we obtain: $\|Pv\|_{C^{\gamma}(Q_0)} \le$

$$\le \max\{\max_{t\in[0,T]}[\tfrac{k_0}{f}t^{\beta_1}(c_1R+c_2R^2)+B_0(t)], \max_{t\in[0,T]}[\tfrac{c_3}{f}t^{\beta_1}(c_1R+c_2R^2)+B_1(t)]\} \quad \forall v\in M_R.$$

In order the inequality

$$\max\{d_0 t^{\beta_1}R^2 + B_0(t), d_1 t^{\beta_1}R^2 + B_1(t), 1\} \le R, \quad \forall t \in [0,T^*,] \tag{17}$$

holds for some $T^* \in (0,T]$, $R \ge 1$ with $d_0 = kk_0$, $d_1 = kc_3$ and $k = \frac{c_1+c_2}{f}$, consider the functions

$$f_j(s) = d_j t^{\beta_1}s^2 - s, \quad s \ge 0, \quad j = 0,1.$$

We find $f_j'(s) = 2d_j t^{\beta_1}s - 1$ and prove that $s_j = [2d_j t^{\beta_1}]^{-1}$ is the point of the minimum of $f_j(s)$, $j = 0,1$. Then the inequality $\|Pv\|_{C^{\gamma}(Q_0^*)} \le R$ for all $v \in M_R^*$ is fulfilled for some $R \ge 1$ if $d_j t^{\beta_1}s_j^2 - s_j \le -B_j(t)$ and $[2d_j t^{\beta_1}]^{-1} \ge 1$ for all $t \in [0,t_j^*]$, $j = 0,1$ and $T^* = \min\{t_0^*, t_1^*, T\}$.

We have $d_j t^{\beta_1}s_j^2 - s_j = -\frac{1}{4d_j t^{\beta_1}}$, $j = 0,1$. There exists $t_j^* > 0$ such that

$$-\frac{1}{4d_j t^{\beta_1}} \le -B_j(t)$$

(that is $4k(k_0t^{\beta_1})^2\|F_0\|_{C(Q_0)} + 4kk_1(k_0t^{\beta_1})\|F_1\|_{C(\bar\Omega_0)} - 1 \le 0$ for all $t \in [0,t_0^*]$ and $4k(c_3t^{\beta_1})^2\|F_0\|_{C(Q_0)} + 4kc_3c_4t^{\beta_1}\|F_1\|_{C(\bar\Omega_0)} - 1 \le 0$ for all $t \in [0,t_1^*]$) and also $[2d_jt^{\beta_1}]^{-1} \ge 1$ for all $t \in [0,t_j^*]$, $j = 0,1$. They are

$$t_0^* = \min\left\{\left[\frac{k_1\|F_1\|_{C(\bar\Omega_0)}}{2k_0\|F_0\|_{C(Q_0)}}\left(\sqrt{1+\frac{f\|F_0\|_{C(Q_0)}}{(c_1+c_2)k_1^2\|F_1\|^2_{C(\bar\Omega_0)}}}-1\right)\right]^{\frac{1}{\beta_1}}, \left[\frac{f}{2(c_1+c_2)k_0}\right]^{\frac{1}{\beta_1}}\right\},$$

$$t_1^* = \min\left\{\left[\frac{c_4\|F_1\|_{C(\bar\Omega_0)}}{2c_3\|F_0\|_{C(Q_0)}}\left(\sqrt{1+\frac{f\|F_0\|_{C(Q_0)}}{(c_1+c_2)c_4^2\|F_1\|^2_{C(\bar\Omega_0)}}}-1\right)\right]^{\frac{1}{\beta_1}}, \left[\frac{f}{2(c_1+c_2)c_3}\right]^{\frac{1}{\beta_1}}\right\}$$

if $\|F_0\|_{C(Q_0)} > 0$,

$$t_0^* = \min\left\{\left[\frac{f}{4(c_1+c_2)k_0k_1\|F_1\|_{C(\bar\Omega_0)}}\right]^{\frac{1}{\beta_1}}, \left[\frac{f}{2(c_1+c_2)k_0}\right]^{\frac{1}{\beta_1}}\right\} \quad \text{and}$$

$$t_1^* = \min\left\{\left[\frac{f}{4(c_1+c_2)c_3c_4\|F_1\|_{C(\bar\Omega_0)}}\right]^{\frac{1}{\beta_1}}, \left[\frac{f}{2(c_1+c_2)c_3}\right]^{\frac{1}{\beta_1}}\right\}$$

in the case $F_0(x,t) \equiv 0$, $(x,t) \in Q_0$.

Note that from (5) and (F) it follows that $\|F_1\|_{C(\bar\Omega_0)} > 0$. We have proven the existence of $R \ge 1$, $T^* > 0$ such that $P : M_R^* \to M_R^*$.

The operator P is continuous on $\tilde{M}_R^* = \{v \in C(Q_0^*) \mid ||v||_{C(Q_0^*)} \le R\}$ (thus, on M_R^*). Namely, for $v_1, v_2 \in \tilde{M}_R^*$

$$||Pv_1 - Pv_2||_{C(Q_0^*)} =$$

$$= \sup_{(x,t)\in Q_0^*} \left| \int_0^t [F(\tau)]^{-1} d\tau \int_{\Omega_0} G_0(x,t,y,\tau) \left[\int_{\Omega_0} v_2(z,\tau)\Delta\varphi_0(z)dz[v_2(y,\tau) - v_1(z,\tau)] \right.\right.$$

$$\left.\left. - \int_{\Omega_0} [v_1(z,\tau) - v_2(z,\tau)]\Delta\varphi_0(z)dz \cdot v_1(y,\tau) \right] dy \right|$$

$$\le c_2 \sup_{(x,t)\in Q_0^*} \int_0^t [F(\tau)]^{-1} d\tau \int_{\Omega_0} G_0(x,t,y,\tau)dy \cdot \left[\|v_1\|_{C(Q_0^*)} + \|v_2\|_{C(Q_0^*)}\right] \|v_1 - v_2\|_{C(Q_0^*)}$$

$$\le \frac{2k_0 c_2 (T^*)^{\beta_1} R}{f} \|v_1 - v_2\|_{C(Q_0^*)}.$$

Similarly we obtain that the operator P is compact on $\tilde{M}_R^*$ (and thus on M_R^*): the uniform boundedness of the set

$$P\tilde{M}_R^* := \{(Pv)(x,t),\ (x,t) \in Q_0^* \mid \quad v \in \tilde{M}_R^*\},$$

was established earlier; in addition, from the properties of Green operators and Remark 2 it follows that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for all $(x,t) \in Q_0^*$, $|\Delta x| < \delta$, $|\Delta t| < \delta$ and for all $v \in \tilde{M}_R^*$

$$\sup_{(x,t)\in Q_0^*} |(Pv)(x+\Delta x, t+\Delta t) - (Pv)(x,t)|$$

$$\le \tfrac{c_1 R + c_2 R^2}{f} \sup_{(x,t)\in Q_0^*} \int_0^t d\tau \int_{\Omega_0} |G_0(x+\Delta x, t+\Delta t, z, \tau) - G_0(x,t,z,\tau)|dz$$

$$+ \sup_{(x,t)\in Q_0^*} |h(x+\Delta x, t+\Delta t) - h(x,t)|$$

$$\le \left[\tfrac{(c_1 R + c_2 R^2) c_3 (T^*)^{\beta_1}}{f} + B_1(T^*)\right] \cdot \left[|\Delta x| + |\Delta t|^{\beta/2}\right]^{\gamma} < \varepsilon.$$

As a result, the operator P is equicontinuous on M_R^*. According to the Schauder principle, there exists the solution $u \in M_R^*$ of equation (16).

Theorem 4. *Under condition $F(t) \neq 0$, $t \in [0,T]$, the solution $(u,b) \in \mathfrak{M}_\beta(Q_0)$ of problem (1)–(4) is unique.*

Proof. Take two solutions $(u_1, b_1), (u_2, b_2) \in \mathfrak{M}_\beta(Q_0)$ of problem (1)–(4) and substitute them into equation (1). We obtain

$$D_t^\beta (u_1 - u_2) = \Delta(u_1 - u_2) + b_1 u_1 - b_2 u_2,$$

$$D_t^\beta (u_1 - u_2) = \Delta(u_1 - u_2) + (b_1 - b_2)u_1 + b_2(u_1 - u_2)$$

and for $u = u_1 - u_2$, $b = b_1 - b_2$

$$D_t^\beta u = \Delta u + b_2 u + b u_1.$$

From the boundary condition it follows that

$$u(x,t) = 0, \quad x \in \bar{\Omega}_1, \quad t \in [0,T]$$

and from the initial condition

$$u(x,0) = 0, \;\; u_t(x,0) = 0, \quad x \in \bar{\Omega}_0.$$

Then, by Theorem 1, the function $u(x,t)$ satisfies the equation

$$u(x,t) = \int\limits_0^t d\tau \int\limits_{\Omega_0} G_0(x,t,y,\tau)(b_2(\tau)u(y,\tau) + b(\tau)u_1(y,\tau))dx, \quad (x,t) \in \bar{Q}_0$$

and belongs to $C_{2,\beta}(\bar{Q}_0)$.

From over-determination condition (4) and from (15) it follows that

$$\int\limits_{\Omega_0} u(x,t)\Delta\varphi_0(x)dx = -b(t)F(t), \quad t \in [0,T]. \tag{18}$$

Then $u(x,t)$ satisfies the integral equation

$$u(x,t) = \int\limits_0^t d\tau \int\limits_{\Omega_0} G_0(x,t,y,\tau)\left(b_2(\tau)u(y,\tau) - \frac{u_1(y,\tau)\int\limits_{\Omega_0} u(z,\tau)\Delta\varphi_0(z)dz}{F(\tau)}\right)d\tau$$

that is

$$u(x,t) = \int\limits_0^t d\tau \int\limits_{\Omega_0} \Bigg[G_0(x,t,z,\tau)b_2(\tau)$$

$$-\frac{1}{F(\tau)}\left(\int\limits_{\Omega_0} G_0(x,t,y,\tau)u_1(y,\tau)dy\right)\Delta\varphi_0(z)\Bigg] u(z,\tau)dz, \quad (x,t) \in \bar{Q}_0.$$

By uniqueness of the solution of the second type linear Volterra integral equation with integrable kernel we obtain $u(x,t) = 0$, $(x,t) \in \bar{Q}_0$. Then from (18) it follows that $b(t)F(t) = 0$, $t \in [0,T]$. Since $F(t) \neq 0$ on $[0,T]$ (under assumption of the theorem), $b(t) \equiv 0$, $t \in [0,T]$.

Remark 3. The obtained result is correct in the case $\beta \in (0,1]$ (without the second initial condition $u_t(x,0) = F_2(x)$, $x \in \bar{\Omega}_0$) if in all formulas we put $F_2(x) \equiv 0$, $x \in \bar{\Omega}_0$.

A similar result holds for inverse problem on determination of a pair of functions: the solution u of the second boundary value problem for such an equation and unknown coefficient $b(t)$ in the young term under the same over-determination condition (4).

We may study the cases $N = 1, 2$ in just the same way.

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ОБЕРНЕНА ЗАДАЧА ДЛЯ РІВНЯННЯ ДИФУЗІЇ З ДРОБОВОЮ ПОХІДНОЮ ТА НЕВІДОМИМ МОЛОДШИМ КОЕФІЦІЄНТОМ

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Досліджено обернену задачу для лінійного неоднорідного рівняння дифузії з регуляризованою похідною дробового порядку $\beta \in (0, 2)$ за часом в обмеженому циліндрі $\Omega_0 \times (0, T]$ — задачу про визначення пари функцій: класичного розв'язку u першої крайової задачі для такого рівняння та невідомого, залежного від часу, неперервного коефіцієнта в молодшому члені рівняння за умови перевизначення

$$\int_{\Omega_0} u(x,t)\varphi(x)dx = F(t), \quad t \in [0, T]$$

з деякими заданими функціями φ і F. Встановлено однозначну розв'язність задачі. Використовуємо метод функції Гріна.

Ключові слова: похідна дробового порядку, обернена крайова задача, вектор-функція Гріна, операторне рівняння.