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ON FEEBLY COMPACT SHIFT-CONTINUOUS TOPOLOGIES ON THE SEMILATTICE $\exp_n \lambda$

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We study feebly compact topologies τ on the semilattice $(\exp_n \lambda, \cap)$ such that $(\exp_n \lambda, \tau)$ is a semitopological semilattice and prove that for any shift-continuous T_1-topology τ on $\exp_n \lambda$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\exp_n \lambda, \tau)$ is an H-closed space.

Key words: topological semilattice, semitopological semilattice, countably pracompact, feebly compact, d-feebly compact, H-closed, semiregular space, regular space.

Dedicated to the memory of Professor Vitaly Sushchanskyy

We shall follow the terminology of [6, 8, 9, 13]. If X is a topological space and $A \subseteq X$, then by $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ we denote the closure and the interior of A in X, respectively. By ω we denote the first infinite cardinal and by $\mathbb{N}$ the set of positive integers.

A subset A of a topological space X is called *regular open* if $\mathrm{int}_X(\mathrm{cl}_X(A)) = A$.

We recall that a topological space X is said to be

- *quasiregular* if for any non-empty open set $U \subset X$ there exists a non-empty open set $V \subset U$ such that $\mathrm{cl}_X(V) \subseteq U$;
- *semiregular* if X has a base consisting of regular open subsets;
- *compact* if each open cover of X has a finite subcover;
- *countably compact* if each open countable cover of X has a finite subcover;
- *countably compact at a subset* $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point x in X;
- *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A;

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- *feebly compact* (or *lightly compact*) if each locally finite open cover of X is finite [3];
- *d-feebly compact* (or *DFCC*) if every discrete family of open subsets in X is finite (see [12]);
- *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded.

According to Theorem 3.10.22 of [8], a Tychonoff topological space X is feebly compact if and only if X is pseudocompact. Also, a Hausdorff topological space X is feebly compact if and only if every locally finite family of non-empty open subsets of X is finite [3]. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is feebly compact (see [2]), and every H-closed space is feebly compact too (see [10]). Also, it is obvious that every feebly compact space is d-feebly compact.

A *semilattice* is a commutative semigroup of idempotents. On a semilattice S there exists a natural partial order: $e \leqslant f$ *if and only if* $ef = fe = e$. For any element e of a semilattice S we put

$$\uparrow e = \{f \in S \colon e \leqslant f\}.$$

A *topological* (*semitopological*) *semilattice* is a topological space together with a continuous (separately continuous) semilattice operation. If S is a semilattice and τ is a topology on S such that (S, τ) is a topological semilattice, then we shall call τ a *semilattice topology* on S, and if τ is a topology on S such that (S, τ) is a semitopological semilattice, then we shall call τ a *shift-continuous topology* on S.

For an arbitrary positive integer n and an arbitrary non-zero cardinal λ we put

$$\exp_n \lambda = \{A \subseteq \lambda \colon |A| \leqslant n\}.$$

It is obvious that for any positive integer n and any non-zero cardinal λ the set $\exp_n \lambda$ with the binary operation $\cap$ is a semilattice. Later in this paper by $\exp_n \lambda$ we shall denote the semilattice $(\exp_n \lambda, \cap)$.

This paper is a continuation of [11] where we study feebly compact topologies τ on the semilattice $\exp_n \lambda$ such that $(\exp_n \lambda, \tau)$ is a semitopological semilattice. Therein, all compact semilattice T_1-topologies on $\exp_n \lambda$ were described. In [11] it was proved that for an arbitrary positive integer n and an arbitrary infinite cardinal λ every T_1-semitopological countably compact semilattice $(\exp_n \lambda, \tau)$ is a compact topological semilattice. Also, there we construct a countably pracompact H-closed quasiregular non-semiregular topology τ_{fc}^2 such that $\left(\exp_2 \lambda, \tau_{\mathsf{fc}}^2\right)$ is a semitopological semilattice with the discontinuous semilattice operation and show that for an arbitrary positive integer n and an arbitrary infinite cardinal λ a semiregular feebly compact semitopological semilattice $\exp_n \lambda$ is a compact topological semilattice.

In this paper we show that for any shift-continuous T_1-topology τ on $\exp_n \lambda$ the following conditions are equivalent: (i) τ is countably pracompact; (ii) τ is feebly compact; (iii) τ is d-feebly compact; (iv) $(\exp_n \lambda, \tau)$ is an H-closed space.

