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ON THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF $\mathbb{N}^2_{\leqslant}$ WITH COFINITE DOMAINS AND IMAGES, II

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Let $\mathbb{N}^2_{\leqslant}$ be the set $\mathbb{N}^2$ with the partial order defined as the product of usual order $\leqslant$ on the set of positive integers $\mathbb{N}$. We study the semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})$ of monotone injective partial selfmaps of $\mathbb{N}^2_{\leqslant}$ having cofinite domain and image. We describe the natural partial order on the semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})$ and show that it coincides with the natural partial order which is induced from symmetric inverse monoid $\mathscr{I}_{\mathbb{N}\times\mathbb{N}}$ over the set $\mathbb{N}\times\mathbb{N}$ onto the semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})$. We proved that the semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})$ is isomorphic to the semidirect product $\mathscr{P\!O}^{+}_{\infty}(\mathbb{N}^2_{\leqslant}) \rtimes \mathbb{Z}_2$ of the monoid $\mathscr{P\!O}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ of orientation-preserving monotone injective partial selfmaps of $\mathbb{N}^2_{\leqslant}$ with cofinite domains and images by the cyclic group $\mathbb{Z}_2$ of the order two. Also we describe the congruence σ on the semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})$ which is generated by the natural order $\preccurlyeq$ on the semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})$: $\alpha\sigma\beta$ if and only if α and β are comparable in $\left(\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant}),\preccurlyeq\right)$. We prove that the quotient semigroup $\mathscr{P\!O}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ is isomorphic to the free commutative monoid $\mathfrak{AM}_{\omega}$ over an infinite countable set and show that the quotient semigroup $\mathscr{P\!O}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ is isomorphic to the semidirect product of the free commutative monoid $\mathfrak{AM}_{\omega}$ by the group $\mathbb{Z}_2$.

Key words: Semigroup of bijective partial transformations, natural partial order, semidirect product, minimum group congruence, free commutative monoid.

We shall follow the terminology of [2] and [10].

In this paper we shall denote the first infinite cardinal by ω and the cardinality of the set A by $|A|$. We shall identify every set X with its cardinality $|X|$. By $\mathbb{Z}_2$ we shall denote the cyclic group of order two. Also, for infinite subsets A and B of an infinite set X we shall write $A\subseteq^{*}B$ if and only if there exists a finite subset A_0 of A such that $A \setminus A_0 \subseteq B$.

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An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$.

If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

If $\alpha\colon X \rightharpoonup Y$ is a partial map, then by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ we denote the domain and the range of α, respectively.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of an infinite set X of cardinality λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \mid y\alpha \in \operatorname{dom}\beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [2, Section 1.9]). The symmetric inverse semigroup was introduced by Vagner [18] and it plays a major role in the theory of semigroups. An element $\alpha \in \mathscr{I}_\lambda$ is called *cofinite*, if the sets $\lambda \setminus \operatorname{dom}\alpha$ and $\lambda \setminus \operatorname{ran}\alpha$ are finite.

Let $(X, \leqslant)$ be a partially ordered set (a poset). For an arbitrary $x \in X$ we denote

$$\uparrow x = \{y \in X \colon x \leqslant y\}.$$

We shall say that a partial map $\alpha\colon X \rightharpoonup X$ is *monotone* if $x \leqslant y$ implies $(x)\alpha \leqslant (y)\alpha$ for $x, y \in \operatorname{dom}\alpha$.

Let $\mathbb{N}$ be the set of positive integers with the usual linear order $\leq$. On the Cartesian product $\mathbb{N} \times \mathbb{N}$ we define the product partial order, i.e.,

$$(i, m) \leqslant (j, n) \qquad \text{if and only if} \qquad (i \leqslant j) \quad \text{and} \quad (m \leqslant n).$$

Later the set $\mathbb{N} \times \mathbb{N}$ with so defined partial order will be denoted by $\mathbb{N}^2_\leqslant$.

By $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we denote the semigroup of injective partial monotone selfmaps of $\mathbb{N}^2_\leqslant$ with cofinite domains and images. Obviously, $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a submonoid of the symmetric inverse semigroup $\mathscr{I}_\omega$ and $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a countable semigroup.

Furthermore, we shall denote the identity of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ by $\mathbb{I}$ and the group of units of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ by $H(\mathbb{I})$.

For any positive integer n and an arbitrary $\alpha \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we denote:

$$\begin{aligned} &\mathsf{V}^n = \{(n, j) \colon j \in \mathbb{N}\}; && \mathsf{H}^n = \{(j, n) \colon j \in \mathbb{N}\}; \\ &\mathsf{V}^n_{\operatorname{dom}\alpha} = \mathsf{V}^n \cap \operatorname{dom}\alpha; && \mathsf{V}^n_{\operatorname{ran}\alpha} = \mathsf{V}^n \cap \operatorname{ran}\alpha; \\ &\mathsf{H}^n_{\operatorname{dom}\alpha} = \mathsf{H}^n \cap \operatorname{dom}\alpha; && \mathsf{H}^n_{\operatorname{ran}\alpha} = \mathsf{H}^n \cap \operatorname{ran}\alpha, \end{aligned}$$

and

$$(i_{\alpha[*,j]}, j_{\alpha[i,*]}) = (i, j)\alpha, \qquad \text{for every} \quad (i, j) \in \operatorname{dom}\alpha.$$

It well known that each partial injective cofinite selfmap f of λ induces a homeomorphism $f^*\colon \lambda^* \to \lambda^*$ of the remainder $\lambda^* = \beta\lambda \setminus \lambda$ of the Stone-Čech compactification of the discrete space λ. Moreover, under some set theoretic axioms (like **PFA** or **OCA**), each homeomorphism of ω^* is induced by some partial injective cofinite selfmap of ω (see [12]–[17]). So, the inverse semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ of injective partial selfmaps of an infinite cardinal λ with cofinite domains and images admits a natural homomorphism $\mathfrak{h}\colon \mathscr{I}_\lambda^{\mathrm{cf}} \to \mathscr{H}(\lambda^*)$ to the homeomorphism group $\mathscr{H}(\lambda^*)$ of λ^* and this homomorphism is surjective under certain set theoretic assumptions.

In the paper [9] algebraic properties of the semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}$ are studied. It is showed that $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a bisimple inverse semigroup and that for every non-empty chain L in $E(\mathscr{I}_\lambda^{\mathrm{cf}})$ there exists an inverse subsemigroup S of $\mathscr{I}_\lambda^{\mathrm{cf}}$ such that S is isomorphic to the bicyclic semigroup and $L \subseteq E(S)$, the Green relations on $\mathscr{I}_\lambda^{\mathrm{cf}}$ are described and it is proved that every non-trivial congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$ is a group congruence. Also, the structure of the quotient semigroup $\mathscr{I}_\lambda^{\mathrm{cf}}/\sigma$ is described, where σ is the least group congruence on $\mathscr{I}_\lambda^{\mathrm{cf}}$.

The semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of injective isotone partial selfmaps with cofinite domains and images of positive integers and integers are studied in [7] and [8], respectively. It was proved that the semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ have similar properties to the bicyclic semigroup: they are bisimple and every non-trivial homomorphic image $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is a group, and moreover the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{N})$ has $\mathbb{Z}(+)$ as a maximal group image and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ has $\mathbb{Z}(+)\times\mathbb{Z}(+)$, respectively.

In the paper [6] we studied the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ of monotone injective partial selfmaps of the set of $L_n \times_{\mathrm{lex}} \mathbb{Z}$ having cofinite domain and image, where $L_n \times_{\mathrm{lex}} \mathbb{Z}$ is the lexicographic product of n-elements chain and the set of integers with the usual linear order. In this paper we described Green's relations on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$, showed that the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is bisimple and established its projective congruences. Also, we proved that $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ is finitely generated, every automorphism of $\mathscr{IO}_\infty(\mathbb{Z})$ is inner and showed that in the case $n \geqslant 2$ the semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ has non-inner automorphisms. In [6] we also proved that for every positive integer n the quotient semigroup $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})/\sigma$, where σ is a least group congruence on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$, is isomorphic to the direct power $(\mathbb{Z}(+))^{2n}$. The structure of the sublattice of congruences on $\mathscr{IO}_\infty(\mathbb{Z}^n_{\mathrm{lex}})$ that are contained in the least group congruence is described in [4].

In the paper [5] we studied algebraic properties of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. We described properties of elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ as monotone partial bijection of $\mathbb{N}^2_\leqslant$ and showed that the group of units of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is isomorphic to the cyclic group of order two. Also in [5] the subsemigroup of idempotents of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and the Green relations on $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ are described. In particular, here we proved that $\mathscr{D} = \mathscr{J}$ in $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

The present paper is a continuation of [5]. We describe the natural partial order $\preccurlyeq$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and show that it coincides with the natural partial order which is induced from symmetric inverse monoid $\mathscr{I}_{\mathbb{N}\times\mathbb{N}}$ over the set $\mathbb{N}\times\mathbb{N}$ onto the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. We proved that the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is isomorphic to the semidirect product $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \rtimes \mathbb{Z}_2$ of the monoid $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ of orientation-preserving monotone injective partial selfmaps of $\mathbb{N}^2_\leqslant$ with cofinite domains and images by the cyclic group $\mathbb{Z}_2$ of the order two. Also we describe the congruence σ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, which is generated by the natural order $\preccurlyeq$ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$: $\alpha\sigma\beta$ if and only if α and β are comparable in $\big(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant), \preccurlyeq\big)$. We prove that the quotient semigroup $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)/\sigma$ is isomorphic to the free commutative monoid $\mathfrak{AM}_\omega$ over an infinite countable set and show that quotient semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma$ is isomorphic to the semidirect product of the free commutative monoid $\mathfrak{AM}_\omega$ by the group $\mathbb{Z}_2$.

