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BOUNDED L-INDEX IN JOINT VARIABLES AND ANALYTIC SOLUTIONS OF SOME SYSTEMS OF PDE's IN BIDISC

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The linear higher-order systems of PDE's with analytic coefficients in bidisc are considered. We study boundedness of the $\mathbf{L}$-index in joint variables of their analytic solutions, where $\mathbf{L}(z_1, z_2) = (l_1(z_1, z_2),\ l_2(z_1, z_2))$, $l_j \colon \mathbb{D}^2 \to \mathbb{R}_+$ is continuous function, $j \in \{1, 2\}$, $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1,\ |z_2| < 1\}$. The main tool of investigations is known Hayman's theorem. Some analytic solutions in bidisc for these system of PDE's are given.

Key words: analytic function, bidisc, bivariate function, bounded $\mathbf{L}$-index in joint variables, linear higher-order systems of PDE, analytic theory of PDE, analytic solution, linear higher-order differential equation.

1. Introduction. Let $\mathbf{L}(z_1, z_2) = (l_1(z_1, z_2),\ l_2(z_1, z_2))$ be a positive continuous vector function in the unit bidisc $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon\ |z_1| < 1,\ |z_2| < 1\}$. Recently, we published few papers [7, 8, 9] which are devoted to the theory of analytic functions in $\mathbb{D}^2$. For these functions there was introduced a concept of bounded $\mathbf{L}$-index in joint variables (see definition below). Also we deduced counterparts of known criteria of boundedness of $\mathbf{L}$-index in joint variables. In this paper, we present an application of obtained theorems to system of PDE's.

The class of functions of bounded index have been used in the theory value distribution and differential equations (see full bibliography in [4]). In particular, every entire function is a function of bounded value distribution if and only if its derivative is a function of bounded index [15]. Similar result for entire bivariate function are deduced by F. Nuray and R. F. Patterson [21]. It is known that every entire solution of the differential equation $f^{(n)}(t) + \sum\limits_{j=0}^{n-1} a_j f^{(j)}(t) = 0$ $(t \in \mathbb{C})$ is a function of bounded index [24]. More general results for entire solutions of PDE's are obtained by A. I. Bandura and O. B. Skaskiv [1, 2, 4, 10]. But there are only two researches [13, 23] where authors study bounded index of entire solutions for systems of PDE's. In particular, M. T. Bordulyak

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[13] considered the next system

$$(1) \qquad a_j(z)f^{(K_j^0)}(z) + \sum\nolimits_{\|K\|\le s-1} g_{K,j}(z)f^{(K)}(z) = h_j(z),\ j\in\{1,\dots,m\}$$

where for all $j\in\{1,\dots,m\}$ $\|K_j^0\| = s$, $\left\{f^{(K_j^0)}(z)\right\}_{j=1}^m$ is the set of all possible s-order partial derivatives of the function f, the entire functions a_j, $g_{K,j}$, h_j are functions with separable variables of the form $g(z) = \prod_{j=1}^n g_j(z_j)$. She obtained sufficient conditions for boundedness of the **L**-index in joint variables for every entire solution of system (1), where $\mathbf{L}(z) = (l_1(|z_1|),\dots,l_n(|z_n|))$ and each function $l_j\colon \mathbb{R}_+\to\mathbb{R}_+$ is continuous. Earlier M. Salmassi [23] established that every entire solution of the system

$$(2) \qquad \begin{cases} a_0 f^{(n_1,0)}(z) + a_1 f^{(n_1-1,0)}(z) + \ldots + a_{n_1} f(z) = g(z),\ a_0\ne 0,\\ b_0 f^{(0,n_2)}(z) + b_2 f^{(0,n_2-1)}(z) + \ldots + b_{n_2} f(z) = h(z),\ b_0\ne 0,\end{cases} \quad z=(z_1,z_2),$$

is a function of bounded index in joint variables, where $a_j\in\mathbb{C}$, $b_i\in\mathbb{C}$, $h(z)$ and $g(z)$ are arbitrary entire functions in $\mathbb{C}^2$ of bounded index in joint variables. Unlike M. T. Bordulyak, he did not suppose that $h(z)$ and $g(z)$ are functions with separable variables.