The proof of the following lemma is similar to Lemma 4.5 of [5] or Proposition 1 from [1].

Lemma 1. *Every Hausdorff d-feebly compact topological space with a dense discrete subspace is countably pracompact.*

We observe that by Proposition 1 from [11] for an arbitrary positive integer n and an arbitrary infinite cardinal λ every shift-continuous T_1-topology τ on $\exp_n \lambda$ is functionally Hausdorff and quasiregular, and hence it is Hausdorff.

Proposition 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for every d-feebly compact shift-continuous T_1-topology τ on $\exp_n \lambda$ the subset $\exp_n \lambda \setminus \exp_{n-1} \lambda$ is dense in $(\exp_n \lambda, \tau)$.*

Proof. Suppose to the contrary that there exists a d-feebly compact shift-continuous T_1-topology τ on $\exp_n \lambda$ such that $\exp_n \lambda \setminus \exp_{n-1} \lambda$ is not dense in $(\exp_n \lambda, \tau)$. Then there exists a point $x \in \exp_{n-1} \lambda$ of the space $(\exp_n \lambda, \tau)$ such that $x \notin \operatorname{cl}_{\exp_n \lambda}(\exp_n \lambda \setminus \exp_{n-1} \lambda)$. This implies that there exists an open neighbourhood $U(x)$ of x in $(\exp_n \lambda, \tau)$ such that $U(x) \cap \big(\exp_n \lambda \setminus \exp_{n-1} \lambda\big) = \varnothing$. The definition of the semilattice $\exp_n \lambda$ implies that every maximal chain in $\exp_n \lambda$ is finite and hence there exists a point $y \in U(x)$ such that $\uparrow y \cap U(x) = \{y\}$. By Proposition 1$(iii)$ from [11], $\uparrow y$ is an open-and-closed subset of $(\exp_n \lambda, \tau)$ and hence $\uparrow y$ is a d-feebly compact subspace of $(\exp_n \lambda, \tau)$.

It is obvious that the subsemilattice $\uparrow y$ of $\exp_n \lambda$ is algebraically isomorphic to the semilattice $\exp_k \lambda$ for some positive integer $k \leqslant n$. This and above arguments imply that without loss of generality we may assume that y is the isolated zero of the d-feebly compact semitopological semilattice $(\exp_n \lambda, \tau)$.

Hence we assume that τ is a d-feebly compact shift-continuous topology on $\exp_n \lambda$ such that the zero 0 of $\exp_n \lambda$ is an isolated point of $(\exp_n \lambda, \tau)$. Next we fix an arbitrary infinite sequence $\{x_i\}_{i\in\mathbb{N}}$ of distinct elements of cardinal λ. For every positive integer j we put

$$a_j = \left\{x_{n(j-1)+1}, x_{n(j-1)+2}, \ldots, x_{nj}\right\}.$$

Then $a_j \in \exp_n \lambda$ and moreover a_j is a greatest element of the semilattice $\exp_n \lambda$ for each positive integer j. Also, the definition of the semilattice $\exp_n \lambda$ implies that for every non-zero element a of $\exp_n \lambda$ there exists at most one element a_j such that $a_j \in \uparrow a$. Then for every positive integer j by Proposition 1(iii) of [11], a_j is an isolated point of $(\exp_n \lambda, \tau)$, and hence the above arguments imply that $\{a_1, a_2, \ldots, a_j, \ldots\}$ is an infinite discrete family of open subset in the space $(\exp_n \lambda, \tau)$. This contradicts the d-feeble compactness of the semitopological semilattice $(\exp_n \lambda, \tau)$. The obtained contradiction implies the statement of our proposition. □

The following example show that the converse statement to Proposition 1 is not true in the case of topological semilattices.

Example 1. Fix an arbitrary cardinal λ and an infinite subset A in λ such that $|\lambda \setminus A| \geqslant \omega$. By $\pi\colon \lambda \to \exp_1 \lambda\colon a \mapsto \{a\}$ we denote the natural embedding of λ into $\exp_1 \lambda$. On $\exp_1 \lambda$ we define a topology τ_{dm} in the following way:

(i) all non-zero elements of the semilattice $\exp_1 \lambda$ are isolated points in $(\exp_1 \lambda, \tau_{\mathsf{dm}})$; and

(ii) the family $\mathscr{B}_{\mathsf{dm}} = \{U_B = \{0\} \cup \pi(B)\colon B \subseteq A \text{ and } A \setminus B \text{ is finite}\}$ is the base of the topology τ_{dm} at zero 0 of $\exp_1 \lambda$.

