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OPTIMAL RESOURCE COEFFICIENT CONTROL IN A DYNAMIC POPULATION MODEL WITHOUT INITIAL CONDITIONS

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An optimal control problem for systems described by Fourier problem (problem without initial conditions) for nonlinear parabolic equations is studied. The control function occurs in the coefficients of the state equations. The existence of the optimal control is proved.

Key words: optimal control, problem without initial conditions, evolution equation.

1. Introduction. Optimal control of determined systems governed by partial differential equations (PDEs) is currently of much interest [1, 4, 6, 7, 17, 18, 19, 20, 21, 22, 25, 26, 27, 29, 30, 31]. Optimal control problems for PDEs are most completely studied for the case in which the control functions occur either on the right-hand sides of the state equations, or the boundary or initial conditions (see for example, [14], [27], [31]). So far, problems in which control functions occur in the coefficients of the state equations are less studied (see for example, [1], [4], [26], [29], [30]). A simple model example of such type problem is the following (see [4]).

Consider the problem of allocating resources to maximize the net benefit in the conservation of the single species while the cost of the resource allocation is minimized. In this case a state of controlled system for given control $v \in U := L^\infty(\Omega \times (0,T))$ is defined by a weak solution $y = y(v) = y(x,t;v)$, $(x,t) \in \Omega \times (0,T)$, from the space $L^2(0,T;H_0^1(\Omega)) \cap L^3(\Omega \times (0,T)) \cap C([0,T];L^2(\Omega))$, of the following problem:

$$y_t - \Delta y + y^2 - vy = 0 \quad \text{in } \ \Omega \times (0,T),$$

$$y\Big|_{\Gamma\times(0,T)} = 0, \quad y\Big|_{t=0} = y_0 \in L^2(\Omega), \quad y \geqslant 0 \ \text{ a.e. on } \Omega \times (0,T).$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ, $T > 0$. The objective is to balance the two features of maximizing the population and minimizing the cost of the control, representing the resources. Therefore the cost functional has the form

$$J(v) := \int_0^T \int_\Omega \big[y(x,t;v) - \rho|v(x,t)|^2\big]\,dxdt \quad \forall v \in U,$$

where y is the population density of a species, v is the resource function, and $\rho > 0$ is given constant. An optimal control problem is to find a function $u \in U_\partial := \big\{v \in U \ : \ 0 \leqslant v \leqslant M \ \text{ a. e. in } \Omega\times(0,T)\big\}$ (here $M = \mathrm{const} > 0$ is given) such that

$$J(u) = \sup_{v\in U_\partial} J(v).$$

This problem is nonlinear, since the dependence between the state and the control is nonlinear.

Various generalizations of this problem were investigated in many papers, including [1], [4], [6], [7], [17]-[20], [21], [25], [26], [29], [30] where the state of controlled system is described by the initial-boundary value problems for parabolic equations.

In [1], [26], [29], [30] the state of controlled system is described by linear parabolic equations and systems, while in [1] and [26] control functions appears as coefficients at lower derivatives, and in [29], [30] the control functions are coefficients at higher derivatives. In [26] the existence and uniqueness of optimal control in the case of final observation was shown and a necessary optimality condition in the form of the generalized rule of Lagrange multipliers was obtained. In paper [1] authors proved the existence of at least one optimal control for system governed by a system of general parabolic equations with degenerate discontinuous parabolicity coefficienti. In papers [29], [30] the authors consider cost function in general form, and as special case it includes different kinds of specific practical optimization problems.

In papers [4], [17]-[20], [21], [25] authors investigate optimal control of systems governed by nonlinear PDEs. In particular, in [4] the problem of allocating resources to maximize the net benefit in the conservation of a single species is studied. The population model is an equation with density dependent growth and spatial-temporal resource control coefficient. Numerical simulations illustrate several cases with Dirichlet and Neumann boundary conditions. In [18] the optimal control problem is converted to an optimization problem which is solved using a penalty function technique. Paper [21] presents analytical and numerical solutions of an optimal control problem for quasilinear parabolic equations. In [22] the authors consider the optimal control of a degenerate parabolic equation governing a diffusive population with logistic growth terms. In paper [25] optimal control for semilinear parabolic equations without Cesari-type conditions is investigated.

In this paper, we study an optimal control problem for systems whose states are described by problems without initial conditions or, other words, Fourier problems for parabolic equations. The model example of considered optimal control problem is a problem which differs from the previous one (see beginning of this section) by the following facts: the initial moment is $-\infty$ and, correspondingly, the state equation and control functions are considered in the domain $\Omega\times(-\infty,T)$, a boundary condition is given on the surface $\Sigma = \partial\Omega\times(-\infty,T)$, while the initial condition is replaced by the condition

$$\lim_{t\to-\infty} \|y(\cdot,t)\|_{L^2(\Omega)} = 0.$$

The problem without initial conditions for evolution equations describes processes that started a long time ago and initial conditions do not affect on them in the actual time moment. Such problem were investigated in the works of many mathematicians (see [5, 12, 28] and bibliography there).

As we know among works devoted to the optimal control problems for PDEs, only in papers [6], [7] the state of controlled system is described by the solution of Fourier problem for parabolic equations. In the current paper, unlike the above two, we consider optimal control problem in case when the control functions occur in the coefficients of the state equations. The main result of this paper is existence of the solution of this problem.

The outline of this paper is as follows. In Section 2, we give notations, definitions of function spaces and auxiliary results. In Section 3, we formulate the optimal control problem. In Section 4, we prove existence and uniqueness of the solutions for the state equations. Furthermore, we obtain estimates for the weak solutions of the state equations. Finally, the existence of the optimal control is presented in Section 5.

2. Preliminaries. Let n be a natural number, $\mathbb{R}^n$ be the linear space of ordered collections $x = (x_1, \ldots, x_n)$ of real numbers with the norm $|x| := (|x_1|^2 + \ldots + |x_n|^2)^{1/2}$. Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ. Set $S := (-\infty, 0]$, $Q := \Omega \times S$, $\Sigma := \Gamma \times S$, $\Omega_t := \Omega \times \{t\}$ for each $t \in \mathbb{R}$.

For any measurable set $G \subset \mathbb{R}^m$, where $m = n$ or $m = n+1$, and arbitrary $q \in [1, \infty]$ we denote by $L^q(G)$ standard Lebesgue space with norm $\|\cdot\|_{L^q(G)}$. Under $L^q_{\mathrm{loc}}(\overline{Q})$, where $q \in [1, \infty]$, we mean the linear space of measurable functions on Q such that their restrictions to any bounded measurable set $Q' \subset Q$ belong to the space $L^q(Q')$.

Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$, $q \in [1, \infty]$. Denote by $L^q_{\mathrm{loc}}(S; X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[a, b] \subset S$ belong to the space $L^q(a, b; X)$.