In the present paper, we consider homogeneous systems of form (1) in two variables with analytic coefficients in the unit bidisc. As M. Salmassi, we do not assume that the coefficients are functions with separable variables. At first, three systems of PDE's with specified coefficients are studied. For each system we find a function $\mathbf{L}=(l_1,l_2)$ such that every its analytic solution has bounded **L**-index in joint variables. Finally, the developed methods helps us to deduce sufficient conditions providing boundedness of **L**-index of analytic solutions for a more general homogeneous system.

Note that the same problem is considered in [12] for analytic functions in the unit ball.

2. Main definitions and notations. We consider two-dimensional complex space $\mathbb{C}^2$. This helps to distinguish main methods of investigation.

We need some standard notations. Denote $\mathbb{R}_+ = [0,+\infty)$, $\mathbf{0} = (0,0)\in\mathbb{R}_+^2$, $\mathbf{1} = (1,1)\in\mathbb{R}_+^2$, $R=(r_1,r_2)\in\mathbb{R}_+^2$, $z=(z_1,z_2)\in\mathbb{C}^2$. For $A=(a_1,a_2)\in\mathbb{R}^2$, $B=(b_1,b_2)\in\mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions

$$AB = (a_1b_1, a_2b_2),\quad A/B = (a_1/b_1, a_2/b_2),\ b_1\ne 0,\ b_2\ne 0,\quad A^B = a_1^{b_1}a_2^{b_2},\ b\in\mathbb{Z}_+^2,$$

and the notation $A<B$ means that $a_j<b_j$, $j\in\{1,2\}$; the relation $A\leqslant B$ is defined similarly. For $K=(k_1,k_2)\in\mathbb{Z}_+^2$ denote $\|K\| = k_1+k_2$, $K! = k_1!k_2!$.

The bidisc $\{z\in\mathbb{C}^2:\ |z_j-z_j^0|<r_j,\ j=1,2\}$ is denoted by $\mathbb{D}^2(z^0,R)$ and the closed bidisc $\{z\in\mathbb{C}^2:\ |z_j-z_j^0|\leqslant r_j,\ j=1,2\}$ is denoted by $\mathbb{D}^2[z^0,R]$, $\mathbb{D}^2=\mathbb{D}^2(\mathbf{0},\mathbf{1})$, $\mathbb{D}=\{z\in\mathbb{C}:\ |z|<1\}$. For p, $q\in\mathbb{Z}_+$ and the partial derivative $\frac{\partial^{p+q}F(z_1,z_2)}{\partial z_1^p\partial z_2^q}$ of analytic function $F(z)$ in $\mathbb{D}^2$ we will use the notation

$$F^{(p,q)}(z) = F^{(p,q)}(z_1,z_2) := \frac{\partial^{p+q}F(z_1,z_2)}{\partial z_1^p\partial z_2^q}.$$

Let $\mathbf{L}(z) = (l_1(z), l_2(z))$, where $l_j(z):\mathbb{D}^2\to\mathbb{R}_+$ is a continuous function such that

$$(\forall z\in\mathbb{D}^2)\colon\ l_j(z) > \beta/(1-|z_j|),\ j\in\{1,2\},$$

where $\beta > 1$ is a some constant. S. N. Strochyk, M. M. Sheremeta, V. O. Kushnir [17], [27] imposed a similar condition for a function $l\colon G \to \mathbb{R}_+$, where G is arbitrary domain in $\mathbb{C}$. A. I. Bandura and O. B. Skaskiv [11, 12] used the similar condition to consider analytic functions in the unit ball of bounded $\mathbf{L}$-index in joint variables.