The following proposition implies that the equations of the form $a \cdot x = b$ and $x \cdot c = d$ in the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ have finitely many solutions. This property holds for the bicyclic monoid, many its generalizations and other semigroups (see corresponding results in [1, 3, 6, 7, 8, 9]).

Proposition 1. *For every $\alpha, \beta \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, both sets*

$$\left\{\chi \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \mid \alpha \cdot \chi = \beta\right\} \qquad \textit{and} \qquad \left\{\chi \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \mid \chi \cdot \alpha = \beta\right\}$$

are finite. Consequently, every right translation and every left translation by an element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a finite-to-one map.

Proof. We consider the case of the equation $\alpha \cdot \chi = \beta$. In the case of the equation $\chi \cdot \alpha = \beta$ the proof is similar.

The definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and the equality $\alpha \cdot \chi = \beta$ imply that $\operatorname{dom} \beta \subseteq \operatorname{dom} \alpha$ and $\operatorname{ran} \chi \subseteq \operatorname{ran} \alpha$. Since any element of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ has a cofinite domain and a cofinite image in $\mathbb{N} \times \mathbb{N}$, we conclude that if an element χ_0 satisfies the equality $\alpha \cdot \chi = \beta$ then for every other root χ of the equation $\alpha \cdot \chi = \beta$ there exist finitely many $(i, j) \in (\mathbb{N} \times \mathbb{N}) \setminus \operatorname{ran} \beta$ such that one of the following conditions holds:

(1) $(i, j)\chi \neq (i, j)\chi_0$;
(2) $(i, j)\chi$ is determined and $(i, j)\chi_0$ is undetermined;
(3) $(i, j)\chi_0$ is determined and $(i, j)\chi$ is undetermined.

This implies that the equation $\alpha \cdot \chi = \beta$ has finitely many solutions, which completes the proof of the proposition. □

Later we shall describe the natural partial order "$\preccurlyeq$" on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. For $\alpha, \beta \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we put

$$\alpha \preccurlyeq \beta \quad \text{if and only if} \quad \alpha = \beta\varepsilon \quad \text{for some} \quad \varepsilon \in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)).$$

We need the following proposition from [11].

Proposition 2 ([11, p. 387, Corollary]). *For any semigroup S and its natural partial order $\preccurlyeq$ the following conditions are equivalent:*

(*i*) *$a \preccurlyeq b$;*
(*ii*) *$a = wb = bz$, $az = a$ for some $w, z \in S^1$;*
(*iii*) *$a = xb = by$, $xa = ay = a$ for some $x, y \in S^1$.*

Proposition 3. *The relation $\preccurlyeq$ is the natural partial order on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.*

Proof. Suppose that $\alpha = \beta\varepsilon$ for some idempotent $\varepsilon \in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$. Then we have that

$$\alpha\varepsilon = (\beta\varepsilon)\varepsilon = \beta(\varepsilon\varepsilon) = \beta\varepsilon = \alpha.$$

Let $\iota \colon \operatorname{dom}(\beta\varepsilon) \to \operatorname{dom}(\beta\varepsilon)$ be the identity map of the set $\operatorname{dom}(\beta\varepsilon)$. Then $\iota \in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ and the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ implies that $\operatorname{dom}(\beta\varepsilon) = \operatorname{dom}(\iota\beta)$, because ε is an idempotent of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. This implies that $(i, j)\iota\beta = (i, j)\beta\varepsilon$ for each $(i, j) \in \operatorname{dom}(\iota\beta)$ and hence we get that $\alpha = \beta\varepsilon = \iota\beta$. Next we apply Proposition 2. □

Remark 1. Proposition 3 implies that the natural partial order on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ coincides with the natural partial order which is induced from symmetric inverse monoid $\mathscr{I}_{\mathbb{N}\times\mathbb{N}}$ over the set $\mathbb{N}\times\mathbb{N}$ onto the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

We define a relation σ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ in the following way:

$$\alpha\sigma\beta \quad \text{if and only if there exists} \quad \varepsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)) \quad \text{such that} \quad \alpha\varepsilon=\beta\varepsilon,$$

for $\alpha,\beta\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

Proposition 4. *For $\alpha,\beta\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ the following conditions are equivalent:*

(*i*) *$\alpha\sigma\beta$;*
(*ii*) *there exist $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ such that $\alpha\varsigma=\beta\upsilon$;*
(*iii*) *there exist $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ such that $\alpha\varsigma=\upsilon\beta$;*
(*iv*) *there exists $\iota\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ such that $\iota\alpha=\iota\beta$;*
(*v*) *there exist $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ such that $\varsigma\alpha=\upsilon\beta$.*

Thus σ is a congruence on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

Proof. Implication $(i)\Rightarrow(ii)$ is trivial.

$(ii)\Rightarrow(i)$ If we have that $\alpha\varsigma=\beta\upsilon$ for some $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ then $\alpha\varsigma(\varsigma\upsilon)=\beta\upsilon(\varsigma\upsilon)$. Since $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a subsemigroup of the symmetric inverse monoid $\mathscr{I}_{|\mathbb{N}\times\mathbb{N}|}$, the idempotents in the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ commute and hence $\alpha(\varsigma\upsilon)=\beta(\varsigma\upsilon)$. This implies that $\alpha\sigma\beta$.

$(ii)\Rightarrow(iii)$ Suppose that $\alpha\varsigma=\beta\upsilon$ for some $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$. Let $\iota\colon \operatorname{dom}(\beta\upsilon)\to\operatorname{dom}(\beta\upsilon)$ be the identity map of the set $\operatorname{dom}(\beta\upsilon)$. Then $\iota\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ and the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ implies that $\operatorname{dom}(\beta\upsilon)=\operatorname{dom}(\iota\beta)$, because υ is an idempotent of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. This implies that $(i,j)\iota\beta=(i,j)\beta\upsilon$ for each $(i,j)\in\operatorname{dom}(\iota\beta)$ and hence we get that $\alpha\varsigma=\beta\upsilon=\iota\beta$.

$(iii)\Rightarrow(ii)$ Suppose that $\alpha\varsigma=\upsilon\beta$ for some $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$. Let $\iota\colon \operatorname{ran}(\upsilon\beta)\to\operatorname{ran}(\upsilon\beta)$ be the identity map of the set $\operatorname{ran}(\upsilon\beta)$. Then $\iota\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ and the definition of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ implies that $\operatorname{ran}(\upsilon\beta)=\operatorname{ran}(\beta\iota)$, because υ is an idempotent of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Since all elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ are partial bijections of $\mathbb{N}\times\mathbb{N}$ we get that $\operatorname{dom}(\upsilon\beta)=\operatorname{dom}(\beta\iota)$. This implies that $(i,j)\beta\iota=(i,j)\upsilon\beta$ for each $(i,j)\in\operatorname{dom}(\beta\iota)$ and hence we get that $\alpha\varsigma=\upsilon\beta=\beta\iota$.

The proofs of equivalences $(iii)\Leftrightarrow(iv)$ and $(iv)\Leftrightarrow(v)$ are similar.

It is obvious that σ is a reflexive and symmetric relation on $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Suppose that $\alpha\sigma\beta$ and $\beta\sigma\gamma$ in $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then there exist $\varsigma,\upsilon\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ such that $\alpha\varsigma=\beta\varsigma$ and $\beta\upsilon=\gamma\upsilon$. This implies that $\alpha\varsigma\upsilon=\beta\varsigma\upsilon$ and $\beta\upsilon\varsigma=\gamma\upsilon\varsigma$, and since the idempotents in $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ commute we get that $\alpha\varsigma\upsilon=\beta\varsigma\upsilon=\beta\upsilon\varsigma=\gamma\upsilon\varsigma$, and hence $\alpha\sigma\gamma$.

Suppose that $\alpha\sigma\beta$ for some $\alpha,\beta\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then by (iv) there exists $\iota\in E(\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant))$ such that $\iota\alpha=\iota\beta$. This implies that $\iota\alpha\gamma=\iota\beta\gamma$ for each $\gamma\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ and hence by item (iv) we get that $(\alpha\gamma)\sigma(\beta\gamma)$. The proof of the statement that $(\gamma\alpha)\sigma(\gamma\beta)$ for each $\gamma\in\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is similar, and hence σ is a congruence on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. □

Corollary 1. *For $\alpha, \beta \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ the following condition are equivalent:*

(*i*) $\alpha\sigma\beta$;
(*ii*) $\alpha\varpi\sigma\beta\varpi$;
(*iii*) $\varpi\alpha\sigma\varpi\beta$.

Proof. (*i*) $\Leftrightarrow$ (*ii*) If $\alpha\sigma\beta$ in $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ then by Proposition 4 there exists $\iota \in E(\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}))$ such that $\iota\alpha = \iota\beta$. This implies that $\iota\alpha\varpi = \iota\beta\varpi$ and hence $(\alpha\varpi)\sigma(\beta\varpi)$. Conversely, if $(\alpha\varpi)\sigma(\beta\varpi)$ then by Proposition 4 we have that $\nu\alpha\varpi = \nu\beta\varpi$ for some $\nu \in E(\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}))$, and hence $\nu\alpha = \nu\alpha\varpi\varpi = \nu\beta\varpi\varpi = \nu\beta$, which implies that $\alpha\sigma\beta$.

The proof of (*i*) $\Leftrightarrow$ (*ii*) is similar. □

Also the definition of the congruence σ on the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ implies the following simple property of σ-equivalent elements of $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$:

Corollary 2. *Let α, β be elements of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ such that $\alpha\sigma\beta$. Then the following assertions hold:*

(*i*) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1$ *if and only if* $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{H}^1$;
(*ii*) $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{V}^1$ *if and only if* $(\mathsf{H}^1_{\operatorname{dom}\beta})\beta \subseteq \mathsf{V}^1$.

We define

$$\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant}) = \left\{\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}) \colon (\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha \subseteq \mathsf{H}^1\right\}.$$

Then Lemma 3 and Theorem 1 from [5] imply that $\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$ is a subsemigroup of $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. The subsemigroup $\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$ is called the *monoid of orientation-preserving monotone injective partial selfmaps of* $\mathbb{N}^2_{\leqslant}$ with cofinite domains and images. Moreover it is obvious that $E(\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant}) = E(\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}))$. Also, later by $\preccurlyeq$ and σ we denote the corresponding induced relations of the relations $\preccurlyeq$ and σ from the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ onto its subsemigroup $\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$.

The proofs of the following propositions are similar to those of Propositions 3 and 4, respectively.

Proposition 5. *The relation $\preccurlyeq$ is the natural partial order on the semigroup $\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$.*

Proposition 6. *The relation σ is a congruence on the semigroup $\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$.*

By ϖ we denote the bijective transformation of $\mathbb{N} \times \mathbb{N}$ defined by the formula $(i, j)\varpi = (j, i)$, for any $(i, j) \in \mathbb{N} \times \mathbb{N}$. It is obvious that ϖ is an element of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ and $\varpi\varpi = \mathbb{I}$.

Remark 2. We observe that

(*i*) $\alpha \in \mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$ if and only if $\alpha\varpi, \varpi\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}) \setminus \mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$;
(*ii*) $\alpha \in \mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$ if and only if $\varpi\alpha\varpi \in \mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$.

We define a map $\mathfrak{h} \colon \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}) \to \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ by the formula $(\alpha)\mathfrak{h} = \varpi\alpha\varpi$, for $\alpha \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$.

Proposition 7. *The map $\mathfrak{h} \colon \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}) \to \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ is an automorphism of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. Moreover its restriction $\mathfrak{h}|_{\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})} \colon \mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant}) \to \mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$ is an automorphism of the subsemigroup $\mathscr{PO}^+_{\infty}(\mathbb{N}^2_{\leqslant})$.*

Proof. First we show that $\mathfrak{h}\colon \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a homomorphism. Fix arbitrary $\alpha, \beta \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then we have that

$$(\alpha\beta)\mathfrak{h} = \varpi(\alpha\beta)\varpi = \varpi(\alpha\mathbb{I}\beta)\varpi = \varpi(\alpha\varpi\varpi\beta)\varpi = (\varpi\alpha\varpi)(\varpi\beta\varpi) = (\alpha)\mathfrak{h}(\beta)\mathfrak{h},$$

and hence $\mathfrak{h}\colon \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a homomorphism.

Fix an arbitrary $\alpha \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then the definition of $\mathfrak{h}$ implies that

$$(\varpi\alpha\varpi)\mathfrak{h} = \varpi\varpi\alpha\varpi\varpi = \mathbb{I}\alpha\mathbb{I} = \alpha,$$

and hence the map $\mathfrak{h}\colon \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is surjective. Suppose that $(\alpha)\mathfrak{h} = (\beta)\mathfrak{h}$ for some $\alpha, \beta \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$. Then

$$\alpha = \mathbb{I}\alpha\mathbb{I} = \varpi\varpi\alpha\varpi\varpi = ((\alpha)\mathfrak{h})\mathfrak{h} = ((\beta)\mathfrak{h})\mathfrak{h} = \varpi\varpi\beta\varpi\varpi = \mathbb{I}\beta\mathbb{I} = \beta,$$

and hence the map $\mathfrak{h}\colon \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is injective. Thus the map $\mathfrak{h}$ is an automorphism of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$.

Now, Remark 2 implies that the restriction $\mathfrak{h}|_{\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)}\colon \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ is an automorphism of the semigroup $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$, too. □

For the automorphism $\mathfrak{h}\colon \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ of the semigroup $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ we have that $\mathfrak{h}^2 = \mathsf{Id}_{\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)}$ is the identity automorphism of $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$. This implies that the element $\mathfrak{h}$ generates the group which is isomorphic to the cyclic group of order two $\mathbb{Z}_2$. By Proposition 4 from [5] the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is isomorphic to $\mathbb{Z}_2$. We define a map $\mathfrak{Q}$ from $H(\mathbb{I})$ into the group $\mathrm{Aut}(\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant))$ of automorphisms of the semigroup $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ in the following way $(\mathbb{I})\mathfrak{Q} = \mathsf{Id}_{\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)}$ and $(\varpi)\mathfrak{Q} = \mathfrak{h}$. It is obvious that so defined map $\mathfrak{Q}\colon H(\mathbb{I}) \to \mathrm{Aut}(\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant))$ is an injective homomorphism.

Let S and T be semigroups and let $\mathfrak{H}$ be a homomorphism from T into the semigroup of endomorphisms $\mathsf{End}(S)$ of S, $\mathfrak{H}\colon t \mapsto \mathfrak{h}_t$. Then the Cartesian product $S \times T$ with the following semigroup operation

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 \cdot (s_2)\mathfrak{h}_{t_1}, t_1 \cdot t_2), \qquad s_1, s_2 \in S,\ t_1, t_2 \in T,$$

is called a *semidirect product* of the semigroup S by T and is denoted by $S \rtimes_{\mathfrak{H}} T$. We remark that if 1_T is the unit of the semigroup T then $(1_T)\mathfrak{H} = \mathfrak{h}_{1_T}$ is the identity homomorphism of S and in the case when T is a group then $(t)\mathfrak{H} = \mathfrak{h}_t$ is an automorphism of S for any $t \in T$.

Theorem 1. *The semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is isomorphic to the semidirect product $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \rtimes_{\mathfrak{Q}} H(\mathbb{I})$ of the semigroup $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ by the group $H(\mathbb{I})$.*

Proof. We define a map $\mathfrak{I}\colon \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \rtimes_{\mathfrak{Q}} H(\mathbb{I}) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ by the formula $(\alpha, g)\mathfrak{I} = \alpha g$. Then for all $\alpha_1, \alpha_2 \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ and $g_1, g_2 \in H(\mathbb{I})$ we have that

$$\begin{aligned}((\alpha_1, g_1) \cdot (\alpha_2, g_2))\,\mathfrak{I} &= (\alpha_1 \cdot (\alpha_2)(g_1)\mathfrak{Q}, g_1 \cdot g_2)\,\mathfrak{I} = (\alpha_1 \cdot g_1 \cdot \alpha_2 \cdot g_1, g_1 \cdot g_2)\,\mathfrak{I} = \\ &= \alpha_1 \cdot g_1 \cdot \alpha_2 \cdot g_1 \cdot g_1 \cdot g_2 = \alpha_1 \cdot g_1 \cdot \alpha_2 \cdot g_2 = \\ &= (\alpha_1, g_1)\mathfrak{I} \cdot (\alpha_2, g_2)\mathfrak{I},\end{aligned}$$

because $g^2 = \mathbb{I}$ for any $g \in H(\mathbb{I})$, and hence the map $\mathfrak{I}\colon \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \rtimes_{\mathfrak{Q}} H(\mathbb{I}) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ is a homomorphism.

By Lemma 3 from [5] for every $\alpha\in\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ there exist $\alpha^{+}\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ and $g_\alpha\in H(\mathbb{I})$ such that $\alpha=\alpha^{+}g_\alpha$. Indeed,

(a) in the case when $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\subseteq\mathsf{H}^1$ we put $\alpha^{+}=\alpha$ and $g_\alpha=\mathbb{I}$;

(b) in the case when $(\mathsf{H}^1_{\operatorname{dom}\alpha})\alpha\subseteq\mathsf{V}^1$ we put $\alpha^{+}=\alpha\omega$ and $g_\alpha=\omega$.

Let $\alpha^{+},\beta^{+}\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ and $g_\alpha,g_\beta\in H(\mathbb{I})$ be such that $\alpha^{+}g_\alpha=(\alpha^{+},g_\alpha)\mathfrak{I}=(\beta^{+},g_\beta)\mathfrak{I}=\beta^{+}g_\beta$. Since $(\mathsf{H}^1_{\operatorname{dom}\alpha^{+}})\alpha^{+}\subseteq\mathsf{H}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\beta^{+}})\beta^{+}\subseteq\mathsf{H}^1$, Lemma 3 from [5] implies that $g_\alpha=g_\beta$. By Proposition 4 from [5] the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ is isomorphic to $\mathbb{Z}_2$ and hence $\alpha^{+}=\alpha^{+}g_\alpha^2=\alpha^{+}g_\alpha g_\beta=\beta^{+}g_\beta^2=\beta^{+}$. Therefore, we get that so defined map $\mathfrak{I}\colon\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})\rtimes_{\mathfrak{Q}}H(\mathbb{I})\to\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ is an isomorphism. □

By Theorem $2(ii_1)$ from [5] for every $\alpha\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ there exists a smallest positive integer n_α such that $(i,j)\alpha=(i,j)$ for each $(i,j)\in\operatorname{dom}\alpha\cap{\uparrow}(n_\alpha,n_\alpha)$.

Lemma 1. *For every $\alpha\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ there exists $\alpha_{\mathfrak{f}}\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ such that the following assertions hold:*

(i) $\alpha\sigma\alpha_{\mathfrak{f}}$;

(ii) $(i+1)_{\alpha_{\mathfrak{f}}[*,j]}-(i+1)=i_{\alpha_{\mathfrak{f}}[*,j]}-i$ *for arbitrary* $(i,j)\in\operatorname{dom}\alpha_{\mathfrak{f}}$ *with* $j<n_\alpha$, $(i,j)\alpha_{\mathfrak{f}}=\big(i_{\alpha_{\mathfrak{f}}[*,j]},j_{\alpha_{\mathfrak{f}}[i,*]}\big)$ *and* $(i+1,j)\alpha_{\mathfrak{f}}=\big((i+1)_{\alpha_{\mathfrak{f}}[*,j]},j_{\alpha_{\mathfrak{f}}[i+1,*]}\big)$, *i.e.,* $\alpha_{\mathfrak{f}}$ *acts as a partial shift on the set* H^j;

(iii) $(j+1)_{\alpha_{\mathfrak{f}}[i,*]}-(j+1)=j_{\alpha_{\mathfrak{f}}[i,*]}-j$ *for arbitrary* $(i,j)\in\operatorname{dom}\alpha_{\mathfrak{f}}$ *with* $i<n_\alpha$, $(i,j)\alpha_{\mathfrak{f}}=\big(i_{\alpha_{\mathfrak{f}}[*,j]},j_{\alpha_{\mathfrak{f}}[i,*]}\big)$ *and* $(i,j+1)\alpha_{\mathfrak{f}}=\big(i_{\alpha_{\mathfrak{f}}[*,j+1]},(j+1)_{\alpha_{\mathfrak{f}}[i,*]}\big)$, *i.e.,* $\alpha_{\mathfrak{f}}$ *acts as a partial shift on the set* V^i.

Moreover, there exist smallest positive integers $\widehat{h}_\alpha,\widehat{v}_\alpha\leqslant n_\alpha$ *such that* $(i,j)\alpha_{\mathfrak{f}}=(i,j)$ *for arbitrary* $(i,j)\in\operatorname{dom}\alpha_{\mathfrak{f}}$ *with* $i\geqslant\widehat{h}_\alpha$ *and* $(k,l)\alpha_{\mathfrak{f}}=(k,l)$ *for arbitrary* $(k,l)\in\operatorname{dom}\alpha_{\mathfrak{f}}$ *with* $l\geqslant\widehat{v}_\alpha$.

Proof. Fix an arbitrary element α of the semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$. Then by Theorem 1(1) from [5] we get that $(\mathsf{H}^n_{\operatorname{dom}\alpha})\alpha\subseteq^{*}\mathsf{H}^n$ and $(\mathsf{V}^n_{\operatorname{dom}\alpha})\alpha\subseteq^{*}\mathsf{V}^n$ for any positive integer n. Also, the definition of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ and Theorem $2(ii_1)$ of [5] imply that there exists a smallest positive integer n_α such that $(i,j)\alpha=(i,j)$ for each $(i,j)\in\operatorname{dom}\alpha\cap{\uparrow}(n_\alpha,n_\alpha)$, and hence for arbitrary positive integers $i,j<n_\alpha$ there exist smallest positive integers h^i_α and v^j_α such that the following conditions hold:

$$\mathsf{H}^i_{\operatorname{ran}\alpha}\cap\big\{(p,i)\colon p\geqslant h^i_\alpha\big\}=\big\{(p,i)\colon p\geqslant h^i_\alpha\big\};$$
$$\mathsf{V}^j_{\operatorname{ran}\alpha}\cap\big\{(j,q)\colon q\geqslant v^j_\alpha\big\}=\big\{(j,q)\colon q\geqslant v^j_\alpha\big\},$$

and

$$(k,i),(j,l)\in\operatorname{dom}\alpha,\qquad (k,i)\alpha\in\mathsf{H}^i,\qquad (j,l)\alpha\in\mathsf{V}^j,$$

for all positive integers $k\geqslant h^i_\alpha$ and $l\geqslant v^j_\alpha$.

We put

$$\bar{h}_\alpha=\max\big\{h^i_\alpha\colon i=1,\ldots,n_\alpha-1\big\}\qquad\text{and}\qquad \bar{v}_\alpha=\max\big\{v^j_\alpha\colon j=1,\ldots,n_\alpha-1\big\}.$$

The above arguments imply that

$$\mathsf{H}^i_{\operatorname{ran}\alpha}\cap\big\{(p,i)\colon p\geqslant\bar{h}_\alpha\big\}=\big\{(p,i)\colon p\geqslant\bar{h}_\alpha\big\};\tag{1}$$

$$\mathsf{V}^{j}_{\operatorname{ran}\alpha}\cap\{(j,q)\colon q\geqslant\bar v_\alpha\}=\{(j,q)\colon q\geqslant\bar v_\alpha\}\,, \qquad (2)$$

and

$$(k,i),(j,l)\in\operatorname{dom}\alpha,\qquad (k,i)\alpha\in\mathsf{H}^i,\qquad (j,l)\alpha\in\mathsf{V}^j,$$

for all positive integers $k\geqslant\bar h_\alpha$ and $l\geqslant\bar v_\alpha$.

Next we put

$$D_\alpha=(\mathbb{N}\times\mathbb{N})\setminus\big(\{(i,j)\colon i\leqslant\bar h_\alpha \text{ and } j\leqslant n_\alpha\}\cup\{(i,j)\colon i\leqslant n_\alpha \text{ and } j\leqslant\bar v_\alpha\}\big). \qquad (3)$$

We define $\alpha_{\mathfrak{f}}=\alpha|_{D_\alpha}$, i.e.,

$$\operatorname{dom}\alpha_{\mathfrak{f}}=D_\alpha,\qquad \operatorname{ran}\alpha_{\mathfrak{f}}=(D_\alpha)\alpha \qquad\text{and}\qquad (i,j)\alpha_{\mathfrak{f}}=(i,j)\alpha \quad\text{for all}\quad (i,j)\in\operatorname{dom}\alpha_{\mathfrak{f}}.$$

Since $\alpha_{\mathfrak{f}}=\varepsilon_\alpha\alpha_{\mathfrak{f}}=\varepsilon_\alpha\alpha$ for the identity partial map $\varepsilon_\alpha\colon\mathbb{N}\times\mathbb{N}\rightharpoonup\mathbb{N}\times\mathbb{N}$ with $\operatorname{dom}\varepsilon_\alpha=\operatorname{ran}\varepsilon_\alpha=D_\alpha$, Proposition 4 implies that $\alpha\sigma\alpha_{\mathfrak{f}}$.

Then condition (1) and the definition of the positive integer $\bar h_\alpha$ imply that

$$\big(\bar h_\alpha+2\big)_{\alpha_{\mathfrak{f}}[*,1]}=\big(\bar h_\alpha+1\big)_{\alpha_{\mathfrak{f}}[*,1]}+1,$$

and by similar arguments and induction we have that $(i+1)_{\alpha_{\mathfrak{f}}[*,1]}=(i,1)_{\alpha_{\mathfrak{f}}[*,1]}+1$ for arbitrary $i\geqslant\bar h_\alpha+1$. Next, if we apply condition (1) and induction for arbitrary $j<n_\alpha$ then we get that $(i+1)_{\alpha_{\mathfrak{f}}[*,j]}=(i)_{\alpha_{\mathfrak{f}}[*,j]}+1$ for arbitrary $i\geqslant\bar h_\alpha+1$. This implies assertion (ii).

The proof of item (iii) is similar to (ii).

The last statement of the lemma follows from the above arguments and Theorem 2(1) from [5]. □

For every positive integer n we define partial maps $\gamma_n\colon\mathbb{N}\times\mathbb{N}\rightharpoonup\mathbb{N}\times\mathbb{N}$ and $\upsilon_n\colon\mathbb{N}\times\mathbb{N}\rightharpoonup\mathbb{N}\times\mathbb{N}$ in the following way:

$$\begin{aligned}&\operatorname{dom}\gamma_n=\mathbb{N}\times\mathbb{N}\setminus\{(1,i)\colon i=1,\ldots,n\}\,,\\&\operatorname{dom}\upsilon_n=\mathbb{N}\times\mathbb{N}\setminus\{(i,1)\colon i=1,\ldots,n\}\,,\\&\operatorname{ran}\gamma_n=\operatorname{ran}\upsilon_n=\mathbb{N}\times\mathbb{N}\end{aligned}$$

and

$$(i,j)\gamma_n=\begin{cases}(i-1,j), & \text{if } j\leqslant n;\\ (i,j), & \text{if } j>n\end{cases}\qquad\text{for}\quad (i,j)\in\operatorname{dom}\gamma_n,$$

$$(i,j)\upsilon_n=\begin{cases}(i,j-1), & \text{if } i\leqslant n;\\ (i,j), & \text{if } i>n\end{cases}\qquad\text{for}\quad (i,j)\in\operatorname{dom}\upsilon_n.$$

Simple verifications show that $\gamma_n,\upsilon_n\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ for every positive integer n, and moreover the subsemigroups $\langle\gamma_k\mid k\in\mathbb{N}\rangle$ and $\langle\upsilon_k\mid k\in\mathbb{N}\rangle$ of the semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$, generated by the sets $\{\gamma_k\colon k\in\mathbb{N}\}$ and $\{\upsilon_k\colon k\in\mathbb{N}\}$, respectively, are isomorphic to the free Abelian semigroup over an infinite countable set.

Lemma 2. *For every $\alpha\in\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ there exist finitely many elements $\gamma_{k_1},\ldots,\gamma_{k_i}$ and $\upsilon_{l_1},\ldots,\upsilon_{l_j}$ of the semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$, with $k_1<\ldots<k_i$, $l_1<\ldots<l_j$, such that*

$$\alpha\sigma\big(\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j}\big), \qquad (4)$$

for some positive integers $p_1,\ldots,p_i,q_1,\ldots,q_j$. Moreover if

$$\alpha\sigma\big(\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j}\big)\qquad\textit{and}\qquad\beta\sigma\big(\gamma_{a_1}^{b_1}\ldots\gamma_{a_i}^{b_i}\upsilon_{c_1}^{d_1}\ldots\upsilon_{c_j}^{d_j}\big)$$

for some $\alpha,\beta\in\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$ *then* $(\alpha,\beta)\notin\sigma$ *if and only if*

$$\iota\gamma_{k_1}^{p_1}\dots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\dots\upsilon_{l_j}^{q_j}\neq\iota\gamma_{a_1}^{b_1}\dots\gamma_{a_i}^{b_i}\upsilon_{c_1}^{d_1}\dots\upsilon_{c_j}^{d_j}$$

for any idempotent $\iota\in\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$.

Proof. Fix an arbitrary element α of the semigroup $\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$. Let $\alpha_{\mathfrak{f}}$ be the element of $\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$ defined in the proof of Lemma 1. By Theorem 3 from [5] and the second statement of Lemma 1 there exist smallest positive integers $\widehat{h}_\alpha,\widehat{v}_\alpha\leqslant n_\alpha$ such that $(i,j)\alpha_{\mathfrak{f}}=(i,j)$ for arbitrary $(i,j)\in\operatorname{dom}\alpha_{\mathfrak{f}}$ with $i\geqslant\widehat{h}_\alpha$ and $(k,l)\alpha_{\mathfrak{f}}=(k,l)$ for arbitrary $(k,l)\in\operatorname{dom}\alpha_{\mathfrak{f}}$ with $l\geqslant\widehat{v}_\alpha$.

By Lemma 1 and Theorem 1(1) of [5] we have that

$$\big(j,\widehat{h}_\alpha-1\big)\alpha_{\mathfrak{f}}=\big(j_{\alpha_{\mathfrak{f}}[*,\widehat{h}_\alpha-1]},\widehat{h}_\alpha-1\big)<(j,\widehat{h}_\alpha-1)\quad\text{and}\quad(j+1)_{\alpha_{\mathfrak{f}}[*,\widehat{h}_\alpha-1]}-j_{\alpha_{\mathfrak{f}}[*,\widehat{h}_\alpha-1]}=1,$$

for arbitrary $\big(j,\widehat{h}_\alpha-1\big),\big(j+1,\widehat{h}_\alpha-1\big)\in\operatorname{dom}\alpha_{\mathfrak{f}}$. Then we put $p_{\widehat{h}_\alpha-1}=j-j_{\alpha_{\mathfrak{f}}[*,\widehat{h}_\alpha-1]}$. Next, for $s=2,\dots,\widehat{h}_\alpha-2$ we define integers $p_{\widehat{h}_\alpha-s},\dots,p_1$ by induction,

$$p_{\widehat{h}_\alpha-s}=j-j_{\alpha_{\mathfrak{f}}[*,\widehat{h}_\alpha-s]}-\Big(p_{\widehat{h}_\alpha-1}+\dots+p_{\widehat{h}_\alpha-s+1}\Big),$$

where $\big(j,\widehat{h}_\alpha-s\big)\alpha_{\mathfrak{f}}=\big(j_{\alpha_{\mathfrak{f}}[*,\widehat{h}_\alpha-s]},\widehat{h}_\alpha-s\big)\leqslant(j,\widehat{h}_\alpha-s)$ for arbitrary $\big(j,\widehat{h}_\alpha-s\big)\in\operatorname{dom}\alpha_{\mathfrak{f}}$.

Similarly, by Lemma 1 and Theorem 1(1) of [5] we have that

$$\big(\widehat{v}_\alpha-1,i\big)\alpha_{\mathfrak{f}}=\big(\widehat{v}_\alpha-1,i_{\alpha_{\mathfrak{f}}[\widehat{v}_\alpha-1,*]}\big)<(\widehat{v}_\alpha-1,i)\quad\text{and}\quad(i+1)_{\alpha_{\mathfrak{f}}[\widehat{v}_\alpha-1,*]}-i_{\alpha_{\mathfrak{f}}[\widehat{v}_\alpha-1,*]}=1,$$

for arbitrary $\big(\widehat{v}_\alpha-1,i\big),\big(\widehat{v}_\alpha-1,i+1\big)\in\operatorname{dom}\alpha_{\mathfrak{f}}$. Then we put $q_{\widehat{v}_\alpha-1}=i-i_{\alpha_{\mathfrak{f}}[\widehat{v}_\alpha-1,*]}$. Next, for $t=2,\dots,\widehat{v}_\alpha-2$ we define integers $q_{\widehat{v}_\alpha-t},\dots,q_1$ by induction

$$q_{\widehat{v}_\alpha-t}=i-i_{\alpha_{\mathfrak{f}}[\widehat{v}_\alpha-t,*]}-(q_{\widehat{v}_\alpha-1}+\dots+q_{\widehat{v}_\alpha-t+1}),$$

where $\big(\widehat{v}_\alpha-t,i\big)\alpha_{\mathfrak{f}}=\big(\widehat{v}_\alpha-t,i_{\alpha_{\mathfrak{f}}[\widehat{v}_\alpha-t,*]}\big)\leqslant\big(\widehat{v}_\alpha-t,i\big)$ for arbitrary $\big(\widehat{v}_\alpha-t,i\big)\in\operatorname{dom}\alpha_{\mathfrak{f}}$.

For any $\alpha\in\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$ put $\varepsilon_\alpha\colon\mathbb{N}\times\mathbb{N}$ be the identity partial map with $\operatorname{dom}\varepsilon_\alpha=\operatorname{ran}\varepsilon_\alpha=D_\alpha$, where the set D_α is defined by formula (3). Simple verification shows that $\varepsilon_\alpha\alpha=\varepsilon_\alpha\big(\gamma_1^{p_1}\dots\gamma_{\widehat{h}_\alpha-1}^{p_{\widehat{h}_\alpha-1}}\upsilon_1^{q_1}\dots\upsilon_{l_j}^{q_{\widehat{v}_\alpha-1}}\big)$ and hence

$$\alpha\sigma\big(\gamma_1^{p_1}\dots\gamma_{\widehat{h}_\alpha-1}^{p_{\widehat{h}_\alpha-1}}\upsilon_1^{q_1}\dots\upsilon_{l_j}^{q_{\widehat{v}_\alpha-1}}\big),$$

which implies that relation (4) holds.

Since $\gamma_m^0=\upsilon_m^0=\mathbb{I}$ for any positive integer m, without loss of generality we may assume that $p_1,\dots,p_i,q_1,\dots,q_j$ are positive integers in formula (4).

Also, the last statement of the lemma follows from the definition of the congruence σ on the semigroup $\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$. □

Lemma 3. *Let be* $\alpha\sigma\big(\gamma_{k_1}^{p_1}\dots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\dots\upsilon_{l_j}^{q_j}\big)$ *for* $\alpha\in\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$ *and positive integers* $p_1,\dots,p_i$, $q_1,\dots,q_j$, $k_1<\dots<k_i$, $l_1<\dots<l_j$. *Then there exists an idempotent* $\widehat{\varepsilon}_\alpha\in\mathscr{PO}_{\infty}^{+}(\mathbb{N}^2_{\leqslant})$ *such that*

$$\widehat{\varepsilon}_\alpha\alpha=\widehat{\varepsilon}_\alpha\gamma_{k_1}^{p_1}\dots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\dots\upsilon_{l_j}^{q_j}=\widehat{\varepsilon}_\alpha\upsilon_{l_1}^{q_1}\dots\upsilon_{l_j}^{q_j}\gamma_{k_1}^{p_1}\dots\gamma_{k_i}^{p_i}.$$

Proof. Put

$$\overline{m}_\alpha = n_\alpha + \bar{h}_\alpha + \bar{v}_\alpha + p_1 + \ldots + p_i + q_1 + \ldots + q_j,$$

where $\bar{h}_\alpha$ and $\bar{v}_\alpha$ are the positive integers defined in the proof of Lemma 1. We define the identity partial map $\widehat{\varepsilon}_\alpha \colon \mathbb{N}\times\mathbb{N} \rightharpoonup \mathbb{N}\times\mathbb{N}$ with $\operatorname{dom}\widehat{\varepsilon}_\alpha = \operatorname{ran}\widehat{\varepsilon}_\alpha = M_\alpha$, where

$$M_\alpha = (\mathbb{N}\times\mathbb{N}) \setminus (\{(i,j) \colon i \leqslant \overline{m}_\alpha \text{ and } j \leqslant \overline{m}_\alpha\}).$$

Then $\widehat{\varepsilon}_\alpha \preccurlyeq \varepsilon_\alpha$ where ε_α is the idempotent of the semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ defined in the proof of Lemma 1. This implies that

$$\widehat{\varepsilon}_\alpha\alpha = \widehat{\varepsilon}_\alpha\varepsilon_\alpha\alpha = \widehat{\varepsilon}_\alpha\varepsilon_\alpha\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j} = \widehat{\varepsilon}_\alpha\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j},$$

and the equlity

$$\widehat{\varepsilon}_\alpha\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j} = \widehat{\varepsilon}_\alpha\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j}\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}$$

follows from the definition of the idempotent $\widehat{\varepsilon}_\alpha \in \mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$. □

The following theorem describes the quotient semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$.

Theorem 2. *The quotient semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ is isomorphic to the free commutative monoid $\mathfrak{AM}_\omega$ over an infinite countable set.*

Proof. Let $X = \{a_i \colon i \in \mathbb{N}\} \cup \{b_j \colon j \in \mathbb{N}\}$ be a countable infinite set.

We define the map $\mathfrak{H}_\sigma \colon \mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant}) \to \mathfrak{AM}_X$ in the following way:

(*a*) if $\alpha\sigma\big(\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j}\big)$ for some positive integers $p_1,\ldots,p_i$, $q_1,\ldots,q_j$, $k_1 < \ldots < k_i$, $l_1 < \ldots < l_j$, then

$$(\alpha)\mathfrak{H}_\sigma = \big(\gamma_{k_1}^{p_1}\ldots\gamma_{k_i}^{p_i}\upsilon_{l_1}^{q_1}\ldots\upsilon_{l_j}^{q_j}\big)\mathfrak{H}_\sigma = a_{k_1}^{p_1}\ldots a_{k_i}^{p_i}b_{l_1}^{q_1}\ldots b_{l_j}^{q_j};$$

(*b*) $(\mathbb{I})\mathfrak{H}_\sigma = e$, where e is the unit of the free commutative monoid $\mathfrak{AM}_X$.

Then Lemmas 2 and 3 imply that $(\alpha)\mathfrak{H}_\sigma = (\beta)\mathfrak{H}_\sigma$ if and only if $\alpha\sigma\beta$ in $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ and hence the quotient semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ is isomorphic to the free commutative monoid $\mathfrak{AM}_X$. □

The following corollary of Theorem 2 shows that the semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ has infinitely many congruences similar as the free commutative monoid $\mathfrak{AM}_\omega$ over an infinite countable set.

Corollary 3. *Every countable (infinite or finite) commutative monoid is a homomorphic image of the semigroup $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$.*

Its obvious that every non-unit element u of the free commutative monoid $\mathfrak{AM}_\omega$ over the infinite countable set $\{a_i \colon i \in \omega\} \cup \{b_j \colon j \in \omega\}$ can be represented in the form $u = a_1^{i_1}\ldots a_k^{i_k}b_1^{j_1}\ldots b_l^{j_l}$, where $i_1,\ldots,i_k,j_1,\ldots,l_l$ are positive integers. We define a map $\mathfrak{f}\colon \mathfrak{AM}_\omega \to \mathfrak{AM}_\omega$ by the formula

$$(a_1^{i_1}\ldots a_k^{i_k}b_1^{j_1}\ldots b_l^{j_l})\mathfrak{f} = a_1^{j_1}\ldots a_l^{j_l}b_1^{i_1}\ldots b_k^{i_k}, \tag{5}$$

for $u = a_1^{i_1}\ldots a_k^{i_k}b_1^{j_1}\ldots b_l^{j_l} \in \mathfrak{AM}_\omega$ and $(e)\mathfrak{f} = e$, for unit element e of $\mathfrak{AM}_\omega$.

Proposition 8. *The map $\mathfrak{f}\colon \mathfrak{AM}_\omega \to \mathfrak{AM}_\omega$ is an automorphism of the free commutative monoid $\mathfrak{AM}_\omega$.*

Proof. First we show that $\mathfrak{f}\colon \mathfrak{AM}_\omega \to \mathfrak{AM}_\omega$ is a homomorphism. Fix arbitrary elements $u, v \in \mathfrak{AM}_\omega$. Without loss of generality we may assume that

$$u = a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p} \qquad \text{and} \qquad v = a_1^{s_1} \ldots a_p^{s_p} b_1^{t_1} \ldots b_p^{t_p}$$

for some non-negative integers $p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p$, where $a^i = b^i = e$ for $i = 0$.

Then we have that

$$\begin{aligned}(uv)\mathfrak{f} &= (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p} a_1^{s_1} \ldots a_p^{s_p} b_1^{t_1} \ldots b_p^{t_p})\mathfrak{f} = \\ &= (a_1^{i_1+s_1} \ldots a_p^{i_p+s_p} b_1^{j_1+t_1} \ldots b_p^{j_p+t_p})\mathfrak{f} = \\ &= a_1^{j_1+t_1} \ldots a_p^{j_p+t_p} b_1^{i_1+s_1} \ldots b_p^{i_p+s_p} = \\ &= a_1^{j_1} \ldots a_p^{j_p} b_1^{i_1} \ldots b_p^{i_p} a_1^{t_1} \ldots a_p^{t_p} b_1^{s_1} \ldots b_p^{s_p} = \\ &= (a_1^{i_1} \ldots a_p^{i_p} b_p^{j_1} \ldots b_l^{j_p})\mathfrak{h}(a_1^{s_1} \ldots a_p^{s_p} b_1^{t_1} \ldots b_p^{t_p})\mathfrak{f} = \\ &= (u)\mathfrak{f}(v)\mathfrak{f}.\end{aligned}$$

It is obvious that $\mathfrak{f}\colon \mathfrak{AM}_\omega \to \mathfrak{AM}_\omega$ is a bijective map and hence $\mathfrak{f}\colon \mathfrak{AM}_\omega \to \mathfrak{AM}_\omega$ is an automorphism. □

The relationships between elements of the subsemigroup $\langle \gamma_k \mid k \in \mathbb{N} \rangle$ and of the subsemigroup $\langle \upsilon_k \mid k \in \mathbb{N} \rangle$ in $\mathscr{PO}_\infty^{+}(\mathbb{N}_{\leqslant}^2)$ is described by the following proposition.

We observe that the cyclic group $\mathbb{Z}_2$ acts on the free commutative monoid $\mathfrak{AM}_\omega$ over the infinite countable set $\{a_i \colon i \in \omega\} \cup \{b_j \colon j \in \omega\}$ in the following way

$$\mathfrak{AM}_\omega \times \mathbb{Z}_2 \to \mathfrak{AM}_\omega \colon (u, g) \mapsto v = \begin{cases} u, & \text{if } g = \bar{0}; \\ (u)\mathfrak{f}, & \text{if } g = \bar{1}, \end{cases}$$

where the map $\mathfrak{f}\colon \mathfrak{AM}_\omega \to \mathfrak{AM}_\omega$ is defined by formula(5). By Proposition 8 the map $\mathfrak{f}$ is an automorphism of the free commutative monoid $\mathfrak{AM}_\omega$.

Proposition 9. *Let $p_1, \ldots, p_i$, $k_1, \ldots, k_i$ be some positive integers such that $k_1 < \ldots < k_i$. Then the following assertions hold:*

(*i*) $\varpi \gamma_{k_1}^{p_1} \ldots \gamma_{k_i}^{p_i} \varpi = \upsilon_{k_1}^{p_1} \ldots \upsilon_{k_i}^{p_i}$;
(*ii*) $\gamma_{k_1}^{p_1} \ldots \gamma_{k_i}^{p_i} \varpi = \varpi \upsilon_{k_1}^{p_1} \ldots \upsilon_{k_i}^{p_i}$;
(*iii*) $\varpi \gamma_{k_1}^{p_1} \ldots \gamma_{k_i}^{p_i} = \upsilon_{k_1}^{p_1} \ldots \upsilon_{k_i}^{p_i} \varpi$;
(*iv*) $\varpi \upsilon_{k_1}^{p_1} \ldots \upsilon_{k_i}^{p_i} \varpi = \gamma_{k_1}^{p_1} \ldots \gamma_{k_i}^{p_i}$.

Proof. Assertion (*i*) follows from the definitions of the elements of the semigroups $\langle \gamma_k \mid k \in \mathbb{N} \rangle$ and $\langle \upsilon_k \mid k \in \mathbb{N} \rangle$. Other assertions follow from (*i*) and the equality $\varpi\varpi = \mathbb{I}$. □

Later we assume that $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

The following theorem describes the quotient semigroup $\mathscr{PO}_\infty(\mathbb{N}_{\leqslant}^2)/\sigma$.

Theorem 3. *The semigroup $\mathscr{PO}_\infty(\mathbb{N}_{\leqslant}^2)/\sigma$ is isomorphic to the semidirect product $\mathfrak{AM}_\omega \rtimes_{\mathfrak{Q}} \mathbb{Z}_2$ of the free commutative monoid $\mathfrak{AM}_\omega$ over an infinite countable set by the cyclic group $\mathbb{Z}_2$.*

Proof. We define a map $\mathfrak{I}\colon \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma \to \mathfrak{AM}_{\omega} \rtimes_{\mathfrak{Q}} \mathbb{Z}_2\colon x \mapsto (u,g)$ in the following way. Let $\mathfrak{P}_{\sigma}\colon \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}) \to \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ be the natural homomorphism generated by the congruence σ on the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. Then for every $x \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ for any $\alpha_x \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ such that $(\alpha_x)\mathfrak{P}_{\sigma} = x$ only one of the following conditions holds:

(1) $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1$;
(2) $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1$.

We put

$$(x)\mathfrak{I} = \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1. \end{cases} \tag{6}$$

for all $\alpha_x \in \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ with $(\alpha_x)\mathfrak{P}_{\sigma} = x$. Then the definition of the congruence σ on the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ and Corollary 2 imply that the map $\mathfrak{I}\colon \mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma \to \mathfrak{AM}_{\omega} \times \mathbb{Z}_2$ is well defined.

We observe that formula (6) implies that $(x_{\mathbb{I}})\mathfrak{I} = (e,\bar{0})$ for $x_{\mathbb{I}} = (\mathbb{I})\mathfrak{P}_{\sigma}$ and $(x_{\varpi})\mathfrak{I} = (e,\bar{1})$ for $x_{\varpi} = (\varpi)\mathfrak{P}_{\sigma}$. Hence we have that

$$\begin{aligned}
(x)\mathfrak{I}\cdot(x_{\mathbb{I}})\mathfrak{I} &= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}, \bar{0})\cdot(e,\bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{1})\cdot(e,\bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}\cdot e, \bar{0}\cdot\bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}\cdot e, \bar{1}\cdot\bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= (x)\mathfrak{I}
\end{aligned}$$

and

$$\begin{aligned}
(x_{\mathbb{I}})\mathfrak{I}\cdot(x)\mathfrak{I} &= \begin{cases} (e,\bar{0})\cdot((\alpha_x)\mathfrak{H}_{\sigma}, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ (e,\bar{0})\cdot((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} (e\cdot(\alpha_x)\mathfrak{H}_{\sigma}, \bar{0}\cdot\bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ (e\cdot(\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{0}\cdot\bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= (x)\mathfrak{I}.
\end{aligned}$$

Also, since σ is congruence on $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$, we get

$$\begin{aligned}
(x)\mathfrak{I}\cdot(x_{\varpi})\mathfrak{I} &= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}, \bar{0})\cdot(e,\bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{1})\cdot(e,\bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}\cdot e, \bar{0}\cdot\bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}\cdot e, \bar{1}\cdot\bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} ((\alpha_x)\mathfrak{H}_{\sigma}, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} = \\
&= \begin{cases} ((\alpha_x\varpi\varpi)\mathfrak{H}_{\sigma}, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_{\sigma}, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1 \end{cases} =
\end{aligned}$$

$$= \begin{cases} (((\alpha_x\varpi)\varpi)\mathfrak{H}_\sigma, \bar{1}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}(\alpha_x\varpi)})\alpha_x\varpi\subseteq\mathsf{V}^1; \\ ((\alpha_x\varpi)\mathfrak{H}_\sigma, \bar{0}), & \text{if } (\mathsf{H}^1_{\operatorname{dom}(\alpha_x\varpi)})\alpha_x\varpi\subseteq\mathsf{H}^1 \end{cases} =$$
$$= (x\cdot x_\varpi)\mathfrak{I}$$

and

(*i*) in the case when $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x\subseteq\mathsf{H}^1$ for $\alpha_x = \gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}$, for some non-negative integers $p, i_1,\ldots,i_p, j_1,\ldots,j_p$, where $\gamma^0=\upsilon^0=\mathbb{I}$, we get that $(\mathsf{H}^1_{\operatorname{dom}(\varpi\alpha_x)})\varpi\alpha_x\subseteq\mathsf{V}^1$,

$$\begin{aligned}(x_\varpi)\mathfrak{I}\cdot(x)\mathfrak{I} &= (e,\bar{1})\cdot((\alpha_x)\mathfrak{H}_\sigma,\bar{0}) = \\ &= (e,\bar{1})\cdot\Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p})\mathfrak{H}_\sigma,\bar{0}\Big) = \\ &= (e,\bar{1})\cdot\Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p},\bar{0}\Big) = \\ &= \Big(e\cdot(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p})\mathfrak{f},\bar{1}\cdot\bar{0}\Big) = \\ &= \Big((a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p})\mathfrak{f},\bar{1}\Big) = \\ &= \Big(a_1^{j_1}\ldots a_p^{j_p}b_1^{i_1}\ldots b_p^{i_p},\bar{1}\Big)\end{aligned}$$

and by Proposition 9,

$$\begin{aligned}(x_\varpi\cdot x)\mathfrak{I} &= ((\varpi\alpha_x\varpi)\mathfrak{H}_\sigma,\bar{1}) = \\ &= \Big((\varpi\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi)\mathfrak{H}_\sigma,\bar{1}\Big) = \\ &= \Big((\upsilon_1^{i_1}\ldots\upsilon_p^{i_p}\varpi\varpi\gamma_1^{j_1}\ldots\gamma_p^{j_p})\mathfrak{H}_\sigma,\bar{1}\Big) = \\ &= \Big((\upsilon_1^{i_1}\ldots\upsilon_p^{i_p}\gamma_1^{j_1}\ldots\gamma_p^{j_p})\mathfrak{H}_\sigma,\bar{1}\Big) = \\ &= \Big(b_1^{i_1}\ldots b_p^{i_p}a_1^{j_1}\ldots a_p^{j_p},\bar{1}\Big) = \\ &= \Big(a_1^{j_1}\ldots a_p^{j_p}b_1^{i_1}\ldots b_p^{i_p},\bar{1}\Big);\end{aligned}$$

(*ii*) in the case when $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x\subseteq\mathsf{V}^1$ we get for $\alpha_x = \gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi$, for some non-negative integers $p, i_1,\ldots,i_p, j_1,\ldots,j_p$, where $\gamma^0=\upsilon^0=\mathbb{I}$, we get that $(\mathsf{H}^1_{\operatorname{dom}(\varpi\alpha_x\varpi)})\varpi\alpha_x\varpi\subseteq\mathsf{H}^1$,

$$\begin{aligned}(x_\varpi)\mathfrak{I}\cdot(x)\mathfrak{I} &= (e,\bar{1})\cdot((\alpha_x\varpi)\mathfrak{H}_\sigma,\bar{1}) = \\ &= (e,\bar{1})\cdot\Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi\varpi)\mathfrak{H}_\sigma,\bar{1}\Big) = \\ &= (e,\bar{1})\cdot\Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p})\mathfrak{H}_\sigma,\bar{1}\Big) = \\ &= (e,\bar{1})\cdot\Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p},\bar{1}\Big) = \\ &= \Big(e\cdot(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p})\mathfrak{f},\bar{1}\cdot\bar{1}\Big) = \\ &= \Big((a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p})\mathfrak{f},\bar{0}\Big) =\end{aligned}$$

$$= \left(a_1^{j_1} \dots a_p^{j_p} b_1^{i_1} \dots b_p^{i_p}, \bar{0}\right)$$

and by Proposition 9,

$$\begin{aligned}
(x_\varpi \cdot x)\mathfrak{I} &= ((\varpi\alpha_x\varpi)\mathfrak{H}_\sigma, \bar{0}) = \\
&= \left((\varpi\gamma_1^{i_1}\dots\gamma_p^{i_p}\upsilon_1^{j_1}\dots\upsilon_p^{j_p}\varpi)\mathfrak{H}_\sigma, \bar{0}\right) = \\
&= \left((\upsilon_1^{i_1}\dots\upsilon_p^{i_p}\varpi\varpi\gamma_1^{j_1}\dots\gamma_p^{j_p})\mathfrak{H}_\sigma, \bar{0}\right) = \\
&= \left((\upsilon_1^{i_1}\dots\upsilon_p^{i_p}\gamma_1^{j_1}\dots\gamma_p^{j_p})\mathfrak{H}_\sigma, \bar{0}\right) = \\
&= \left(b_1^{i_1}\dots b_p^{i_p} a_1^{j_1}\dots a_p^{j_p}, \bar{0}\right) = \\
&= \left(a_1^{j_1}\dots a_p^{j_p} b_1^{i_1}\dots b_p^{i_p}, \bar{0}\right),
\end{aligned}$$

which implies that $(x_\varpi \cdot x)\mathfrak{I} = (x_\varpi)\mathfrak{I} \cdot (x)\mathfrak{I}$.

Therefore we have showed that $(x_{\mathbb{I}})\mathfrak{I}$ is the identity element of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma$ and $(x_\varpi)\mathfrak{I} \cdot (x_\varpi)\mathfrak{I} = (x_{\mathbb{I}})\mathfrak{I}$.

Next we shall show that so defined map $\mathfrak{I}$ is a homomorphism from $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma$ into the semigroup $\mathfrak{AM}_\omega \rtimes_{\mathfrak{Q}} \mathbb{Z}_2$. Fix arbitrary elements x and y of $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma$. We consider the following four possible cases:

(*i*) $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\alpha_y})\alpha_y \subseteq \mathsf{H}^1$ for any $\alpha_x, \alpha_y \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(\alpha_x)\mathfrak{P}_\sigma = x$ and $(\alpha_y)\mathfrak{P}_\sigma = y$;

(*ii*) $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\alpha_y})\alpha_y \subseteq \mathsf{H}^1$ for any $\alpha_x, \alpha_y \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(\alpha_x)\mathfrak{P}_\sigma = x$ and $(\alpha_y)\mathfrak{P}_\sigma = y$;

(*iii*) $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{H}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\alpha_y})\alpha_y \subseteq \mathsf{V}^1$ for any $\alpha_x, \alpha_y \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(\alpha_x)\mathfrak{P}_\sigma = x$ and $(\alpha_y)\mathfrak{P}_\sigma = y$;

(*iv*) $(\mathsf{H}^1_{\operatorname{dom}\alpha_x})\alpha_x \subseteq \mathsf{V}^1$ and $(\mathsf{H}^1_{\operatorname{dom}\alpha_y})\alpha_y \subseteq \mathsf{V}^1$ for any $\alpha_x, \alpha_y \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ such that $(\alpha_x)\mathfrak{P}_\sigma = x$ and $(\alpha_y)\mathfrak{P}_\sigma = y$.

Assume that (*i*) hods. Then we have that $\alpha_x, \alpha_y, \alpha_x\alpha_y \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$. Since σ is a congruence on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, we may choose an element $\alpha_{xy} = \alpha_x\alpha_y \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$. Then $(\alpha_{xy})\mathfrak{P}_\sigma = xy$ Also, since $\mathfrak{P}_\sigma \colon \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant) \to \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma$ is the natural homomorphism generated by the congruence σ on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$ we get that

$$\begin{aligned}
(xy)\mathfrak{I} &= ((\alpha_{xy})\mathfrak{P}_\sigma)\mathfrak{I} = ((\alpha_{xy})\mathfrak{H}_\sigma, \bar{0}) = ((\alpha_x\alpha_y)\mathfrak{H}_\sigma, \bar{0}) = ((\alpha_x)\mathfrak{H}_\sigma \cdot (\alpha_y)\mathfrak{H}_\sigma, \bar{0}\cdot\bar{0}) = \\
&= ((\alpha_x)\mathfrak{H}_\sigma, \bar{0}) \cdot ((\alpha_y)\mathfrak{H}_\sigma, \bar{0}) = (x)\mathfrak{I} \cdot (y)\mathfrak{I}.
\end{aligned}$$

If (*ii*) hods then by Propositions 1 and 3 from [5], $\alpha_x\varpi, \alpha_y, \alpha_x\alpha_y\varpi \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ and by Lemma 2 without loss of generality we may assume that

$$\alpha_x = \gamma_1^{i_1}\dots\gamma_p^{i_p}\upsilon_1^{j_1}\dots\upsilon_p^{j_p}\varpi \qquad \text{and} \qquad \alpha_y = \gamma_1^{s_1}\dots\gamma_p^{s_p}\upsilon_1^{t_1}\dots\upsilon_p^{t_p},$$

for some non-negative integers $p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p$, where $\gamma^0 = \upsilon^0 = \mathbb{I}$. This and the fact that σ is a congruence on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, Proposition 9 imply that

$$
\begin{aligned}
(xy)\mathfrak{I} &= ((\alpha_x\alpha_y\varpi)\mathfrak{H}_\sigma, \bar{1}) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi)\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\upsilon_1^{s_1}\ldots\upsilon_p^{s_p}\varpi\varpi\gamma_1^{t_1}\ldots\gamma_p^{t_p})\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\upsilon_1^{s_1}\ldots\upsilon_p^{s_p}\gamma_1^{t_1}\ldots\gamma_p^{t_p})\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}b_1^{s_1}\ldots b_p^{s_p}a_1^{t_1}\ldots a_p^{t_p}, \bar{1}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}(a_1^{s_1}\ldots a_p^{s_p}b_1^{t_1}\ldots b_p^{t_p})\mathfrak{f}, \bar{1}\cdot\bar{0}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}, \bar{1}\Big)\cdot\big(a_1^{s_1}\ldots a_p^{s_p}b_1^{t_1}\ldots b_p^{t_p}, \bar{0}\big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p})\mathfrak{H}_\sigma, \bar{1}\Big)\cdot\big((\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p})\mathfrak{H}_\sigma, \bar{0}\big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi\varpi)\mathfrak{H}_\sigma, \bar{1}\Big)\cdot\big((\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p})\mathfrak{H}_\sigma, \bar{0}\big) = \\
&= ((\alpha_x\varpi)\mathfrak{H}_\sigma, \bar{1})\cdot((\alpha_y)\mathfrak{H}_\sigma, \bar{0}) = \\
&= (x)\mathfrak{I}\cdot(y)\mathfrak{I}.
\end{aligned}
$$

If (iii) hods then by Propositions 1 and 3 from [5], $\alpha_x, \alpha_y\varpi, \alpha_x\alpha_y\varpi \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ and by Lemma 2 without loss of generality we may assume that

$$
\alpha_x = \gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p} \qquad \text{and} \qquad \alpha_y = \gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi,
$$

for some non-negative integers $p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p$, where $\gamma^0 = \upsilon^0 = \mathbb{I}$. Since σ is a congruence on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, this and Proposition 9 imply that

$$
\begin{aligned}
(xy)\mathfrak{I} &= ((\alpha_x\alpha_y\varpi)\mathfrak{H}_\sigma, \bar{1}) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi\varpi)\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p})\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}a_1^{s_1}\ldots a_p^{s_p}b_1^{t_1}\ldots b_p^{t_p}, \bar{0}\cdot\bar{1}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}, \bar{0}\Big)\cdot\big(a_1^{s_1}\ldots a_p^{s_p}b_1^{t_1}\ldots b_p^{t_p}, \bar{1}\big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p})\mathfrak{H}_\sigma, \bar{0}\Big)\cdot\big((\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p})\mathfrak{H}_\sigma, \bar{1}\big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p})\mathfrak{H}_\sigma, \bar{0}\Big)\cdot\big((\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi\varpi)\mathfrak{H}_\sigma, \bar{1}\big) = \\
&= ((\alpha_x)\mathfrak{H}_\sigma, \bar{0})\cdot((\alpha_y\varpi)\mathfrak{H}_\sigma, \bar{1}) = \\
&= (x)\mathfrak{I}\cdot(y)\mathfrak{I}.
\end{aligned}
$$

Assume that (*iv*) hods. Then by Propositions 1 and 3 from [5] we have that $\alpha_x\varpi, \alpha_y\varpi, \alpha_x\alpha_y,\ \alpha_x\varpi\alpha_y\varpi \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ and by Lemma 2 without loss of generality we may assume that

$$\alpha_x = \gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi \qquad \text{and} \qquad \alpha_y = \gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi,$$

for some non-negative integers $p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p$, where $\gamma^0 = \upsilon^0 = \mathbb{I}$. Since σ is a congruence on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)$, this and Proposition 9 imply that

$$\begin{aligned}
(xy)\mathfrak{I} &= ((\alpha_x\alpha_y)\mathfrak{H}_\sigma, \bar{0}) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi)\mathfrak{H}_\sigma, \bar{0}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\upsilon_1^{s_1}\ldots\upsilon_p^{s_p}\varpi\varpi\gamma_1^{t_1}\ldots\gamma_p^{t_p})\mathfrak{H}_\sigma, \bar{0}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\upsilon_1^{s_1}\ldots\upsilon_p^{s_p}\gamma_1^{t_1}\ldots\gamma_p^{t_p})\mathfrak{H}_\sigma, \bar{0}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}b_1^{s_1}\ldots b_p^{s_p}a_1^{t_1}\ldots a_p^{t_p}, \bar{0}\Big) = \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}\cdot(a_1^{s_1}\ldots a_p^{s_p}b_1^{t_1}\ldots b_p^{t_p})\mathfrak{f}, \bar{1}\cdot\bar{1}\Big) \\
&= \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}\ldots b_p^{j_p}, \bar{1}\Big)\cdot\Big(a_1^{s_1}\ldots a_p^{s_p}b_1^{t_1}\ldots b_p^{t_p}, \bar{1}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p})\mathfrak{H}_\sigma, \bar{1}\Big)\cdot\Big((\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p})\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= \Big((\gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}\varpi\varpi)\mathfrak{H}_\sigma, \bar{1}\Big)\cdot\Big((\gamma_1^{s_1}\ldots\gamma_p^{s_p}\upsilon_1^{t_1}\ldots\upsilon_p^{t_p}\varpi\varpi)\mathfrak{H}_\sigma, \bar{1}\Big) = \\
&= ((\alpha_x\varpi)\mathfrak{H}_\sigma, \bar{1})\cdot((\alpha_y\varpi)\mathfrak{H}_\sigma, \bar{1}) = \\
&= (x)\mathfrak{I}\cdot(y)\mathfrak{I}.
\end{aligned}$$

Thus the map $\mathfrak{I}\colon \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma \to \mathfrak{AM}_\omega \rtimes_{\mathfrak{Q}} \mathbb{Z}_2$ is a homomorphism. Also, since $(x_\mathbb{I})\mathfrak{I} = (e, \bar{0})$, $(x_\varpi)\mathfrak{I} = (e, \bar{1})$ and for any $\alpha_x = \gamma_1^{i_1}\ldots\gamma_p^{i_p}\upsilon_1^{j_1}\ldots\upsilon_p^{j_p}$, where $p, i_1, \ldots, i_p, j_1, \ldots, j_p$ are some positive integers, our above arguments imply that

$$(x)\mathfrak{I} = \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}, \bar{0}\Big) \qquad \text{and} \qquad (y)\mathfrak{I} = \Big(a_1^{i_1}\ldots a_p^{i_p}b_1^{j_1}, \bar{1}\Big),$$

where $x = (\alpha_x)\mathfrak{P}_\sigma$ and $y = (\alpha_x\varpi)\mathfrak{P}_\sigma$. This implies that the homomorphism $\mathfrak{I}$ is surjective.

Now suppose that $(x)\mathfrak{I} = (y)\mathfrak{I} = (u, g)$ for some $x, y \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma$. Then there exist $\alpha_x, \alpha_y \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ such that $(\alpha_x)\mathfrak{P}_\sigma = x$ and $(\alpha_y)\mathfrak{P}_\sigma = y$ in the case when $g = \bar{0}$, and $(\alpha_x\varpi)\mathfrak{P}_\sigma = x$ and $(\alpha_y\varpi)\mathfrak{P}_\sigma = y$ in the case when $g = \bar{1}$. If $g = \bar{0}$ then $x, y \in \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ and the condition $\alpha_x\sigma\alpha_y$ in $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ implies the equality $x = y$. Similarly, if $g = \bar{1}$ then $x, y \in \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant) \setminus \mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ and the condition $\alpha_x\varpi\sigma\alpha_y\varpi$ in $\mathscr{PO}^+_\infty(\mathbb{N}^2_\leqslant)$ implies the equality $x = y$. Hence $\mathfrak{I}\colon \mathscr{PO}_\infty(\mathbb{N}^2_\leqslant)/\sigma \to \mathfrak{AM}_\omega \rtimes_{\mathfrak{Q}} \mathbb{Z}_2$ is an isomorphism. □

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ПРО МОНОЇД МОНОТОННИХ ІН'ЄКТИВНИХ ЧАСТКОВИХ ПЕРЕТВОРЕНЬ МНОЖИНИ $\mathbb{N}^2_{\leqslant}$ З КОСКІНЧЕННИМИ ОБЛАСТЯМИ ВИЗНАЧЕНЬ І ЗНАЧЕНЬ, II

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Нехай $\mathbb{N}^2_{\leqslant}$ — множина $\mathbb{N}^2$ з частковим порядком, визначеним як добуток звичайного лінійного порядку $\leqslant$ на множині натуральних чисел $\mathbb{N}$. Вивчаємо напівгрупу $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ монотонних ін'єктивних часткових перетворень частково впорядкованої множини $\mathbb{N}^2_{\leqslant}$, які мають коскінченні області визначення та значення. Описуємо природний частковий порядок на напівгрупі $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ і доводимо, що він збігається з природним частковим порядком, який індукується з симетичного інверсного моноїда $\mathscr{I}_{\mathbb{N}\times\mathbb{N}}$ над множиною $\mathbb{N}\times\mathbb{N}$ на напівгрупу $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$. Доводимо, що напівгрупа $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$ ізоморфна напівпрямому добутку $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant}) \rtimes \mathbb{Z}_2$ моноїда $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})$ орієнтованих монотонних ін'єктивних часткових перетворень частково впорядкованої множини $\mathbb{N}^2_{\leqslant}$, які мають коскінченні області визначення та значення, циклічною групою $\mathbb{Z}_2$ другого порядку. Також описуємо конгруенцію σ на напівгрупі $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$, яка породжується природним частковим порядком $\preccurlyeq$ на напівгрупі $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})$: $\alpha\sigma\beta$ тоді і лише тоді, коли α та β є порівняльними в $\left(\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant}), \preccurlyeq\right)$. Доводимо, що фактор-напівгрупа $\mathscr{PO}^{+}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ ізоморфна вільному комутативному моноїду $\mathfrak{AM}_{\omega}$ над нескінченною зліченною множиною і, що фактор-напівгрупа $\mathscr{PO}_{\infty}(\mathbb{N}^2_{\leqslant})/\sigma$ ізоморфна напівпрямому добутку вільного комутативного моноїда $\mathfrak{AM}_{\omega}$ групою $\mathbb{Z}_2$.

Ключові слова: напівгрупа часткових біекцій, монотонне часткове відображення, природний частковий порядок, напівпрямий добуток, найменша групова конгруенція, вільний комутативний моноїд.