Let $\nu \in \mathbb{R}, q \in [1, \infty)$ and let X be as above. Put by definition

$$L^q_\nu(S; X) := \Big\{ f \in L^q_{\mathrm{loc}}(S; X) \ \Big| \ \int\limits_S e^{-2\nu t} \|f(t)\|^q_X dt < \infty \Big\}.$$

This space is a Banach space with respect to the norm

$$\|f\|_{L^q_\nu(S;X)} := \Big(\int\limits_S e^{-2\nu t} \|f(t)\|^q_X \, dt \Big)^{1/q}.$$

If X is a Hilbert space with the scalar product $(\cdot, \cdot)_X$ then the space $L^2_\nu(S; X)$ is also a Hilbert space with the scalar product

$$(f, g)_{L^2_\nu(S;X)} = \int\limits_S e^{-2\nu t} (f(t), g(t))_X \, dt.$$

Denote by $C^1_c(I)$, where $I \subset \mathbb{R}$ is an interval, the linear space of continuously differentiable functions on I with compact supports (if $I = (t_1, t_2)$, then we will write $C^1_c(t_1, t_2)$ instead of $C^1_c((t_1, t_2))$). Under $C(I; X)$, where $I \subset \mathbb{R}$ is an interval and X is an arbitrary Banach space, we mean the linear space of continuous functions defined on I with values in X. If I is a bounded closed interval then $C(I; X)$ is Banach space with a norm $\|z\|_{C(I;X)} = \max\limits_{t\in I}\|z(t)\|_X$. In case when I is an open interval we say that $z_m \underset{m\to\infty}{\longrightarrow} z$ in $C(I;X)$ if for each $\tau_1,\tau_2 \in I$ $(\tau_1<\tau_2)$ we have $\|z-z_m\|_{C([\tau_1,\tau_2];X)} \underset{m\to\infty}{\longrightarrow} 0$.

Let $H^1(\Omega) := \{v \in L_2(\Omega) \mid v_{x_i} \in L_2(\Omega)\ (i=\overline{1,n})\}$ be a Sobolev space, which is a Hilbert space with respect to the scalar product $(v,w)_{H^1(\Omega)} := \int\limits_\Omega \big\{\nabla v \nabla w + vw\big\}\,dx$ and the corresponding norm $\|v\|_{H^1(\Omega)} := \big(\int\limits_\Omega \big\{|\nabla v|^2+|v|^2\big\}\,dx\big)^{1/2}$, where $\nabla v := (v_{x_1},\dots,v_{x_n})$, $|\nabla v|^2 := \sum\limits_{i=1}^n |v_{x_i}|^2$. Under $H^1_0(\Omega)$ we mean the closure in $H^1(\Omega)$ of the space $C_c^\infty(\Omega)$ consisting of infinitely differentiable functions on Ω with compact supports. Denote

$$K := \inf_{v\in H^1_0(\Omega),\ v\neq 0} \frac{\int\limits_\Omega |\nabla v|^2\,dx}{\int\limits_\Omega |v|^2\,dx}. \tag{1}$$

Taking into account inequality (1), we define a norm $\|v\|_{H^1_0(\Omega)} = \Big(|\nabla v|^2\,dx\Big)^{1/2}$ on $H^1_0(\Omega)$, which is equivalent to the standard norm on $H^1(\Omega)$.

It is well known that the constant K is finite and coincides with the first eigenvalue of the following eigenvalue problem:

$$-\Delta v = \lambda v, \quad v|_{\partial\Omega} = 0. \tag{2}$$

From (1) it clearly follows the Friedrichs inequality

$$\int\limits_\Omega |\nabla v|^2\,dx \geqslant K \int\limits_\Omega |v|^2\,dx \quad \forall\, v\in H^1_0(\Omega). \tag{3}$$

Let $q>1$ be a real number and $q' := \frac{q}{q-1}$, that is, $\frac{1}{q}+\frac{1}{q'}=1$. We denote

$$V^q(\Omega) := H^1_0(\Omega)\cap L^q(\Omega).$$

It is well known that

$$\big(V^q(\Omega)\big)' := H^{-1}(\Omega) + L^{q'}(\Omega).$$

Also we denote $\int\limits_{\Omega_t} z\,dx := \int\limits_\Omega z(x,t)\,dx$ for each $z\in L^1_{\mathrm{loc}}(S;L^1(\Omega))$ and for a.e. $t\in S$.

Proposition 1. *(Aubin theorem, see [2] and [3, p. 393]). If $q>1, r>1$ are any real numbers, $t_1,t_2\in\mathbb{R}$ $(t_1<t_2)$, $\mathcal{W},\mathcal{L},\mathcal{B}$ are any Banach spaces such that $\mathcal{W} \overset{K}{\subset} \mathcal{L} \circlearrowleft \mathcal{B}$ (here $\overset{K}{\subset}$ means compact embedding and $\circlearrowleft$ means continuous embedding), then*

$$\{w\in L^q(t_1,t_2;\mathcal{W}) \mid w'\in L^r(t_1,t_2;\mathcal{B})\} \overset{K}{\subset} \Big(L^q(t_1,t_2;\mathcal{L})\cap C([t_1,t_2];\mathcal{B})\Big),$$

that is, if sequence $\{w_m\}_{m\in\mathbb{N}}$ is bounded in the space $L^q(t_1,t_2;\mathcal{W})$ and sequence $\{w'_m\}_{m\in\mathbb{N}}$ is bounded in the space $L^r(t_1,t_2;\mathcal{B})$, then there exists a subsequence $\{w_{m_j}\}_{j\in\mathbb{N}}\subset\{w_m\}_{m\in\mathbb{N}}$ and function $w\in L^q(t_1,t_2;\mathcal{L})\cap C([t_1,t_2];\mathcal{B})$ such that $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ strongly in $L^q(t_1,t_2;\mathcal{L})$ and in $C([t_1,t_2];\mathcal{B})$.

3. Formulation of the main problem and results. Let U be a closed linear subspace of $L^\infty(Q)$, for example, $U := L^\infty(Q)$ or $U := \{u \in L^\infty(Q) \mid v(x,t) = 0$ for a.e. $(x,t) \in Q \setminus Q_{t^*,0}\}$, where $t^* < 0$ is arbitrary fixed. Assume that U is the space of controls and for given $M = \text{const} > 0$ the set $U_\partial := \Big\{ v \in U \,\Big|\, 0 \leqslant v \leqslant M \quad \text{a.e. in} \quad Q \Big\}$ be the set of admissible controls.

We assume that the state of the investigated evolutionary system for a given control $v \in U_\partial$ is described by a weak solution of problem

$$y_t - \sum_{i,j=1}^{n} (a_{ij}(x) y_{x_i})_{x_j} + c(x)|y|^{p-2}y - v(x,t)y = f(x,t), \quad (x,t) \in Q, \tag{4}$$

$$y\big|_\Sigma = 0, \tag{5}$$

$$\lim_{t\to-\infty} e^{-\lambda t} \|y(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{6}$$

where $\lambda \in \mathbb{R}$ is given.

Before defining the weak solution of problem (4)-(6), we make some assumptions:

($\mathcal{A}$): $a_{ij} \in L^\infty(\Omega)$ $(i,j = \overline{1,n})$, there exists $\mu =$const> 0 such that $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \mu \sum\limits_{k=1}^{n} |\xi_k|^2$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, and

$$M - \mu K > 0;$$

($\mathcal{C}$): $c \in L^\infty(\Omega)$, $c(x) \geqslant c_0 = \text{const} > 0$ for a.e. $(x,t) \in Q$;
($\mathcal{F}$): $f \in L^2_{\rm loc}(S; L^2(\Omega))$;
($\mathcal{P}$): $p > 2$.