An analytic function $F\colon \mathbb{D}^2 \to \mathbb{C}$ [7, 8, 9] is called a function of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z = (z_1, z_2) \in \mathbb{D}^2$ and for all $(p_1, p_2) \in \mathbb{Z}_+^2$

$$(3) \qquad \frac{1}{p_1!p_2!}\frac{|F^{(p_1,p_2)}(z)|}{l_1^{p_1}(z)l_2^{p_2}(z)} \leqslant \max\left\{\frac{1}{k_1!k_2!}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} : 0 \leqslant k_1 + k_2 \leqslant n_0\right\}.$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* $F(z)$ and is denoted by $N(F, \mathbf{L}, \mathbb{D}^2) = n_0$. This is an analog of the definition of entire function of bounded $\mathbf{L}$-index or bounded index ($\mathbf{L} \equiv 1$) in joint variables in $\mathbb{C}^2$ (see [5], [20, 21, 22]) and the definition of analytic in a domain function of bounded index [16]. Note that a primary definition of entire in $\mathbb{C}$ function of bounded index was supposed by B. Lepson [19]. Other approach (so-called L-index in a direction) is considered in [4, 1, 6].

By $Q(\mathbb{D}^2)$ we denote the class of functions $\mathbf{L}$, which satisfy the condition

$$(4) \qquad (\forall r_j \in [0, \beta],\ j \in \{1, 2\})\colon\ 0 < \lambda_{1,j}(R) \leqslant \lambda_{2,j}(R) < \infty,$$

where

$$\lambda_{1,j}(R) = \inf_{z^0\in\mathbb{D}^2} \inf\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$$
$$\lambda_{2,j}(R) = \sup_{z^0\in\mathbb{D}^2} \sup\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$$

It is easy to prove that the function $L_1(z_1, z_2) = (\beta'/(1 - |z_1|), \beta'/(1 - |z_2|))$ belongs to $Q(\mathbb{D}^2)$, where $\beta' > \beta$. We need the following theorem from [8].

Теорема 1 ([8])**.** *Let* $\mathbf{L} \in Q(\mathbb{D}^2)$*. An analytic function* F *in* $\mathbb{D}^2$ *has bounded* $\mathbf{L}$*-index in joint variables if and only if there exist* $p \in \mathbb{Z}_+$ *and* $c \in \mathbb{R}_+$ *such that for each* $z = (z_1, z_2) \in \mathbb{D}^2$ *the next inequality*

$$\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)} : j_1 + j_2 = p + 1\right\} \leqslant c \cdot \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} : k_1 + k_2 \leqslant p\right\}$$

holds.

This proposition and its generalizations [26, 18, 5, 8] are analogs of known Hayman's Theorem [15] in theory of functions of bounded index. M.T. Bordulyak [14] apllied the theorem to investigate l-index boundedness of entire solutions of linear high-order differential equations. Later, A. I. Bandura and O. B. Skaskiv [2] used the same method to study bounded L-index in direction of entire solutions of PDE's. We will apply Theorem 1 to systems of PDE's.

3. General result. Let us consider the following system of PDE's

$$\begin{cases} a_{1,k,0}(z)F^{(k,0)}(z) + \sum\limits_{0\leqslant j_1+j_2\leqslant k-1} a_{1,j_1,j_2}(z)F^{(j_1,j_2)}(z) = 0, \\ a_{2,k-1,1}(z)F^{(k-1,1)}(z) + \sum\limits_{0\leqslant j_1+j_2\leqslant k-1} a_{2,j_1,j_2}(z)F^{(j_1,j_2)}(z) = 0 \\ \dots \\ a_{k+1,0,k}(z)F^{(0,k)}(z) + \sum\limits_{0\leqslant j_1+j_2\leqslant k-1} a_{k+1,j_1,j_2}(z)F^{(j_1,j_2)}(z) = 0. \end{cases} \tag{5}$$

Theorem 1. *Let* $\mathbf{L} \in Q(\mathbb{D}^2)$, $a_{i,j_1,j_2}(z)$ *be analytic functions in* $\mathbb{D}^2$, *which satisfy the following conditions for all* $z \in \mathbb{D}^2$:

$$|a_{i,j_1,j_2}(z)|l_1^{j_1}(z)l_2^{j_2}(z) \leqslant Cl_1^{k+1-i}(z)l_2^{i-1}(z)|a_{i,k+1-i,i-1}(z)|, \quad a_{i,k+1-i,i-1}(z) \neq 0, \tag{6}$$

where $0 \le j_1 + j_2 \le k-1$, $i \in \{1,\dots,k+1\}$ *and* $C > 0$ *is some constant. If analytic function* $F(z_1,z_2)$ *in* $\mathbb{D}^2$ *is a solution of* (5) *then* $F(z_1,z_2)$ *is of bounded* **L**-*index in joint variables.*

Proof. From the first equation we have:

$$|F^{(k,0)}(z)| \leqslant \frac{\Big|\sum\limits_{0\leqslant j_1+j_2\leqslant k-1} a_{1,j_1,j_2}(z)F^{(j_1,j_2)}(z)\Big|}{|a_{1,k,0}(z)|} \leqslant$$

$$\leqslant \sum_{0\leqslant j_1+j_2\leqslant k-1} \frac{|a_{1,j_1,j_2}(z)|}{|a_{1,k,0}(z)|}|F^{(j_1,j_2)}(z)| \leqslant \sum_{0\leqslant j_1+j_2\leqslant k-1} Cl_1^{k-j_1}(z)l_2^{-j_2}(z)|F^{(j_1,j_2)}(z)|,$$

whence

$$\frac{|F^{(k,0)}(z)|}{l_1^k(z)l_2^0(z)} \leqslant \sum_{0\leqslant j_1+j_2\leqslant k-1} Cl_1^{-j_1}(z)l_2^{-j_2}(z)|F^{(j_1,j_2)}(z)| \leqslant C_1 \max_{0\leqslant j_1+j_2\leqslant k-1} \frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)}$$

Similarly, for i-th equation of system (5) it can be proved that

$$\frac{|F^{(k-i,i)}(z)|}{l_1^{k-i}(z)l_2^i(z)} \leqslant C_i \max_{0\leqslant j_1+j_2\leqslant k-1} \frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)}$$

Finally, we obtain

$$\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)} : k_1+k_2=k\right\} \leqslant \max_{1\le i\le k+1} C_i \cdot \max_{0\leqslant j_1+j_2\leqslant k-1} \frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)}.$$

By Theorem 1 this means that $F(z_1,z_2)$ is of bounded **L**-index in joint variables. □

To formulate one-dimensional corollary, we consider the following linear higher-order differential equation:

$$a_k(t)f^{(k)}(t) + \sum_{j=0}^{k-1} a_j(t)f^{(j)}(t) = 0. \tag{7}$$

Corollary 1. *Let* $l \in Q(\mathbb{D})$ *and analytic functions* $a_j(t)$ *in* $\mathbb{D}$ *satisfy the conditions:*

$$(\forall t \in \mathbb{D}) \quad (\forall j \in \{0,1,\dots,k-1\}) \quad |a_j(t)|l^j(t) \leqslant Cl^k(t)|a_k(t)|, \quad a_k(t) \neq 0, \tag{8}$$

where $C > 0$ is a some constant. If an analytic function f in $\mathbb{D}$ satisfies equation (7) *then f has bounded l-index.*

M. Sheremeta [25] considered equation (7) with coefficients which are analytic functions of bounded l-index. As M. Bordulyak in [14] for entire functions, we replaced the restriction by inequality (8). Thus, Corollary 1 is a new proposition for analytic functions in the unit disc.

4. Examples of linear higher-order systems of PDE.

Corollary 2. *Every analytic function $F(z_1, z_2)$ in $\mathbb{D}^2$, satisfying the following system of partial differential equations*

$$(9)\qquad \begin{cases} 2F^{(2,0)}(z_1,z_2) + \frac{3}{(z_1-1)} \cdot F^{(1,0)}(z_1,z_2) + \frac{1}{2} \cdot \frac{1}{(z_1-1)^3(z_2+1)} \cdot F(z_1,z_2) = 0, \\ 2F^{(0,2)}(z_1,z_2) + \frac{3}{(z_2+1)} \cdot F^{(0,1)}(z_1,z_2) + \frac{1}{2} \cdot \frac{1}{(z_1-1)(z_2+1)^3} \cdot F(z_1,z_2) = 0, \\ 2F^{(1,1)}(z_1,z_2) + \frac{1}{(z_2+1)} \cdot F^{(1,0)}(z_1,z_2) + \frac{1}{2} \cdot \frac{1}{(z_1-1)^2(z_2+1)^2} \cdot F(z_1,z_2) = 0, \end{cases}$$

is a function of bounded **L**-*index in joint variables, where*

$$(10)\quad \mathbf{L}(z) = (l_1(z), l_2(z)) = \left(\frac{1}{\sqrt{|1-z_1||1+z_2|}(1-|z_1|)}, \frac{1}{\sqrt{|1-z_1||1+z_2|}(1-|z_2|)} \right).$$

Proof. Using (4) it is easy to show that the function **L** belongs to $Q(\mathbb{D}^2)$ (see similar propositions in [3]). Now we check validity of (6) for system (9):

$$\frac{3l_1(z)}{|z_1-1|} = \frac{3}{|z_1-1|\sqrt{|1-z_1||1+z_2|}(1-|z_1|)} \le \frac{6}{(\sqrt{|1-z_1||1+z_2|}(1-|z_1|))^2} \le 2Ml_1^2(z),$$

$$\frac{3l_2(z)}{|z_2+1|} = \frac{3}{|z_2+1|\sqrt{|1-z_1||1+z_2|}(1-|z_2|)} \le \frac{6}{(\sqrt{|1-z_1||1+z_2|}(1-|z_2|))^2} \le 2Ml_2^2(z),$$

$$\frac{l_1(z)}{|z_2+1|} = \frac{1}{|z_2+1|\sqrt{|1-z_1||1+z_2|}(1-|z_1|)} \le \frac{2}{|1-z_1||1+z_2|(1-|z_1|)(1-|z_2|)} \le$$
$$\le 2Ml_1(z)l_2(z),$$

$$\frac{1}{2|z_1-1|^3|z_2+1|} \le \frac{1}{2|1-z_1||1+z_2|(1-|z_1|)^2} \le 2Ml_1^2(z),$$

$$\frac{1}{2|z_1-1||z_2+1|^3} \le \frac{1}{2|1-z_1||1+z_2|(1-|z_2|)^2} \le 2Ml_2^2(z),$$

$$\frac{1}{2|z_1-1|^2|z_2+1|^2} \le \frac{1/2}{|1-z_1||1+z_2|(1-|z_1|)(1-|z_2|)} \le 2Ml_1(z)l_2(z).$$

Hence, by Theorem 1 the function F has bounded **L**-index in joint variables. □

Example 1. The function

$$F(z_1,z_2) = \cos\frac{1}{\sqrt{(z_1-1)(z_2+1)}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!((z_1-1)(z_2+1))^n},$$

is a solution of system (9). Indeed, formally find all first and second order partial derivatives

$$F^{(1,0)}(z_1,z_2)=\sin\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{1}{2(z_1-1)\sqrt{(z_1-1)(z_2+1)}},$$

$$F^{(0,1)}(z_1,z_2)=\sin\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{1}{2\sqrt{(z_1-1)(z_2+1)}(z_2+1)},$$

$$F^{(2,0)}(z_1,z_2)=-\cos\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{1}{4(z_1-1)^3(z_2+1)}-$$
$$-\sin\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{3}{4(z_1-1)^2\sqrt{(z_1-1)(z_2+1)}},$$
$$F^{(0,2)}(z_1,z_2)=-\cos\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{1}{4(z_1-1)(z_2+1)^3}-$$
$$-\sin\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{3}{4\sqrt{(z_1-1)(z_2+1)}(z_2+1)^2},$$
$$F^{(1,1)}(z_1,z_2)=-\cos\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{1}{4(z_1-1)^2(z_2+1)^2}-$$
$$-\sin\frac{1}{\sqrt{(z_1-1)(z_2+1)}}\cdot\frac{1}{4(\sqrt{(z_1-1)(z_2+1)})^3}.$$

We choose a branch of square root such that $\sqrt{1}=1$ and take into account that $\frac{1}{\sqrt{w}}\sin\frac{1}{\sqrt{w}}$ can be extended to an entire function. Substituting these derivatives in system (9), it is easy to check that the function $F(z_1,z_2)$ is its solution. Then by Corollary 2 the function F has bounded **L**-index in joint variables with **L** given in (10).

Corollary 3. *Every analytic function $F(z_1,z_2)$ in $\mathbb{D}^2$, satisfying the following system of partial differential equations*

$$(11)\qquad \begin{cases} F^{(2,0)}(z_1,z_2)+\frac{1}{(z_1+1)^2(z_2-1)}\cdot F^{(1,0)}(z_1,z_2)-\frac{2}{(z_1+1)^3(z_2-1)}\cdot F(z_1,z_2)=0\\ F^{(0,2)}(z_1,z_2)+\frac{1}{(z_1+1)(z_2-1)^2}\cdot F^{(0,1)}(z_1,z_2)-\frac{2}{(z_1+1)(z_2-1)^3}\cdot F(z_1,z_2)=0\\ F^{(1,1)}(z_1,z_2)+\frac{1}{(z_1+1)(z_2-1)^2}\cdot F^{(1,0)}(z_1,z_2)-\frac{1}{(z_1+1)^2(z_2-1)^2}\cdot F(z_1,z_2)=0, \end{cases}$$

is a function of bounded **L***-index in joint variables, where*

$$(12)\qquad \mathbf{L}(z)=(l_1(z),l_2(z))=\left(\frac{1}{|1+z_1||1-z_2|(1-|z_1|)},\frac{1}{|1+z_1||1-z_2|(1-|z_2|)}\right).$$

To prove Corollary 3 it is sufficient to check assumptions of Theorem 1.

Example 2. Analytic function in the unit bidisc

$$F(z_1,z_2)=\exp\left(\frac{1}{(z_1+1)(z_2-1)}\right),$$

satisfies system of partial differential equations (11). Hence, by Corollary 3 the function F is of bounded **L**-index in joint variables, where **L** is given in (12).

Corollary 4. *An analytic function $F(z_1,z_2)$ in $\mathbb{D}^2$, satisfying the following system of partial differential equations*

$$
(13)\qquad \begin{cases} F^{(2,0)}(z_1,z_2)+\frac{2}{(z_1+1)}\cdot F^{(1,0)}(z_1,z_2)-\frac{1}{(z_2-1)^2}\cdot F(z_1,z_2)=0\\ F^{(0,2)}(z_1,z_2)+\frac{2+z_1}{z_2-1}\cdot F^{(0,1)}(z_1,z_2)-\frac{2+z_1}{(z_2-1)^2}\cdot F(z_1,z_2)=0\\ F^{(1,1)}(z_1,z_2)+\frac{1}{(z_1+1)}\cdot F^{(0,1)}(z_1,z_2)+\frac{z_1}{(z_2-1)^3}\cdot F(z_1,z_2)=0, \end{cases}
$$

has bounded $\mathbf{L}$*-index in joint variables, where*

$$
(14)\qquad \mathbf{L}(z)=(l_1(z),l_2(z))=\left(\frac{1+\frac{1}{|z_2-1|}}{1-|z_1|};\frac{1+\frac{|z_1|}{|z_2-1|}}{1-|z_2|}\right).
$$

In view of Theorem 1 it is necessary to check validity of (6) (see a scheme in Corollary 2).

Example 3. Analytic function in the unit bidisc

$$
F(z_1,z_2)=\frac{z_2-1}{z_1+1}\exp\frac{z_1}{z_2-1},
$$

is a solution of system of partial differential equations (13). By Corollary 4 the function F has bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}$ is given in (14).

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ОБМЕЖЕНІСТЬ L-ІНДЕКСУ ЗА СУКУПНІСТЮ ЗМІННИХ ТА АНАЛІТИЧНІ У БІКРУЗІ РОЗВ'ЯЗКИ ДЕЯКИХ СИСТЕМ РЧП

Наталія ПЕТРЕЧКО

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Розглянуто системи лінійних рівнянь з частинними похідними вищих порядків з аналітичними у бікрузі коефіцієнтами. Досліджено обмеженість **L**-індексу за сукупністю змінних їхніх аналітичних розв'язків, де $\mathbf{L}(z_1, z_2) = (l_1(z_1, z_2), \ l_2(z_1, z_2))$, $l_j : \mathbb{D}^2 \to \mathbb{R}_+$ — неперервна функція, $j \in \{1, 2\}$, $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2\colon \ |z_1| < 1, \ |z_2| < 1\}$. