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p-ELLIPTIC FUNCTIONS

Andriy KONDRATYUK,
Vasylyna KHOROSHCHAK, Dzvenyslava LUKIVSKA

Ivan Franko National University of Lviv,
Universytetska Str., 1, Lviv, 79000
e-mails: v.khoroshchak@gmail.com
d.lukivska@gmail.com

We investigate p-elliptic functions (meromorphic in $\mathbb{C}$ functions satisfying the conditions $g(u+\omega_1) = g(u)$, $g(u+\omega_2) = pg(u)$, $\omega_1, \omega_2 \in \mathbb{C}$, $p \in \mathbb{C}\backslash\{0\}$, $Im\frac{\omega_2}{\omega_1} > 0$). In the case $p = 1$ this is the classical theory of elliptic functions. p-Elliptic functions generate so-called p-loxodromic functions and vice versa. We generalize the elliptic Weierstrass $\wp$-function and find the corresponding p-loxodromic function in the case $|p| = 1$.

Key words: p-elliptic function, the Weierstrass $\wp$-function, p-loxodromic function, generalized Weierstrass $\wp$-function.

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2. Introduction

As usual, $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Definition 1. *Let ω_1, ω_2 be complex numbers such that $Im\frac{\omega_2}{\omega_1} > 0$. A meromorphic in $\mathbb{C}$ function g is called p-**elliptic**, if there exists $p \in \mathbb{C}^*$ such that for every $u \in \mathbb{C}$*

$$g(u+\omega_1) = g(u), \quad g(u+\omega_2) = pg(u). \tag{1}$$

The second property is called the **multi p-periodicity of period** ω_2. Note that (1) implies $g(u+m\omega_1+n\omega_2)=p^n g(u)$, where $m,n\in\mathbb{Z}$.

Denote the class of p-elliptic functions by $\mathcal{E}_p$.

If $p=1$ we obtain the classical theory of elliptic functions, which are well known due to the works of K. Jacobi, N. Abel, K. Weierstrass. The theories of loxodromic (multiplicatively periodic) and elliptic functions are dual (see [1], [2], [3]).

3. Relation between p-loxodromic and p-elliptic functions

We are going to show that the p-elliptic functions generate the so-called p-loxodromic functions and vice versa.

Definition 2. *Let $q\in\mathbb{C}^*$, $0<|q|<1$. A meromorphic in $\mathbb{C}^*$ function f is said to be* **p-loxodromic of multiplicator** *q if there exists $p\in\mathbb{C}^*$, $p\neq 1$, such that for every $z\in\mathbb{C}^*$*

$$f(qz)=pf(z). \tag{2}$$

Let $\mathcal{L}_{qp}$ denote the class of p-loxodromic functions of multiplicator q.

If f is meromorphic in $\mathbb{C}^*$ and p-loxodromic of multiplicator $q=e^{2\pi i\frac{\omega_2}{\omega_1}}$, $Im\frac{\omega_2}{\omega_1}>0$, that is $f\in\mathcal{L}_{qp}$, then the function

$$g(u)=f\left(e^{2\pi i\frac{u}{\omega_1}}\right)$$

is meromorphic in $\mathbb{C}$ and p-elliptic of periods ω_1,ω_2. Indeed, for all $u\in\mathbb{C}$ we have

$$g(u+m\omega_1+n\omega_2)=f\left(e^{2\pi i\frac{u+m\omega_1+n\omega_2}{\omega_1}}\right)=f\left(e^{2\pi in\frac{\omega_2}{\omega_1}}e^{2\pi i\frac{u}{\omega_1}}\right)=$$

$$=f\left(q^n e^{2\pi i\frac{u}{\omega_1}}\right)=p^n f\left(e^{2\pi i\frac{u}{\omega_1}}\right)=p^n g(u).$$

Hence, $g\in\mathcal{E}_p$.

Conversely, if $g\in\mathcal{E}_p$ and $z\in\mathbb{C}^*$, then the function

$$f(z)=g\left(\frac{\omega_1}{2i\pi}\log z\right)$$

is well defined because g admits the period ω_1 and $\log z\in\mathbb{C}/2i\pi\mathbb{Z}$. In other words we have here that the composition of a multivalent mapping and a univalent one is a univalent function. Hence, if we set $q=e^{2\pi i\frac{\omega_2}{\omega_1}}$, $Im\frac{\omega_2}{\omega_1}>0$, we obtain

$$f(qz)=g\left(\frac{\omega_1}{2i\pi}\log(qz)\right)=g\left(\omega_2+\frac{\omega_1}{2i\pi}\log z\right)=$$

$$=pg\left(\frac{\omega_1}{2i\pi}\log z\right)=pf(z).$$

Thus, $f\in\mathcal{L}_{qp}$.

4. Generalization of the Weierstrass $\wp$-function

Taking into account the fact that $\wp$ and $\wp'$ generate the field of elliptic functions, it will be interesting to obtain a counterpart of $\wp$ in the theory of p-elliptic functions.

Therefore, in this section we generalize the elliptic Weierstrass $\wp$-function

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right),$$

where $\omega_1, \omega_2 \in \mathbb{C}$, $Im\frac{\omega_2}{\omega_1} > 0$, $\omega = m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$. The proposed generalized function will be a p-elliptic function.

Let $p = e^{i\alpha}$ and

$$g_\alpha(u) = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{e^{in\alpha}}{(u-\omega)^2} - \frac{e^{in\alpha}}{\omega^2}\right).$$

If $p = 1$, we have

$$g_0(u) = \wp(u).$$

We suppose further $p \neq 1$.

Definition 3. *Let $p = e^{i\alpha}$, $p \neq 1$. The function of the form*

$$\wp_\alpha(u) = g_\alpha(u) + C_\alpha,$$

where

$$C_\alpha = \frac{g_\alpha\left(\frac{\omega_2}{2}\right) - e^{i\alpha} g_\alpha\left(-\frac{\omega_2}{2}\right)}{e^{i\alpha} - 1},$$

is called **the generalized Weierstrass $\wp$-function.**

We prove the following theorem.

Theorem 1. *The generalized Weierstrass $\wp$-function $\wp_\alpha$ belongs to $\mathcal{E}_p$ with $p = e^{i\alpha} \neq 1$.*

Proof. Let us consider the derivative of g_α,

$$g'_\alpha(u) = -2\sum_{\omega}\frac{e^{in\alpha}}{(u-\omega)^3}.$$

We have

$$g'_\alpha(u+\omega_2) = -2\sum_{m,n\in\mathbb{Z}}\frac{e^{in\alpha}}{(u+\omega_2 - m\omega_1 - n\omega_2)^3} = -2\sum_{m,n\in\mathbb{Z}}\frac{e^{in\alpha}}{(u - m\omega_1 - (n-1)\omega_2)^3} =$$

$$= -2e^{i\alpha}\sum_{m,n\in\mathbb{Z}}\frac{e^{i(n-1)\alpha}}{(u - m\omega_1 - (n-1)\omega_2)^3} = e^{i\alpha} g'_\alpha(u).$$

Thus, we obtain

$$g'_\alpha(u+\omega_2) - e^{i\alpha} g'_\alpha(u) = 0. \tag{3}$$

Note that the function $(g_\alpha + C)$ satisfies (3) for any $C \in \mathbb{C}$. Put

$$C = C_\alpha.$$

Then relation (3) implies

$$g_\alpha(u+\omega_2) + C_\alpha - e^{i\alpha}\big(g_\alpha(u) + C_\alpha\big) = A,$$

where A is a constant. Let us define A. Setting $u = -\frac{\omega_2}{2}$ in the preceding equality, we obtain

$$g_\alpha\left(\frac{\omega_2}{2}\right) - e^{i\alpha} g_\alpha\left(-\frac{\omega_2}{2}\right) + (1-e^{i\alpha})C_\alpha = A.$$

Taking into account the choice of C_α, we conclude that $A = 0$. Hence, we have

$$g_\alpha(u+\omega_2) + C_\alpha = e^{i\alpha}\big(g_\alpha(u) + C_\alpha\big), \tag{4}$$

that is we have shown that the function $\wp_\alpha = g_\alpha + C_\alpha$ is multi p-periodic of period ω_2.

It remains to prove the uniqueness of C_α. We suppose that there is a constant C such that the function $\big(g_\alpha + C\big)$ is multi p-periodic of period ω_2, that is

$$g_\alpha(u+\omega_2) + C = e^{i\alpha}\big(g_\alpha(u) + C\big).$$

Using (4), we obtain $C - C_\alpha = e^{i\alpha}\big(C - C_\alpha\big)$, which implies $C = C_\alpha$. Let us now consider the period ω_1. We have

$$g'_\alpha(u+\omega_1) = -2\sum_{m,n\in\mathbb{Z}} \frac{e^{in\alpha}}{(u+\omega_1 - m\omega_1 - n\omega_2)^3} =$$

$$= -2\sum_{m,n\in\mathbb{Z}} \frac{e^{in\alpha}}{(u-(m-1)\omega_1 - n\omega_2)^3} = g'_\alpha(u).$$

Hence, $g'_\alpha(u+\omega_1) = g'_\alpha(u)$. We can deduce from this the following

$$g_\alpha(u+\omega_1) + C_\alpha = g_\alpha(u) + C_\alpha + B, \tag{5}$$

where B is some constant.

Let us now define B. Using equalities (4) and (5), we obtain

$$g_\alpha(u+\omega_2+\omega_1) + C_\alpha = g_\alpha(u+\omega_2) + C_\alpha + B,$$

$$g_\alpha(u+\omega_1+\omega_2) + C_\alpha = e^{i\alpha}\big(g_\alpha(u+\omega_1) + C_\alpha\big) = e^{i\alpha}\big(g_\alpha(u) + C_\alpha + B\big).$$

We can write B in the form

$$B = \frac{g_\alpha(u+\omega_2) - e^{i\alpha} g_\alpha(u) + C_\alpha(1-e^{i\alpha})}{e^{i\alpha}-1}.$$

Setting $u = -\dfrac{\omega_2}{2}$, we have

$$B = \frac{g_\alpha\left(\frac{\omega_2}{2}\right) - e^{i\alpha} g_\alpha\left(-\frac{\omega_2}{2}\right)}{e^{i\alpha}-1} - C_\alpha.$$

According to the definition of C_α, we can conclude $B = C_\alpha - C_\alpha = 0$. Since $B = 0$, equalities (4), (5) imply that the function $\wp_\alpha = g_\alpha + C_\alpha$ belongs to $\mathcal{E}_p$ with $p = e^{i\alpha}$, $p \neq 1$, which completes the proof.

5. Generalization of the Weierstrass ζ and σ functions

Let us now consider the function

$$\zeta_\alpha(u) = \frac{1}{u} + \sum_{\omega\neq 0}\left(\frac{e^{in\alpha}}{u-\omega} + \frac{e^{in\alpha}}{\omega} + \frac{ue^{in\alpha}}{\omega^2}\right),$$

where $\omega_1, \omega_2 \in \mathbb{C}, Im\frac{\omega_2}{\omega_1} > 0, \omega = m\omega_1 + n\omega_2,\ m, n \in \mathbb{Z}$. The remainders of the series converge uniformly on the compact subsets of $\mathbb{C}$, see [4].

Differentiating ζ_α we obtain $g_\alpha(u) = -\zeta'_\alpha(u)$. Hence, $\wp_\alpha(u) = g_\alpha(u) + C_\alpha = C_\alpha - \zeta'_\alpha(u)$. We can rewrite ζ_α as follows

$$\zeta_\alpha(u) = \frac{1}{u} + \sum_{n\in\mathbb{Z}} e^{in\alpha} \sum_{m\in\mathbb{Z}}\left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right), \quad m^2+n^2 \neq 0.$$

Fix $n \in \mathbb{Z}$ and denote

$$\chi_0(u) = \frac{1}{u} + \sum_{m\neq 0}\left(\frac{1}{u-m\omega_1} + \frac{1}{m\omega_1} + \frac{u}{m^2\omega_1^2}\right),$$

$$\chi_n(u) = \sum_{m\in\mathbb{Z}}\left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right), \quad n \neq 0.$$

Then, ζ_α can be rewritten as follows

$$\zeta_\alpha(u) = \sum_{n\in\mathbb{Z}} e^{in\alpha}\chi_n(u). \tag{6}$$

Let $f(u) = 1 - \frac{u}{\omega}$. We have $\frac{f'(u)}{f(u)} = \frac{1}{u-\omega}$. Hence, we obtain

$$\int_0^u \frac{d\zeta}{\zeta-\omega} = \int_0^u \frac{f'(\zeta)}{f(\zeta)}d\zeta = \log f(u) - \log f(0)$$

for any branch of $\log f$ and specifically for the branch defined by the condition $\log f(0) = \log 1 = 0$. Thus, we have

$$\int_0^u \frac{d\zeta}{\zeta-\omega} = \log\left(1 - \frac{u}{\omega}\right).$$

By A^* denote $\mathbb{C}$ with radial slits from ω to ∞. Integrating $\left(\chi_0(t) - \frac{1}{t}\right)$ and $\chi_n(t)$ along a path in A^* which connects the points 0 and u, we obtain

$$\int_0^u \left(\chi_0(t) - \frac{1}{t}\right)dt = \sum_{m\neq 0, n=0}\left(\log\left(1-\frac{u}{\omega}\right) + \frac{u}{\omega} + \frac{u^2}{2\omega^2}\right), \tag{7}$$

$$\int_0^u \chi_n(t)dt = \sum_{m\in\mathbb{Z}}\left(\log\left(1-\frac{u}{\omega}\right) + \frac{u}{\omega} + \frac{u^2}{2\omega^2}\right), \quad n \neq 0. \tag{8}$$

Let us consider the entire functions

$$\sigma_0(u) = u \prod_{m\neq 0, n=0} \left(1 - \frac{u}{\omega}\right) e^{\frac{u}{\omega} + \frac{u^2}{2\omega^2}},$$

$$\sigma_n(u) = \prod_{m\in\mathbb{Z}} \left(1 - \frac{u}{\omega}\right) e^{\frac{u}{\omega} + \frac{u^2}{2\omega^2}}, \quad n \neq 0.$$

Using these functions, we can rewrite (7) and (8) in the form

$$\int_0^u \left(\chi_0(t) - \frac{1}{t}\right) dt = \log \frac{\sigma_0(u)}{u}, \quad \int_0^u \chi_n(t) dt = \log \sigma_n(u).$$

If we differentiate these relations, then we obtain

$$\chi_0(u) = \frac{\sigma_0'(u)}{\sigma_0(u)}, \quad \chi_n(u) = \frac{\sigma_n'(u)}{\sigma_n(u)}.$$

Taking into account such representations of $\chi_n(u)$, $n \in \mathbb{Z}$, we can rewrite (6) as follows

$$\zeta_\alpha(u) = \sum_{n\in\mathbb{Z}} e^{in\alpha} \frac{\sigma_n'(u)}{\sigma_n(u)}.$$

Hence, $\wp_\alpha$ can be rewritten in the next form

$$\wp_\alpha(u) = C_\alpha + \sum_{n\in\mathbb{Z}} e^{in\alpha} \frac{\sigma_n'^2(u) - \sigma_n''(u)\sigma_n(u)}{\sigma_n^2(u)}.$$

Remark 1. If we consider the product $\prod_{n\in\mathbb{Z}} \sigma_n(u)$, then we obtain the Weierstrass σ-function. If $\alpha = 0$, then ζ_0 is the Weierstrass ζ-function.

6. p-loxodromic function that corresponds to the generalized Weierstrass $\wp$-function

Let us consider the function

$$\rho_p(z) = \sum_{n\in\mathbb{Z}} \frac{(pq)^n z}{(z - q^n)^2}, \quad |q| < 1,\ |q| < |p| < \frac{1}{|q|}.$$

Since $|pq| < 1$, $q^n \to 0$ as $n \to +\infty$, and $\left|\frac{q}{p}\right| < 1$, the remainders of the series converge uniformly on the compact subsets of $\mathbb{C}^*$.

The function ρ_p belongs to $\mathcal{L}_{qp}$. Indeed,

$$\rho_p(qz) = \sum_{n\in\mathbb{Z}} \frac{(pq)^n qz}{(qz - q^n)^2} = \sum_{n\in\mathbb{Z}} \frac{p^n q^{n-1} z}{(z - q^{n-1})^2} =$$

$$= p \sum_{n\in\mathbb{Z}} \frac{(pq)^{n-1} z}{(z - q^{n-1})^2} = p\rho_p(z).$$

The following theorem holds.

Theorem 2. *If* $q = e^{2\pi i \frac{\omega_2}{\omega_1}}$, $Im \frac{\omega_2}{\omega_1} > 0$, *then* $\rho_p\left(e^{2\pi i \frac{u}{\omega_1}}\right) = -\frac{\omega_1^2}{4\pi^2} \wp_\alpha(u)$, $p = e^{i\alpha} \neq 1$.

Proof. Let us consider the function

$$f(z) = \sum_{n=0}^{+\infty} \frac{(pq)^n}{z - q^n} + \sum_{k=1}^{+\infty} \frac{1}{p^k} \left(\frac{1}{q^k z - 1} + 1 \right).$$

The remainders of the first series converge if $|p| < \frac{1}{|q|}$, and of the second if $|q| < |p|$. Hence, if $|q| < |p| < \frac{1}{|q|}$, the function f is meromorphic in $\mathbb{C}^*$. It is easy to verify that the functions f and ρ_p are connected as follows

$$\rho_p(z) = -zf'(z). \tag{9}$$

All points satisfying the equation

$$e^{2\pi i \frac{u}{\omega_1}} = q^n, \quad n \in \mathbb{Z} \tag{10}$$

are simple poles of the function $f\left(e^{2\pi i \frac{u}{\omega_1}}\right)$. If u satisfies relation (10), then $(u + m\omega_1)$, $m \in \mathbb{Z}$, satisfy it as well. Thus, $f\left(e^{2\pi i \frac{u}{\omega_1}}\right)$ has the poles at the points $\omega = m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$.

Let now calculate the residues of $f\left(e^{2\pi i \frac{u}{\omega_1}}\right)$ at the points ω. If $n \geqslant 0$, then

$$\lim_{u\to\omega} (u-\omega) f\left(e^{2\pi i \frac{u}{\omega_1}}\right) = \lim_{u\to\omega} (pq)^n \frac{u-\omega}{e^{2\pi i \frac{u}{\omega_1}} - e^{2\pi i \frac{\omega}{\omega_1}}} = \lim_{u\to\omega} p^n \frac{u-\omega}{e^{\frac{2\pi i}{\omega_1}(u-\omega)} - 1} = \frac{\omega_1}{2\pi i} p^n.$$

Similarly, if $n < 0, n = -k$, then we obtain

$$\lim_{u\to\omega} (u-\omega) f\left(e^{2\pi i \frac{u}{\omega_1}}\right) = \lim_{u\to\omega} p^n (u-\omega) \left(\frac{1}{e^{-2\pi i n \frac{\omega_2}{\omega_1}} e^{-2\pi i n \frac{u}{\omega_1}} - 1} + 1 \right) =$$

$$= \lim_{u\to\omega} \frac{p^n (u-\omega)}{e^{\frac{2\pi i}{\omega_1}(u-\omega)} - 1} = \frac{\omega_1}{2\pi i} p^n.$$

Thus, the principal parts corresponding to each pole ω take the form $\frac{\omega_1}{2\pi i} \frac{p^n}{u-\omega}$.

Since $f\left(e^{2\pi i \frac{u}{\omega_1}}\right)$ is a meromorphic in $\mathbb{C}$ function of variable u, in virtue of the Mittag-Leffler theorem [4] there exists a meromorphic function $F(u)$ with the same poles and principal parts. That is there exists an entire function $G(u)$ such that

$$f\left(e^{2\pi i \frac{u}{\omega_1}}\right) = G(u) + F(u).$$

Applying the theorem of expansion into the simple fraction [4] to the function $F(u)$, we obtain

$$F(u) = \frac{\omega_1}{2\pi i} \left(\frac{1}{u} + \sum_{\omega \neq 0} \frac{u^2}{\omega^2} \frac{p^n}{u - \omega} \right).$$

Since the double series $\sum_{\omega \neq 0} \frac{1}{|\omega|^3}$ is convergent (see [3], [4]), the series on the right hand side of preceding equality is uniformly convergent on the compact subsets of $\mathbb{C}$.

Hence, we obtain

$$f\left(e^{2\pi i \frac{u}{\omega_1}}\right) = G(u) + \frac{\omega_1}{2\pi i} \left(\frac{1}{u} + \sum_{\omega \neq 0} \frac{u^2}{\omega^2} \frac{p^n}{u - \omega} \right). \tag{11}$$

Relation (9) implies
$$\rho_p\left(e^{2\pi i\frac{u}{\omega_1}}\right) = -e^{2\pi i\frac{u}{\omega_1}} f'\left(e^{2\pi i\frac{u}{\omega_1}}\right).$$
Differentiating equality (11), we have
$$-\rho_p\left(e^{2\pi i\frac{u}{\omega_1}}\right) = \frac{\omega_1}{2\pi i}G'(u) - \frac{\omega_1^2}{4\pi^2}\left(-\frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{p^n}{\omega^2} - \frac{p^n}{(u-\omega)^2}\right)\right).$$

According to the definition of $\wp_\alpha$ we can deduce
$$-\rho_p\left(e^{2\pi i\frac{u}{\omega_1}}\right) = \frac{\omega_1}{2\pi i}G'(u) + \frac{\omega_1^2}{4\pi^2}\left(\wp_\alpha(u) - C_\alpha\right)$$
or this can be rewritten in the form
$$\frac{\omega_1^2}{4\pi^2}\wp_\alpha(u) + \rho_p\left(e^{2\pi i\frac{u}{\omega_1}}\right) = \frac{\omega_1^2}{4\pi^2}C_\alpha - \frac{\omega_1}{2\pi i}G'(u). \tag{12}$$

The function on the left hand side of equality (12) is p-elliptic as the sum of two p-elliptic functions. Thus, an entire function on the right hand side of (12) is p-elliptic. Since $|p| = 1$ then (1) implies that every entire p-elliptic function is bounded in $\mathbb{C}$. Thus, by the Liouville theorem it is constant. The only constant function $g \in \mathcal{E}_p$ in the case $p \neq 1$ is $g \equiv 0$. Hence, we can conclude from (12) the equality
$$\rho_p\left(e^{2\pi i\frac{u}{\omega_1}}\right) = -\frac{\omega_1^2}{4\pi^2}\wp_\alpha(u).$$
This completes the proof.

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p-ЕЛІПТИЧНІ ФУНКЦІЇ

Андрій КОНДРАТЮК,

Василина ХОРОЩАК, Дзвенислава ЛУКІВСЬКА

Львівський національний університет імені Івана Франка,
вул. Університетська, 1, Львів, 79000
e-mails: v.khoroshchak@gmail.com
d.lukivska@gmail.com

Вивчено p-еліптичні функції (мероморфні у $\mathbb{C}$ функції, що задовольняють умови $g(u+\omega_1)=g(u)$, $g(u+\omega_2)=pg(u)$, $\omega_1,\omega_2\in\mathbb{C}$, $Im\frac{\omega_2}{\omega_1}>0$, $p\in\mathbb{C}\backslash\{0\}$). У випадку $p=1$ — це класична теорія еліптичних функцій. Доведено зв'язок p-еліптичних функцій з p-локсодромними. Узагальнено еліптичну $\wp$-функцію Веєрштрасса. Знайдено p-локсодромну функцію, яка відповідає узагальненій $\wp$-функції Веєрштрасса у випадку $|p|=1$.

Ключові слова: p-еліптична функція, $\wp$-функція Веєрштрасса, p-локсодромна функція, узагальнена $\wp$-функція Веєрштрасса.