Denote $p' = \frac{p}{p-1}$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 1. *The function $y \in L^2_{loc}(S; H^1_0(\Omega)) \cap L^p_{loc}(S; L^p(\Omega)) \cap C(S; L^2(\Omega))$ is called a weak solution of problem* (4)–(6) *if its derivative y_t belongs to $L^2_{loc}(S; L^2(\Omega))$, and the following integral equality holds*

$$\int\limits_{\Omega_t} \Big\{ y_t\psi + \sum_{i,j=1}^{n} a_{ij} y_{x_i} \psi_{x_j} + (c|y|^{p-2}y - vy)\psi \Big\}\, dx$$

$$= \int\limits_{\Omega_t} f\psi\, dx \quad \text{for a.e. } t \in S \text{ and for all } \psi \in V^p(\Omega). \tag{7}$$

A weak solution y of problem (4)–(6) will be denoted by y, or $y(v)$, or $y(x,t)$, $(x,t) \in Q$, or $y(x,t;v)$, $(x,t) \in Q$.

Remark 1. Research methodology of problems similar to problem (4)–(6) is quite well developed, in particular, in papers of one of the authors [9]-[11],[12]. But exactly the same problem as considered here, more precisely, Fourier problem for semilinear parabolic equation in bounded spatial variables domains, is not investigated in literature. Moreover, estimates of the weak solution are important for us. So, for a complete presentation of the material, in Section 3 we give full proof of existence and uniqueness of the weak solution of problem (4)–(6) (for a given $v \in U_\partial$) and its estimates.

Hereafter we assume that $\lambda > 0$ and the cost functional has the form

$$J(v) = \iint\limits_Q \left[\, |y(x,t;v)| - \rho(x,t)|v|^2 \,\right] dxdt, \quad v \in U, \tag{8}$$

where $\rho \in L^1(Q)$ is given.

Remark 2. If $\lambda > 0$ and (6) hold, then functional J is well defined. Indeed, (6) implies that $\|y(\cdot,t)\|_{L^2(\Omega)} \leqslant \widetilde{C}e^{\lambda t}$ $\forall t \in S$, where $\widetilde{C} = \mathrm{const} > 0$. Hence, Cauchy-Schwarz inequality yields $\iint\limits_Q |y(x,t)|\,dxdt = \int\limits_S dt \int\limits_\Omega |y(x,t)|\,dx \leqslant (\mathrm{mes}_n\Omega)^{1/2} \int\limits_S \|y(\cdot,t)\|_{L^2(\Omega)}\,dt$ $\leqslant \widetilde{C}(\mathrm{mes}_n\Omega)^{1/2} \int\limits_S e^{\lambda t}dt < \infty$. Hence, the first term of functional J is well defined. As well $\rho \in L^1(Q),\ v \in L^\infty(Q)$, so the second term of functional J is also well defined.

We consider the following **optimal control problem**: find a control $u \in U_\partial$ such that

$$J(u) = \sup_{v\in U_\partial} J(v). \tag{9}$$

We briefly call this problem (9), and its solutions will be called *optimal controls*.

The main result of this paper is the following theorem.

Theorem 1. *Let conditions* $(\mathcal{A})$, $(\mathcal{C})$, $(\mathcal{P})$, $\lambda > M - \mu K$ *hold and*

$$f \in L^2_\lambda(S; L^2(\Omega)). \tag{10}$$

Then problem (9) has a solution.

Remark 3. In real processes function y describes the density of population. In this cases the additional condition $y \geqslant 0$ is required. This condition is satisfied if $f \geqslant 0$ (see Lemma 2).

4. Well-posedness of the problem without initial conditions (Fourier problem) for nonlinear parabolic equations.

Lemma 1. *If conditions* $(\mathcal{A})$, $(\mathcal{C})$, $(\mathcal{F})$, $(\mathcal{P})$ *and* $\lambda \geqslant M - \mu K$ *hold, then problem* (4)–(6) *has at most one weak solution.*

Proof. Assume the opposite. Let y_1, y_2 be two weak solutions of problem (4)–(6). Substituting them one by one into integral identity (7) and subtracting the obtained equalities, for the difference $z = y_1 - y_2$ we obtain

$$\int\limits_{\Omega_t} \Big[z_t\psi + \sum_{i,j=1}^n a_{ij} z_{x_i} \psi_{x_j} + c(|y_1|^{p-2}y_1 - |y_2|^{p-2}y_2)\psi$$
$$-vz\psi \Big]\, dx = 0 \quad \text{for every } \psi \in V^p(\Omega) \ \text{ and for a.e. } t \in S. \tag{11}$$

From (6) it follows the following condition

$$e^{-2\lambda t}\int\limits_{\Omega_t}|z|^2\,dx\to 0\quad\text{as}\quad t\to-\infty. \tag{12}$$

Taking in (11) $\psi(\cdot)=z(\cdot,t)$, we get

$$\int\limits_{\Omega_t}\Big[z_t z+\sum_{i,j=1}^{n}a_{ij}z_{x_i}z_{x_j} \tag{13}$$

$$+c(|y_1|^{p-2}y_1-|y_2|^{p-2}y_2)(y_1-y_2)-v|z|^2\Big]\,dx=0\quad\text{and for a.e. } t\in S.$$

Let us take arbitrary numbers $\tau_1,\tau_2\in S$ $(\tau_1<\tau_2)$. Multiplying identity (13) by $2e^{-2\lambda t}$, integrating from τ_1 to τ_2 and using the integration-by-parts formula, we obtain

$$e^{-2\lambda t}\int\limits_{\Omega}|z(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2}+2\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}e^{-2\lambda t}\Big[\sum_{i,j=1}^{n}a_{ij}z_{x_i}z_{x_j}$$

$$+c(|y_1|^{p-2}y_1-|y_2|^{p-2}y_2)(y_1-y_2)+(\lambda-v)|z|^2\Big]\,dxdt=0.$$

Thus, taking into account that $c\geqslant 0$, $(|s_1|^{p-2}s_1-|s_2|^{p-2}s_2)(s_1-s_2)\geqslant 0\ \forall s_1,s_2\in\mathbb{R}$, and using $(\mathcal{A})$ and (3), we obtain

$$e^{-\lambda\tau_2}\int\limits_{\Omega}|z(x,\tau_2)|^2\,dx\ -\ e^{-\lambda\tau_1}\int\limits_{\Omega}|z(x,\tau_1)|^2\,dx$$

$$+\ 2\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}e^{-2\lambda t}(\lambda+\mu K-M)|z|^2\,dxdt\leqslant 0. \tag{14}$$

Since $\lambda\geqslant M-\mu K$, from (14) we obtain

$$e^{-2\lambda\tau_2}\int\limits_{\Omega}|z(x,\tau_2)|^2\,dx\leqslant e^{-2\lambda\tau_1}\int\limits_{\Omega}|z(x,\tau_1)|^2\,dx. \tag{15}$$

In (15) we fix τ_2 and pass to the limit as $\tau_1\to-\infty$. According to condition (12) we obtain the equality $e^{-2\lambda\tau_2}\int\limits_{\Omega}|z(x,\tau_2)|^2\,dx=0$. Since $\tau_2\in S$ is an arbitrary number, we have $z(x,t)=0$ for a. e. $(x,t)\in Q$, that is, $y_1(x,t)=y_2(x,t)=0$ for a. e. $(x,t)\in Q$. The resulting contradiction proves our statement. $\square$

Remark 4. In case $M-\mu K\leqslant 0$, there is no need to require additional condition on solutions behavior on infinity (like condition (6)) to insure uniqueness of solution of problem (4)–(6) (see [13]).

Lemma 2. *Let conditions* $(\mathcal{A})$, $(\mathcal{C})$, $(\mathcal{F})$, $(\mathcal{P})$, $f\geqslant 0$ *and* $\lambda\geqslant M-\mu K$ *are satisfied. Then the weak solution of problem* (4)–(6) *is nonnegative.*

Proof. We denote $y^-(x,t) := \begin{cases} y(x,t), & \text{if } y(x,t) \leqslant 0, \\ 0, & \text{if } y(x,t) > 0, \end{cases}$ for a.e $(x,t) \in Q$. Let us consider integral identity (7). In this identity for a.e. $t \in S$ we take $\psi(\cdot) = y^-(\cdot,t)$. Then

$$\int\limits_{\Omega_t} \Big\{ (y^-)_t y^- + \sum_{i,j=1}^n a_{ij}(y^-)_{x_i}(y^-)_{x_j} + c|y^-|^p - v|y^-|^2 \Big\}\, dx$$

$$= \int\limits_{\Omega_t} f y^- \, dx \quad \text{for a.e. } t \in S. \tag{16}$$

Multiplying identity (16) by $e^{-2\lambda t}$, integrating from τ_1 to τ_2 ($\tau_1, \tau_2 \in S$ arbitrary numbers, $\tau_1 < \tau_2$) and using the integration-by-parts formula, we obtain

$$\frac{1}{2}e^{-2\lambda\tau_2}\int\limits_{\Omega} |y^-(x,\tau_2)|^2\, dx - \frac{1}{2}e^{-2\lambda\tau_1}\int\limits_{\Omega} |y^-(x,\tau_1)|^2\, dx + \lambda \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega} e^{-2\lambda t}|y^-|^2\, dxdt$$

$$+ \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega} e^{-2\lambda t}\big[\sum_{i,j=1}^n a_{ij}(y^-)_{x_i}(y^-)_{x_j} + c|y^-|^p - v|y^-|^2 \big]\, dxdt = \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega} f y^- e^{-2\lambda t}\, dxdt. \tag{17}$$

Since $f \geqslant 0$ and condition $(\mathcal{A})$ hold, we obtain

$$\frac{1}{2}e^{-2\lambda\tau_2}\int\limits_{\Omega} |y^-(x,\tau_2)|^2\, dx - \frac{1}{2}e^{-2\lambda\tau_1}\int\limits_{\Omega} |y^-(x,\tau_1)|^2\, dx$$

$$+ \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega} e^{-2\lambda t}\big[(\lambda + \mu K - M)|y^-|^2 \big]\, dxdt \leqslant 0.$$

Since $\lambda \geqslant M - \mu K$, then from previous inequality we obtain

$$e^{-2\lambda\tau_2}\int\limits_{\Omega} |y^-(x,\tau_2)|^2\, dx \leqslant e^{-2\lambda\tau_1}\int\limits_{\Omega} |y^-(x,\tau_1)|^2\, dx. \tag{18}$$

We pass to the limit when $\tau_1 \to -\infty$ in (18). Taking into account that $\tau_2 \in S$ is arbitrary in (18) we have $e^{-2\lambda\tau_2}\int\limits_{\Omega} |y^-(x,\tau_2)|^2\, dx \leqslant 0$, we conclude that $|y^-(x,t)|_{L^2(\Omega)} = 0$ for a.e. $t \in S$, what yields that $y^-(x,t) = 0$ a.e. in Q. □

Theorem 2. *Suppose that conditions $(\mathcal{A})$, $(\mathcal{C})$, $(\mathcal{P})$, (10) and $\lambda > M - \mu K$ hold. Then problem (4)–(6) has a unique weak solution y, and $y \in L^2_\lambda(S; H^1_0(\Omega)) \cap L^p_\lambda(S; L^p(\Omega))$, $y_t \in L^2_\lambda(S; L^2(\Omega))$. Moreover, the following estimates hold:*

$$e^{-2\lambda t}\|y(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C_1 \int\limits_{-\infty}^{t} e^{-2\lambda s}\|f(\cdot,s)\|^2_{L^2(\Omega)}\, ds, \ t \in S, \tag{19}$$

$$\|y\|^2_{L^2_\lambda(S;H^1_0(\Omega))} + \|y_t\|^2_{L^2_\lambda(S;L^2(\Omega))} + \|y\|^p_{L^p_\lambda(S;L^p(\Omega))} \leqslant C_2\|f\|^2_{L^2_\lambda(S;L^2(\Omega))}, \tag{20}$$

where C_1, C_2 are positive constants depending on M, K, μ and λ only.

Proof of Theorem 2. First, for each $m \in N$ we define $Q_m := \Omega \times (-m, 0]$, $\Sigma_m := \Gamma \times (-m, 0]$, $f_m(\cdot, t) := f(\cdot, t)$, if $-m < t \leqslant 0$, and $f_m(\cdot, t) := 0$, if $t \leqslant -m$.

Consider the problem of finding a function y_m satisfying (in some sense) the equation

$$y_{m,t} - \sum_{i,j=1}^{n} (a_{ij}(x) y_{m,x_i})_{x_j} + c(x)|y_m|^{p-2} y_m - v(x,t) y_m = f_m(x,t), \quad (x,t) \in Q_m, \tag{21}$$

boundary condition

$$y\big|_{\Sigma_m} = 0, \tag{22}$$

and initial condition

$$y_m(x, -m) = 0, \quad x \in \Omega. \tag{23}$$

A *weak solution of problem* (21)–(23) is a function $y_m \in L^2(-m, 0; H_0^1(\Omega)) \cap L^p(-m, 0; L^p(\Omega)) \cap C([-m, 0]; L^2(\Omega))$, whose derivative $y_{m,t} \in L^2(-m, 0; L^2(\Omega))$, and which satisfies condition (23) and the following integral identity

$$\int\limits_{\Omega_t} \Big\{ y_{m,t}\psi + \sum_{i,j=1}^{n} a_{ij} y_{m,x_i} \psi_{x_j} + (c|y_m|^{p-2} y_m - v y_m)\psi \Big\}\, dx$$
$$= \int\limits_{\Omega_t} f_m \psi \, dxdt \quad \text{for a.e. } t \in [-m, 0] \text{ and for all } \psi \in V^p(\Omega)\ . \tag{24}$$

Lemma 3. *Let conditions* $(\mathcal{A})$, $(\mathcal{C})$, $(\mathcal{F})$ *and* $(\mathcal{P})$ *hold. Then problem* (21)–(23) *has unique weak solution* y_m. *Moreover, for any* $\lambda > M - \mu K$ *this solution satisfies following estimates:*

$$e^{-2\lambda t} \int\limits_{\Omega_t} |y_m|^2 \, dx \leqslant C_1 \int\limits_{-m}^{t} \int\limits_{\Omega} e^{-2\lambda s} |f(x,s)|^2 \, dxds, \quad t \in [-m, 0], \tag{25}$$

$$\iint\limits_{Q_m} e^{-2\lambda t} \Big[|\nabla y_m|^2 + |y_{m,t}|^2 + |y_m|^p \Big] \, dxdt \leqslant C_2 \iint\limits_{Q_m} e^{-2\lambda t} |f_m|^2 \, dxdt, \tag{26}$$

where C_1, C_2 *are positive constants depending on* M, K, μ *and* λ *only.*

The proof of Lemma 3 is given later in this section.