Simple verifications show that τ_{dm} is a Hausdorff locally compact semilattice topology on $\exp_1 \lambda$ which is not compact and hence by Corollary 8 of [11] it is not feebly compact.

Remark 1. We observe that in the case when $\lambda=\omega$ by Proposition 13 of [11] the topological space $(\exp_1\lambda,\tau_{\mathsf{dm}})$ is collectionwise normal and it has a countable base, and hence $(\exp_1\lambda,\tau_{\mathsf{dm}})$ is metrizable by the Urysohn Metrization Theorem [14]. Moreover, if $|B|=\omega$ then the space $(\exp_1\lambda,\tau_{\mathsf{dm}})$ is metrizable for any infinite cardinal λ, as a topological sum of the metrizable space $(\exp_1\omega,\tau_{\mathsf{dm}})$ and the discrete space of cardinality λ.

Remark 2. If n is an arbitrary positive integer $\geqslant 3$, λ is any infinite cardinal and τ_{c}^n is the unique compact semilattice topology on the semilattice $\exp_n\lambda$ defined in Example 4 of [11], then we construct more stronger topology τ_{dm}^n on $\exp_n\lambda$ them τ_{c}^n in the following way. Fix an arbitrary element $x\in\exp_n\lambda$ such that $|x|=n-1$. It is easy to see that the subsemilattice ${\uparrow}x$ of $\exp_n\lambda$ is isomorphic to $\exp_1\lambda$, and by $h\colon\exp_1\lambda\to{\uparrow}x$ we denote this isomorphism.

Fix an arbitrary subset A in λ such that $|\lambda\setminus A|\geqslant\omega$. For every zero element $y\in\exp_n\lambda\setminus{\uparrow}x$ we assume that the base $\mathscr{B}_{\mathsf{dm}}^n(y)$ of the topology τ_{dm}^n at the point y coincides with the base of the topology τ_{c}^n at y, and assume that ${\uparrow}x$ is an open-and-closed subset and the topology on ${\uparrow}x$ is generated by the map $h\colon\big(\exp_2\lambda,\tau_{\mathsf{fc}}^2\big)\to{\uparrow}x$. We observe that $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$ is a Hausdorff locally compact topological space, because it is the topological sum of a Hausdorff locally compact space ${\uparrow}x$ (which is homeomorphic to the Hausdorff locally compact space $(\exp_1\lambda,\tau_{\mathsf{dm}})$ from Example 1) and an open-and-closed subspace $\exp_n\lambda\setminus{\uparrow}x$ of $(\exp_n\lambda,\tau_{\mathsf{c}}^n)$. It is obvious that the set $\exp_n\lambda\setminus\exp_{n-1}\lambda$ is dense in $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$. Also, since ${\uparrow}x$ is an open-and-closed subsemilattice with zero x of $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$, the continuity of the semilattice operations in $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$ and $(\exp_n\lambda,\tau_{\mathsf{c}}^n)$ and the property that the topology τ_{dm}^n is more stronger them τ_{c}^n, imply that $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$ is a topological semilattice. Moreover, the space $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$ is not d-feebly compact, because it contains an open-and-closed non-d-feebly compact subspace ${\uparrow}x$.

Arguments presented in the proof of Proposition 1 and Proposition 1(iii) of [11] imply the following corollary.

Corollary 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for every d-feebly compact shift-continuous T_1-topology τ on $\exp_n\lambda$ a point x is isolated in $(\exp_n\lambda,\tau)$ if and only if $x\in\exp_n\lambda\setminus\exp_{n-1}\lambda$.*

Remark 3. We observe that the example presented in Remark 2 implies there exists a locally compact non-d-feebly compact semitopological semilattice $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$ with the following property: *a point x is isolated in $(\exp_n\lambda,\tau_{\mathsf{dm}}^n)$ if and only if $x\in\exp_n\lambda\setminus\exp_{n-1}\lambda$.*

The following proposition gives an amazing property of the system of neighbourhoodd of zero in a T_1-feebly compact semitopological semilattice $\exp_n\lambda$.

Proposition 2. *Let n be an arbitrary positive integer, λ be an arbitrary infinite cardinal and τ be a shift-continuous feebly compact T_1-topology on the semilattice $\exp_n\lambda$. Then for every open neighbourhood $U(0)$ of zero 0 in $(\exp_n\lambda,\tau)$ there exist finitely many $x_1,\ldots,x_m\in\lambda$ such that*

$$\exp_n\lambda\setminus\operatorname{cl}_{\exp_n\lambda}(U(0))\subseteq{\uparrow}x_1\cup\cdots\cup{\uparrow}x_m.$$

Proof. Suppose to the contrary that there exists an open neighbourhood $U(0)$ of zero in a Hausdorff feebly compact semitopological semilattice $(\exp_n \lambda, \tau)$ such that

$$\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0)) \nsubseteq {\uparrow}x_1 \cup \cdots \cup {\uparrow}x_m$$

for any finitely many $x_1, \ldots, x_m \in \lambda$.

We fix an arbitrary $y_1 \in \lambda$ such that $\left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap {\uparrow}y_1 \neq \varnothing$. By Proposition 1$(iii)$ of [11] the set ${\uparrow}y_1$ is open in $(\exp_n \lambda, \tau)$ and hence the set $\left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap {\uparrow}y_1$ is open in $(\exp_n \lambda, \tau)$ too. Then by Proposition 1 there exists an isolated point $m_1 \in \exp_n \lambda \setminus \exp_{n-1} \lambda$ in $(\exp_n \lambda, \tau)$ such that $m_1 \in \left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap {\uparrow}y_1$. Now, by the assumption there exists $y_2 \in \lambda$ such that

$$\left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap ({\uparrow}y_2 \setminus {\uparrow}y_1) \neq \varnothing.$$

Again, since by Proposition 1(iii) of [11] both sets ${\uparrow}y_1$ and ${\uparrow}y_2$ are open-and-closed in $(\exp_n \lambda, \tau)$, Proposition 1 implies that there exists an isolated point $m_2 \in \exp_n \lambda \setminus \exp_{n-1} \lambda$ in $(\exp_n \lambda, \tau)$ such that

$$m_2 \in \left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap ({\uparrow}y_2 \setminus {\uparrow}y_1).$$

Hence by induction we can construct a sequence $\{y_i \colon i = 1, 2, 3, \ldots\}$ of distinct points of λ and a sequence of isolated points $\{m_i \colon i = 1, 2, 3, \ldots\} \subset \exp_n \lambda \setminus \exp_{n-1} \lambda$ in $(\exp_n \lambda, \tau)$ such that for any positive integer k the following conditions hold:

(i) $\left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap ({\uparrow}y_k \setminus ({\uparrow}y_1 \cup \cdots \cup {\uparrow}y_{k-1})) \neq \varnothing$; and

(ii) $m_k \in \left(\exp_n \lambda \setminus \mathrm{cl}_{\exp_n \lambda}(U(0))\right) \cap ({\uparrow}y_k \setminus ({\uparrow}y_1 \cup \cdots \cup {\uparrow}y_{k-1}))$.

Then similar arguments as in the proof of Proposition 1 imply that the following family

$$\{\{m_i\} : i = 1, 2, 3, \ldots\}$$

is infinite and locally finite, which contradicts the feeble compactness of $(\exp_n \lambda, \tau)$. The obtained contradiction implies the statement of the proposition. □

Proposition 1(iii) of [11] implies that for any element $x \in \exp_n \lambda$ the set ${\uparrow}x$ is open-and-closed in a T_1-semitopological semilattice $(\exp_n \lambda, \tau)$ and hence by Theorem 14 from [3] we have that for any $x \in \exp_n \lambda$ the space ${\uparrow}x$ is feebly compact in a feebly compact T_1-semitopological semilattice $(\exp_n \lambda, \tau)$. Hence Proposition 2 implies the following proposition.

Proposition 3. *Let n be an arbitrary positive integer, λ be an arbitrary infinite cardinal and τ be a shift-continuous feebly compact T_1-topology on the semilattice $\exp_n \lambda$. Then for any point $x \in \exp_n \lambda$ and any open neighbourhood $U(x)$ of x in $(\exp_n \lambda, \tau)$ there exist finitely many $x_1, \ldots, x_m \in {\uparrow}x \setminus \{x\}$ such that*

$$ {\uparrow}x \setminus \mathrm{cl}_{\exp_n \lambda}(U(x)) \subseteq {\uparrow}x_1 \cup \cdots \cup {\uparrow}x_m.$$

The main results of this paper is the following theorem.

Theorem 1. *Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for any shift-continuous T_1-topology τ on $\exp_n \lambda$ the following conditions are equivalent:*

(i) *τ is countably pracompact;*

(ii) *τ is feebly compact;*

(iii) *τ is d-feebly compact;*

(*iv*) *the space* $(\exp_n \lambda, \tau)$ *is* H*-closed.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial and implication $(iii) \Rightarrow (i)$ follows from Proposition 1 of [11], Lemma 1 and Proposition 1.

Implication $(iv) \Rightarrow (ii)$ follows from Proposition 4 of [10].

$(ii) \Rightarrow (iv)$ We shall prove this implication by induction.

By Corollary 2 from [11] every feebly compact T_1-topology τ on the semilattice $\exp_1 \lambda$ such that $(\exp_1 \lambda, \tau)$ is a semitopological semilattice, is compact, and hence $(\exp_1 \lambda, \tau)$ is an H-closed topological space.

Next we shall show that if our statements holds for all positive integers $j < k \leqslant n$ then it holds for $j = k$. Suppose that a feebly compact T_1-semitopological semilattice $(\exp_k \lambda, \tau)$ is a subspace of Hausdorff topological space X. Fix an arbitrary point $x \in X$ and an arbitrary open neighbourhood $V(x)$ of x in X. Since X is Hausdorff, there exist disjoint open neighbourhoods $U(x) \subseteq V(x)$ and $U(0)$ of x and zero 0 of the semilattice $\exp_k \lambda$ in X, respectively. Then $\operatorname{cl}_X(U(0)) \cap U(x) = \varnothing$ and hence by Proposition 2 there exists finitely many $x_1, \ldots, x_m \in \lambda$ such that

$$\exp_k \lambda \cap U(x) \subseteq {\uparrow}x_1 \cup \cdots \cup {\uparrow}x_m.$$

But for any $x \in \lambda$ the subsemilattice ${\uparrow}x$ of $\exp_k \lambda$ is algebraically isomorphic to the semilattice $\exp_{k-1} \lambda$. Then by Proposition 1(iii) of [11] and Theorem 14 from [3], ${\uparrow}x$ is a feebly compact T_1-semitopological semilattice, and the assumption of our induction implies that ${\uparrow}x_1, \cdots, {\uparrow}x_m$ are closed subsets of X. This implies that

$$W(x) = U(x) \setminus ({\uparrow}x_1 \cup \cdots \cup {\uparrow}x_m)$$

is an open neighbourhood of x in X such that $W(x) \cap \exp_k \lambda = \varnothing$. Thus, $(\exp_k \lambda, \tau)$ is an H-closed space. This completes the proof of the requested implication. □

The following theorem gives a sufficient condition when a d-feebly compact space is feebly compact.

Theorem 2. *Every quasiregular d-feebly compact space is feebly compact.*

Proof. Suppose to the contrary that there exists a quasiregular d-feebly compact space X which is not feebly compact. Then there exists an infinite locally finite family $\mathscr{U}_0$ of non-empty open subsets of X.

By induction we shall construct an infinite discrete family of non-empty open subsets of X.

Fix an arbitrary $U_1 \in \mathscr{U}_0$ and an arbitrary point $x_1 \in U_1$. Since the family $\mathscr{U}_0$ is locally finite there exists an open neighbourhood $U(x_1) \subseteq U_1$ of the point x_1 in X such that $U(x_1)$ intersects finitely many elements of $\mathscr{U}_0$. Also, the quasiregularity of X implies that there exists a non-empty open subset $V_1 \subseteq U(x_1)$ such that $\operatorname{cl}_X(V_1) \subseteq U(x_1)$. Put

$$\mathscr{U}_1 = \{U \in \mathscr{U}_0 \colon U(x_1) \cap U = \varnothing\}.$$

Since the family $\mathscr{U}_0$ is locally finite and infinite, so is $\mathscr{U}_1$. Fix an arbitrary $U_2 \in \mathscr{U}_1$ and an arbitrary point $x_2 \in U_2$. Since the family $\mathscr{U}_1$ is locally finite, there exists an open neighbourhood $U(x_2) \subseteq U_2$ of the point x_2 in X such that $U(x_2)$ intersects finitely many elements of $\mathscr{U}_1$. Since X is quasiregular, there exists a non-empty open subset $V_2 \subseteq U(x_2)$

such that $\mathrm{cl}_X(V_2) \subseteq U(x_2)$. Our construction implies that the closed sets $\mathrm{cl}_X(V_1)$ and $\mathrm{cl}_X(V_2)$ are disjoint and hence so are V_1 and V_2. Next we put

$$\mathscr{U}_2 = \{U \in \mathscr{U}_1 \colon U(x_2) \cap U = \varnothing\}.$$

Also, we observe that it is obvious that $U(x_1) \cap U = \varnothing$ for each $U \in \mathscr{U}_1$.

Suppose for some positive integer $k > 1$ we construct:

(a) a sequence of infinite locally finite subfamilies $\mathscr{U}_1, \ldots, \mathscr{U}_{k-1}$ in $\mathscr{U}_0$ of non-empty open subsets in the space X;

(b) a sequence of open subsets $U_1, \ldots, U_k$ in X;

(c) a sequence of points $x_1, \ldots, x_k$ in X and a sequence of their corresponding open neighbourhoods $U(x_1), \ldots, U(x_k)$ in X;

(d) a sequence of disjoint non-empty subsets $V_1, \ldots, V_k$ in X

such that the following conditions hold:

(i) $\mathscr{U}_i$ is a proper subfamily of $\mathscr{U}_{i-1}$;

(ii) $U_i \in \mathscr{U}_{i-1}$ and $U_i \cap U = \varnothing$ for each $U \in \mathscr{U}_j$ with $i \leqslant j \leqslant k$;

(iii) $x_i \in U_i$ and $U(x_i) \subseteq U_i$;

(iv) V_i is an open subset of U_i with $\mathrm{cl}_X(V_i) \subseteq U(x_i)$,

for all $i = 1, \ldots, k$, and

(v) $\mathrm{cl}_X(V_1), \ldots, \mathrm{cl}_X(V_k)$ are disjoint.

Next we put

$$\mathscr{U}_k = \{U \in \mathscr{U}_{k-1} \colon U(x_1) \cap U = \ldots = U(x_k) \cap U = \varnothing\}.$$

Since the family $\mathscr{U}_{k-1}$ is infinite and locally finite, there exists a subfamily $\mathscr{U}_k$ in $\mathscr{U}_{k-1}$ which is infinite and locally finite. Fix an arbitrary $U_{k+1} \in \mathscr{U}_k$ and an arbitrary point $x_{k+1} \in U_{k+1}$. Since the family $\mathscr{U}_k$ is locally finite, there exists an open neighbourhood $U(x_{k+1}) \subseteq U_{k+1}$ of the point x_{k+1} in X such that $U(x_{k+1})$ intersects finitely many elements of $\mathscr{U}_k$. Since the space X is quasiregular, there exists a non-empty open subset $V_{k+1} \subseteq U(x_{k+1})$ such that $\mathrm{cl}_X(V_{k+1}) \subseteq U(x_{k+1})$. Simple verifications show that the conditions $(i) - (iv)$ hold in the case of the positive integer $k + 1$.

Hence by induction we construct the following two infinite countable families of open non-empty subsets of X:

$$\mathscr{U} = \{U_i \colon i = 1, 2, 3, \ldots\} \qquad \text{and} \qquad \mathscr{V} = \{V_i \colon i = 1, 2, 3, \ldots\}$$

such that $\mathrm{cl}_X(V_i) \subseteq U_i$ for each positive integer i. Since $\mathscr{U}$ is a subfamily of $\mathscr{U}_0$ and $\mathscr{U}_0$ is locally finite in X, $\mathscr{U}$ is locally finite in X as well. Also, above arguments imply that $\mathscr{V}$ and

$$\overline{\mathscr{V}} = \{\mathrm{cl}_X(V_i) \colon i = 1, 2, 3, \ldots\}$$

are locally finite families in X too.

Next we shall show that the family $\mathscr{V}$ is discrete in X. Indeed, since the family $\overline{\mathscr{V}}$ is locally finite in X, by Theorem 1.1.11 of [8] the union $\bigcup \overline{\mathscr{V}}$ is a closed subset of X, and hence any point $x \in X \setminus \bigcup \overline{\mathscr{V}}$ has an open neighbourhood $O(x) = X \setminus \bigcup \overline{\mathscr{V}}$ which does not intersect the elements of the family $\mathscr{V}$. If $x \in \mathrm{cl}_X(V_i)$ for some positive integer i, then our construction implies that $U(x_i)$ is an open neighbourhood of x which intersects only the set $V_i \in \mathscr{V}$. Hence X has an infinite discrete family $\mathscr{V}$ of non-empty open subsets in

X, which contradicts the assumption that the space X is d-feebly compact. The obtained contradiction implies the statement of the theorem. □

We finish this note by some simple remarks about dense embedding of an infinite semigroup of matrix units and a polycyclic monoid into d-feebly compact topological semigroups which follow from the results of the paper [5].

Let λ be a non-zero cardinal. On the set $B_\lambda = (\lambda \times \lambda) \cup \{0\}$, where $0 \notin \lambda \times \lambda$, we define the semigroup operation " $\cdot$ " as follows

$$(a,b)\cdot(c,d) = \begin{cases} (a,d), & \text{if } b=c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a,b)\cdot 0 = 0\cdot(a,b) = 0\cdot 0 = 0$ for $a,b,c,d \in \lambda$. The semigroup B_λ is called the *semigroup of $\lambda\times\lambda$-matrix units* (see [7]).

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$ [7]. For a non-zero cardinal λ, the polycyclic monoid P_λ on λ generators is the semigroup with zero given by the presentation:

$$P_\lambda = \left\langle \{p_i\}_{i\in\lambda}, \{p_i^{-1}\}_{i\in\lambda} \mid p_i p_i^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle$$

(see [5]). It is obvious that in the case when $\lambda = 1$ the semigroup P_1 is isomorphic to the bicyclic semigroup with adjoined zero.

By Theorem 4.4 from [5] for every infinite cardinal λ the semigroup of $\lambda\times\lambda$-matrix units B_λ does not densely embed into a Hausdorff feebly compact topological semigroup, and by Theorem 4.5 from [5] for arbitrary cardinal $\lambda \geqslant 2$ there exists no Hausdorff feebly compact topological semigroup which contains the λ-polycyclic monoid P_λ as a dense subsemigroup. These theorems and Lemma 1 imply the following two corollaries.

Corollary 2. *For every infinite cardinal λ the semigroup of $\lambda\times\lambda$-matrix units B_λ does not densely embed into a Hausdorff d-feebly compact topological semigroup.*

Corollary 3. *For arbitrary cardinal $\lambda \geqslant 2$ there exists no Hausdorff d-feebly compact topological semigroup which contains the λ-polycyclic monoid P_λ as a dense subsemigroup.*

The proof of the following corollary is similar to Theorem 5.1(5) from [4].

Corollary 4. *There exists no Hausdorff topological semigroup with the d-feebly compact square which contains the bicyclic monoid $\mathscr{C}(p,q)$ as a dense subsemigroup.*

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ПРО СЛАБКО КОМПАКТНІ ТОПОЛОГІЇ, СТОСОВНО ЯКИХ НАПІҐРАТКА $\exp_n \lambda$ МАЄ НЕПЕРЕРВНІ ЗСУВИ

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Вивчаємо слабко компактні топології τ на напівґратці $(\exp_n \lambda, \cap)$ такі, що $(\exp_n \lambda, \tau)$ є напівтопологічною напівґраткою і доведено, що для довільної T_1-топології τ на $\exp_n \lambda$, стосовно якої зсуви в $(\exp_n \lambda, \tau)$ є неперервними, такі умови еквівалентні: (i) τ – зліченно пракомпактна; (ii) τ – слабко компактна; (iii) τ – d-слабко компактна; (iv) $(\exp_n \lambda, \tau)$ – H-замнений простір.

Ключові слова: топологічна напівґратка, напівтопологічна напівґратка, зліченно пракомпактний. слабко компактний, d-слабко компактний, H-замкнений простір, напіврегулярний простір, регулярний простір.