For every $m \in \mathbb{N}$ we extend y_m by zero for the entire set Q and keep the same notation y_m for this extension. Note that for each $m \in N$, the function y_m belongs to $L^2(S; H_0^1(\Omega)) \cap L^p(S; L^p(\Omega)) \cap C(S; L^2(\Omega))$, its derivative $y_{m,t}$ belongs to $L^2(-m, 0; L^2(\Omega))$, and y_m satisfies integral identity (7) with f_m substituted for f, i.e.,

$$\int\limits_{\Omega_t} \Big\{ y_{m,t}\psi + \sum_{i,j=1}^{n} a_{ij} y_{x_i} \psi_{x_j} + (c|y_m|^{p-2} y_m - v y_m)\psi \Big\}\, dx = \int\limits_{\Omega_t} f_m \psi \, dx$$
$$\text{for a.e. } t \in S \text{ and for all } \psi \in V^p(\Omega). \tag{27}$$

Thus, y_m is a weak solution of problem (4)–(6) with f_m substituted for f, and according to Lemma 3 and condition (10), for y_m we obtain estimates

$$e^{-2\lambda t}\|y_m(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C_1 \int\limits_{-\infty}^{t} e^{-2\lambda s}\|f(\cdot,s\|^2_{L^2(\Omega)}\,ds, \quad t\in S, \tag{28}$$

$$\|y_m\|^2_{L^2_\lambda(S;H^1_0(\Omega))} + \|y_{m,t}\|^2_{L^2_\lambda(S;L^2(\Omega))} + \|y_m\|^p_{L^p_\lambda(S;L^p(\Omega))} \leqslant C_2\|f\|^2_{L^2_\lambda(S;L^2(\Omega))}. \tag{29}$$

According to Proposition 1 and the compactness of the embedding $H^1_0(\Omega)\subset L^2(\Omega)$, estimate (29), we obtain that there exist a subsequence of the sequence $\{y_m\}$ (still denoted by $\{y_m\}$ for simplicity) and the function $y\in L^2_\lambda(S;H^1_0(\Omega))\cap L^p_\lambda(S;L^p(\Omega))\cap C(S;L^2(\Omega))$ such that $y_t\in L^2_\lambda(S;L^2(\Omega))$ and

$$y_m \underset{m\to\infty}{\longrightarrow} y \qquad \text{weakly in} \quad L^2_\lambda(S;H^1_0(\Omega)), \tag{30}$$

$$y_{m,t} \underset{m\to\infty}{\longrightarrow} y_t \qquad \text{weakly in} \quad L^2_\lambda(S;L^2(\Omega)), \tag{31}$$

$$y_m \underset{m\to\infty}{\longrightarrow} y \qquad \text{weakly in} \quad L^p_\lambda(S;L^p(\Omega)), \tag{32}$$

$$y_m \underset{m\to\infty}{\longrightarrow} y \quad \text{in} \quad C(S;L^2(\Omega)), \tag{33}$$

$$y_m \underset{m\to\infty}{\longrightarrow} y \quad \text{a.e. in} \quad Q, \tag{34}$$

$$|y_m|^{p-2}y_m \underset{m\to\infty}{\longrightarrow} |y|^{p-2}y \quad \text{weakly in} \quad L^{p'}_\lambda(Q). \tag{35}$$

From (35) we obtain

$$\iint\limits_Q c|y_m|^{p-2}y_m\psi\varphi\,dxdt \underset{m\to\infty}{\longrightarrow} \iint\limits_Q c|y|^{p-2}y\psi\varphi\,dxdt \quad \forall\,\psi\in V^p(\Omega),\ \forall\,\varphi\in C^1_c(-\infty,0). \tag{36}$$

Let us show that the function y is a weak solution of problem (4)–(6). To do this, we multiply identity (24) by arbitrary $\varphi\in C^1_c(-\infty,0)$ and integrate over $t\in S$

$$\iint\limits_Q \Big\{y_{m,t}\psi\varphi + \sum_{i,j=1}^n a_{ij}y_{x_i}\psi_{x_j}\varphi + (c|y_m|^{p-2}y_m - vy_m)\psi\varphi\Big\}\,dxdt = \iint\limits_Q f_m\psi\varphi\,dxdt,$$
$$\psi\in V^p(\Omega), \quad \varphi\in C^1_c(-\infty,0). \tag{37}$$

Now we let $m\to\infty$ in identity (37), taking into account (30), (31), (36) and the definition of the function f_m. From the obtained integral identity, taking into account Du Bois-Reymond lemma, we get identity (7). Next, taking into account (33), we let $m\to+\infty$ in (28). From the resulting inequality and condition (10), we obtain condition (6). Hence, we have proven that y is a weak solution of problem (4)–(6). And from estimate (29) and convergence (30)–(32) we obtain estimate (20). Estimate (19) easily follows from (28) and (33). $\square$

Proof of Lemma 3. We fix arbitrary $m\in\mathbb{N}$ and, for simplicity, for the weak solution y_m of problem (21)–(23) we use notation z.

To prove our statement we use Galerkin's method. Let $\{w_l\,|\,l\in\mathbb{N}\}$ be a linear independent set of functions from $V^p(\Omega)$, which is complete in $V^p(\Omega)$, that is, the set of all its finite linear combinations is dense in $V^p(\Omega)$. According to Galerkin's method, for every $r \in \mathbb{N}$ we put

$$z_r(x,t) = \sum_{k=1}^{r} c_{r,k}(t) w_k(x), \quad (x,t) \in \overline{Q_m},$$

where $c_{r,1}, \dots, c_{r,r}$ are absolutely continuous functions, which are solutions of the Cauchy problem for the system of ordinary differential equations

$$\int\limits_{\Omega_t} z_{r,t} w_l \, dx + \int\limits_{\Omega_t} \Big\{ \sum_{i,j=1}^{n} a_{ij} z_{r,x_i} w_{l,x_j} + c|z_r|^{p-2} z_r w_l - v z_r w_l \Big\} \, dx$$
$$= \int\limits_{\Omega_t} f w_l \, dx, \quad t \in [-m, 0], \quad l = \overline{1,r}, \tag{38}$$
$$c_{r,l}(-m) = 0, \quad l = \overline{1,r}. \tag{39}$$

The linear independence of functions $w_1, \dots, w_r$ yields that the matrix $(b^r_{k,l})^r_{k,l=1}$ is positive-definite, where $b^r_{k,l} = \int_\Omega w_k w_l \; dx$ $(k,l = \overline{1,r})$. Thus the system of ordinary differential equations (38) can be transformed to the normal form. Hence, according to the theorems of existence and extension of the solution to this problem (see [16]), there exists the global solution $c_{r,1}, \dots.., c_{r,r}$ of problem (38), (39), defined on $[-m, \bar{t} >$, where $\bar{t} \in (-m, 0]$, ">" means either ")" or "]". Later we will show that $[-m, \bar{t} >= [-m, 0]$.

Multiply the equation of system (38) with number $l \in \{1, \dots, r\}$ by $e^{-2\lambda t} c_{r,l}$ and sum over $l \in \{1, \dots, r\}$. Integrating the obtained equality over $t \in [-m, \tau] \subset [-m, \bar{t} >$, we have

$$\int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} z_{r,t} z_r \; dxdt + \int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} \Big[\sum_{i,j=1}^{n} a_{ij} z_{r,x_i} z_{r,x_j}$$
$$+ c|z_r|^p - v|z_r|^2 \Big] \; dxdt = \int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} f z_r \, dxdt. \tag{40}$$

From (40), using (39), Cauchy inequality and the integration-by-parts formula, we obtain

$$\frac{1}{2} \int\limits_{\Omega} e^{-2\lambda \tau} |z_r(x,\tau)|^2 \; dx + \lambda \int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} |z_r|^2 \; dxdt$$
$$+ \int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} \Big[\sum_{i,j=1}^{n} a_{ij} z_{r,x_i} z_{r,x_j} + c|z_r|^p - v|z_r|^2 \Big] \; dxdt \tag{41}$$
$$\leqslant \frac{\varepsilon_1}{2} \int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} |z_r|^2 \, dxdt + \frac{1}{2\varepsilon_1} \int\limits_{-m}^{\tau} \int\limits_{\Omega} e^{-2\lambda t} |f|^2 \, dxdt, \quad \tau \in [-m, \bar{t} >,$$

where $\varepsilon_1 > 0$ is arbitrary number.

Since $v(x,t) \leqslant M$ for a.e. $(x,t) \in Q$, using (3) and condition $(\mathcal{A})$, from (41), we have

$$\frac{1}{2}\int\limits_{\Omega} e^{-2\lambda\tau}|z_r(x,\tau)|^2\,dx + (\lambda - M + \mu K(1-\varepsilon_2) - \frac{\varepsilon_1}{2})\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}|z_r|^2\,dxdt$$
$$+\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}\big[\varepsilon_2\mu|\nabla z_r|^2 + c_0|z_r|^p\big]\,dxdt \leqslant \frac{1}{2\varepsilon_1}\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}|f|^2\,dxdt, \quad \tau \in [-m,\bar{t}> . \tag{42}$$

Since $\lambda \geqslant M - \mu K$, then one can easily choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\lambda - M + \mu K(1-\varepsilon_2) - \frac{\varepsilon_1}{2} > 0$ (for example, $\varepsilon_2 = \frac{\lambda - M + \mu K}{4\mu K} > 0$ and $\varepsilon_1 = \frac{\lambda - M + \mu K}{2} > 0$). This implies the following inequality

$$\int\limits_{\Omega} e^{-2\lambda\tau}|z_r(x,\tau)|^2\,dx + C_3\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}\big[|\nabla z_r|^2 + |z_r|^2 + |z_r|^p\big]\,dxdt \tag{43}$$
$$\leqslant C_4\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}|f|^2\,dxdt, \quad \tau \in [-m,\bar{t}>,$$

where positive constants C_3, C_4 do not depend on m and r.

From (43) we get the following estimates

$$e^{-2\lambda\tau}\int\limits_{\Omega}|z_r(x,\tau)|^2\,dx \leqslant C_1\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}|f|^2\,dxdt, \quad \tau \in [-m,\bar{t}>, \tag{44}$$

$$\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}\big[|\nabla z_r|^2 + |z_r|^2 + |z_r|^p\big]\,dxdt \leqslant C_2\int\limits_{-m}^{\tau}\int\limits_{\Omega} e^{-2\lambda t}|f|^2\,dxdt, \quad \tau \in [-m,\bar{t}> . \tag{45}$$

Estimate (44) yields that the sequence $\Big\{ \operatorname*{ess\,sup}_{t\in[-m,\bar{t}>} \|z_r(\cdot,t)\|^2_{L^2(\Omega)}\Big\}$ is bounded by a constant, which is independent on $\bar{t}$. This yields that $[-m,\bar{t}>=[-m,0]$.

Multiply the equation of system (38) with number $l \in \{1,\dots,r\}$ by $e^{-2\lambda t}c'_{r,l}(t)$ and sum over $l \in \{1,\dots,r\}$. Integrating the obtained equality over $t \in [-m,0]$, we obtain

$$\iint\limits_{Q_m} e^{-2\lambda\tau}|z_{r,t}|^2\,dxdt + \iint\limits_{Q_m} e^{-2\lambda t}\Big[\sum_{i,j=1}^{n} a_{ij}z_{r,x_i}z_{r,x_j,t}$$
$$+c|z_r|^{p-2}z_r z_{r,t} - v z_r z_{r,t}\Big]\,dxdt = \iint\limits_{Q_m} e^{-2\lambda t} f z_{r,t}\,dxdt. \tag{46}$$

From (46), using (39) and the integration-by-parts formula and the fact that in our case

$$|z_r|^{p-2}z_r z_{r,t} = \frac{1}{p}(|z_r|^p)_t,$$

we obtain

$$\iint\limits_{Q_m} e^{-2\lambda t}|z_{r,t}|^2\,dxdt + \frac{1}{2}\int\limits_{\Omega}\sum_{i,j=1}^{n} a_{ij}z_{r,x_i}(x,0)z_{r,x_j}(x,0)\,dx$$
$$+\lambda\iint\limits_{Q_m} e^{-2\lambda t}\sum_{i,j=1}^{n} a_{ij}z_{r,x_i}z_{r,x_j}\,dxdt + \frac{1}{p}\int\limits_{\Omega} c(x)|z_r(x,0)|^p\,dx$$
$$+\frac{2\lambda}{p}\iint\limits_{Q_m} e^{-2\lambda t}c|z_r|^p\,dxdt = \iint\limits_{Q_m} e^{-2\lambda t}fz_{r,t}\,dxdt + \iint\limits_{Q_m} e^{-2\lambda t}vz_rz_{r,t}\,dxdt. \tag{47}$$

Using conditions $(\mathcal{A})$, $(\mathcal{C})$ and Cauchy inequality from (47) we obtain

$$\iint\limits_{Q_m} e^{-2\lambda t}|z_{r,t}|^2\,dxdt + \lambda\mu\iint\limits_{Q_m} e^{-2\lambda t}|\nabla z_r|^2\,dxdt$$
$$+\frac{2\lambda c_0}{p}\iint\limits_{Q_m} e^{-2\lambda t}|z_r|^p\,dxdt \leqslant \frac{1}{2\varepsilon_2}\iint\limits_{Q_m} e^{-2\lambda t}|f|^2\,dxdt$$
$$+\frac{M}{2\varepsilon_1}\iint\limits_{Q_m} e^{-2\lambda t}|z_r|^2\,dxdt + \Big(\frac{\varepsilon_1 M}{2} + \frac{\varepsilon_2}{2}\Big)\iint\limits_{Q_m} e^{-2\lambda t}|z_{r,t}|^2\,dxdt. \tag{48}$$

From (48), using (45) and taking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $1 - \frac{\varepsilon_1 M}{2} - \frac{\varepsilon_2}{2} > 0$, we get the following estimate

$$\iint\limits_{Q_m} e^{-2\lambda t}|z_{r,t}|^2\,dxdt \leqslant C_5\iint\limits_{Q_m} e^{-2\lambda t}|f|^2\,dxdt, \tag{49}$$

where constant $C_5 > 0$ is independent on m and r.

Estimates (44), (45), (49) yield that sequence $\{z_r\}_{r=1}^{\infty}$ is bounded in the spaces $L^2(-m,0;H_0^1(\Omega))$, $L^\infty(-m,0;L^2(\Omega))$ and $L^p(-m,0;L^p(\Omega))$, and $z_{r,t}$ is bounded in $L^2(-m,0;L^2(\Omega))$. Consequently, taking into account Proposition 1, we obtain existence of the subsequence of $\{z_r\}_{r=1}^{\infty}$ and the function $z \in L^2(-m,0;H_0^1(\Omega)) \cap L^\infty(-m,0;L^2(\Omega)) \cap L^p(-m,0;L^p(\Omega))$ such that $z_t \in L^2(-m,0;L^2(\Omega))$ and

$$z_r \underset{r\to\infty}{\longrightarrow} z \quad \text{weakly in} \quad L^2(-m,0;H_0^1(\Omega)), \tag{50}$$

$$z_{r,t} \underset{r\to\infty}{\longrightarrow} z_t \quad \text{weakly in} \quad L^2(-m,0;L^2(\Omega)), \tag{51}$$

$$z_r \underset{r\to\infty}{\longrightarrow} z \quad \text{weakly in} \quad L^p(-m,0;L^p(\Omega)), \tag{52}$$

$$z_r \underset{r\to\infty}{\longrightarrow} z \quad \text{strongly in} \quad L^2(Q_m),\ \text{and in}\ C([-m,0];L^2(\Omega)), \tag{53}$$

$$z_r \underset{r\to\infty}{\longrightarrow} z \quad \text{a.e. in} \quad Q, \tag{54}$$

$$|z_r|^{p-2}z_r \underset{r\to\infty}{\longrightarrow} |z|^{p-2}z \quad \text{weakly in} \quad L_\lambda^{p'}(Q). \tag{55}$$

From (54), (55), similar to the convergence (36), we have convergence

$$\iint\limits_{Q_m} c|z_r|^{p-2} z_r \psi\varphi\, dxdt \underset{r\to\infty}{\longrightarrow} \iint\limits_{Q_m} c|z|^{p-2} z\psi\varphi\, dxdt. \tag{56}$$

Let $\nu_1, ..., \nu_k$ $(k \in \mathbb{N})$ are any real numbers and $\varphi \in C_c^1(-m, 0)$ is arbitrary function. For every $j \in \{1, ..., k\}$ we multiply the equation of system (38) with number $j \in \{1, \dots, r\}$ by ν_j, summarizing obtained equations and pass to the limit as $r \to \infty$, denoting $\psi = \sum\limits_{j=1}^{k} \nu_j w_j$ and integrating resulting equality over $t \in [-m, 0]$, we get

$$\iint\limits_{Q_m} z_t\psi\varphi\; dxdt + \iint\limits_{Q_m} \Big\{ \sum_{i,j=1}^{n} a_{ij} z_{x_i} \psi_{x_j} + c|z|^{p-2} z\psi - vz\psi \Big\}\varphi\, dxdt$$

$$= \iint\limits_{Q_m} f\psi\varphi\; dxdt \quad \forall\, \varphi \in C_c^1(-m, 0). \tag{57}$$

Since the set $\{\nu_1 w_1 + ... + \nu_k w_k \mid k \in \mathbb{N}, \nu_1, ..., \nu_k \in \mathbb{R}\}$ is dense in $V^p(\Omega)$, then (57) yields the equality

$$\iint\limits_{Q_m} z_t\psi\varphi\; dxdt + \iint\limits_{Q_m} \Big\{ \sum_{i,j=1}^{n} a_{ij} z_{x_i} \psi_{x_j} + c|z|^{p-2} z\psi - vz\psi \Big\}\varphi\, dxdt$$

$$= \iint\limits_{Q_m} f\psi\varphi\; dxdt, \quad \psi \in V^p(\Omega), \quad \varphi \in C_c^1(-m, 0). \tag{58}$$

Using Du Bois-Reymond lemma we obtain identity (24). Thus, we have shown that problem (21)–(23) has a solution $z = y_m$. From (44), (45) and (49), taking into account (50) – (53), we obtain that function y_m satisfies estimates (25), (26). □

5. Proof of the main result.

Proof of Theorem 3. Existence of the solution. Since the cost functional J is bounded above, there exists a maximizing sequence $\{v_k\}$ in U_∂: $J(v_k) \underset{k\to\infty}{\longrightarrow} \sup\limits_{v\in U_\partial} J(v)$. The sequence $\{v_k\}$ is bounded in the space $L^\infty(Q)$, that is

$$0 \leqslant v_k(x, t) \leqslant M \quad \text{for a.e.} \quad (x, t) \in Q. \tag{59}$$

Since for each $k \in \mathbb{N}$ the function $y_k := y(v_k)$ $(k \in \mathbb{N})$ is a weak solution of problem (4)–(6) for $v = v_k$, then the following identity holds:

$$\iint\limits_{Q} \Big\{ y_{k,t}\psi\varphi + \sum_{i,j=1}^{n} a_{ij} y_{k,x_i} \psi_{x_j}\varphi + (c|y_k|^{p-2} y_k - v_k y_k)\psi\varphi \Big\}\, dxdt$$

$$= \iint\limits_{Q} f\psi\varphi\, dxdt, \quad \psi \in V^p(\Omega),\; \varphi \in C_c^1(-\infty, 0). \tag{60}$$

According to Theorem 2 we have the estimates

$$e^{-2\lambda t}\|y_k(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C_1 \int\limits_{-\infty}^{t} e^{-2\lambda s}\|f(\cdot,s)\|^2_{L^2(\Omega)}\,ds,\ t\in S, \tag{61}$$

$$\|y_k\|^2_{L^2_\lambda(S;H^1_0(\Omega))} + \|y_{k,t}\|^2_{L^2_\lambda(S;L^2(\Omega))} + \|y_k\|^p_{L^p_\lambda(S;L^p(\Omega))} \leqslant C_2\|f\|^2_{L^2_\lambda(S;L^p(\Omega))}. \tag{62}$$

Taking into account estimate (62) for arbitrary $\tau_1,\tau_2\in S$ $(\tau_1<\tau_2)$ we obtain

$$\int\limits_{\tau_1}^{\tau_2} \|y_{k,t}\|^2_{L^2(\Omega_t)}\,dt \leqslant C_6, \tag{63}$$

where $C_6>0$ is a constant which depends on τ_1 and τ_2, but does not depend on k.

Since $\rho\in L^1(Q)$, using (59), we get that sequence $\{\sqrt{\rho}v_k\}_{k=1}^\infty$ is bounded in $L^2(Q)$. Since $V^p(\Omega) \circlearrowleft H^1_0(\Omega)\overset{K}{\subset} L^2(\Omega)$ (see [23] c. 245), then $V^p(\Omega)\overset{K}{\subset} L^2(\Omega)$. According to Theorem 1 with $\mathcal{W}=V^p(\Omega)$, $\mathcal{L}=L^2(\Omega)$, $\mathcal{B}=L^2(\Omega)$, $q=2$, $r=2$, estimates (59), (62), (63) yield that there exists a subsequence of the sequence $\{v_k,y_k\}$ (still denoted by $\{v_k,y_k\}$) and functions $u\in U_\partial$, $\zeta\in L^2(Q)$, $y\in L^2_\lambda(S;H^1_0(\Omega))\cap L^p_\lambda(S;L^p(\Omega))$, $y_t\in L^2_\lambda(S;L^2(\Omega))$ such that

$$v_k \underset{k\to\infty}{\longrightarrow} u \quad *\text{-weakly in}\quad L^\infty(Q), \tag{64}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in}\quad L^2_\lambda(S;H^1_0(\Omega)), \tag{65}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in}\quad L^p_\lambda(S;L^p(\Omega)), \tag{66}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{in}\quad C(S;L^2(\Omega)),\ \text{and strongly in}\ L^2_{\rm loc}(S;L^2(\Omega)), \tag{67}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{a.e. on}\quad Q, \tag{68}$$

$$y_{k,t} \underset{k\to\infty}{\longrightarrow} y_t \quad \text{weakly in}\quad L^2_\lambda(S;L^2(\Omega)) \tag{69}$$

$$|y_k| \underset{k\to\infty}{\longrightarrow} |y| \quad \text{weakly in}\quad L^2_\lambda(S;L^2(\Omega)). \tag{70}$$

Note that (65) implies the following

$$y_k \underset{k\to\infty}{\longrightarrow} y,\quad y_{k,x_i} \underset{k\to\infty}{\longrightarrow} y_{x_i}\ (i=\overline{1,n})\quad \text{weakly in}\quad L^2_{\rm loc}(S;L^2(\Omega)). \tag{71}$$

As in (56), from (62), (67) and [?, Lemma 2.2], we obtain

$$c|y_k|^{p-2}y_k \underset{k\to\infty}{\longrightarrow} c|y|^{p-2}y \quad \text{weakly in}\quad L^{p'}_{\rm loc}(\overline{Q}). \tag{72}$$

Let us show that (64) and (67) yield

$$\iint\limits_Q y_k v_k \psi\varphi\,dxdt \underset{k\to\infty}{\longrightarrow} \iint\limits_Q yu\psi\varphi\,dxdt \quad \forall\ \psi\in V^p(\Omega), \forall\ \varphi\in C^1_c(-\infty,0). \tag{73}$$

Indeed, let $g:=\psi\varphi$, $t_1,t_2\in S$ be such that $\operatorname{supp}\varphi\subset[t_1,t_2]$. Then we have

$$\iint\limits_Q y_k v_k g\,dxdt = \int\limits_{t_1}^{t_2}\int\limits_\Omega (y_k v_k - y v_k + y v_k) g\,dxdt$$

$$= \int\limits_{t_1}^{t_2}\int\limits_{\Omega} y v_k g \, dxdt + \int\limits_{t_1}^{t_2}\int\limits_{\Omega} (y_k - y) v_k g \, dxdt. \tag{74}$$

From (59) and (67) it follows

$$\Big|\int\limits_{t_1}^{t_2}\int\limits_{\Omega} (y_k - y) v_k g \, dxdt\Big| \leqslant \Big(\int\limits_{t_1}^{t_2}\int\limits_{\Omega} |v_k g|^2 \, dxdt\Big)^{1/2} \Big(\int\limits_{t_1}^{t_2}\int\limits_{\Omega} |y_k - y|^2 \, dxdt\Big)^{1/2} \underset{k\to\infty}{\longrightarrow} 0. \tag{75}$$

Thus, using (64) and (75), (74) implies (73).

Using (71) and (73), and letting $k \to \infty$ in (60), we obtain

$$\iint\limits_Q \Big\{ y_t \psi\varphi + \sum_{i,j=1}^n a_{ij} y_{x_i} \psi_{x_j} \varphi + (c|y|^{p-2} y - uy)\psi\varphi \Big\} \, dxdt$$

$$= \iint\limits_Q f\psi\varphi \, dxdt \quad \forall\, \psi \in V^p(\Omega) \quad \forall\, \varphi \in C_c^1(-\infty, 0). \tag{76}$$

According to Du Bois-Reymond lemma, identity (76) implies that the function $y = y(u)$ satisfies integral identity (7). Let us show that y satisfies condition (6).

Taking into account (67), we pass to the limit in (61) as $k \to \infty$. The resulting inequality, according to condition (10), implies

$$\lim_{t\to-\infty} e^{-2\lambda t} \int\limits_{\Omega} |y(x,t)|^2 \, dx = 0. \tag{77}$$

Hence, we have shown that $y = y(u) = y(x,t;u)$, $(x,t) \in Q$, is the state of the controlled system for the control u.

It remains to prove that u is a maximizing element of the functional J. Indeed, from (64) we get

$$\sqrt{\rho} v_k \underset{k\to\infty}{\longrightarrow} \sqrt{\rho} u \quad \text{weakly in} \quad L^2(Q). \tag{78}$$

According to [15, p. 58, Proposition 3.5] we obtain

$$\liminf_{k\to\infty} \|\sqrt{\rho} v_k\|^2_{L^2(Q)} \geqslant \|\sqrt{\rho} u\|^2_{L^2(Q)}. \tag{79}$$

One can check that the functional $w \mapsto \iint\limits_Q w \, dxdt \; : \; L^2_\lambda(S; L^2(\Omega)) \to \mathbb{R}$ is well defined. Indeed,

$$\Big|\iint\limits_Q w \, dxdt\Big| \leqslant \iint\limits_Q |w| \, dxdt = \iint\limits_Q e^{-\lambda t} e^{\lambda t} |w| \, dxdt$$

$$\leqslant \Big(\iint\limits_Q e^{-2\lambda t} |w| \, dxdt\Big)^{1/2} \Big(\iint\limits_Q e^{2\lambda t} \, dxdt\Big)^{1/2} = C_7 \|w\|_{L^2_\lambda(S; L^2(\Omega))}, \tag{80}$$

where $C_7 > 0$ is some constant.

We denote this functional by $\mathbb{I}$. It belongs to $\big(L^2_\lambda(S; L^2(\Omega))\big)'$. Actually, the linearity of $\mathbb{I}$ is trivial. And estimate (80) implies that $\mathbb{I}$ is bounded. Hence, according to (70) we have

$$\iint\limits_Q |y_k|\,dxdt = < \mathbb{I}, |y_k| > \underset{k\to\infty}{\longrightarrow} < \mathbb{I}, |y| > = \iint\limits_Q |y|\,dxdt. \tag{81}$$

It follows easily from (8), (79) and (81) that

$$\lim_{k\to\infty} J(v_k) = \lim_{k\to\infty}\Big[\iint\limits_Q |y_k|\,dxdt - \iint\limits_Q \rho|v_k|^2\,dxdt\Big]$$

$$\leqslant \lim_{k\to\infty}\iint\limits_Q |y_k|\,dxdt - \liminf_{k\to\infty}\|\sqrt{\rho}v_k\|^2_{L^2(Q)} \leqslant \iint\limits_Q |y|\,dxdt - \|\sqrt{\rho}u\|^2_{L^2(Q)} = J(u).$$

Thus, we have shown that u is a solution of problem (9). □

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ЗАДАЧА ОПТИМАЛЬНОГО КЕРУВАННЯ РЕСУРСНИМ КОЕФІЦІЄНТОМ ПОПУЛЯЦІЙНОЇ МОДЕЛІ, ЩО ОПИСУЄТЬСЯ ЗАДАЧЕЮ БЕЗ ПОЧАТКОВИХ УМОВ ДЛЯ ПАРАБОЛІЧНОГО РІВНЯННЯ

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Вивчено задачу оптимального керування системами, що описуються задачею Фур'є для нелінійних параболічних рівнянь. Керування є коефіцієнтом рівняння стану. Доведено існування оптимального управління.

Ключові слова: оптимальне керування, задачі без початкових умов, еволюційні рівняння.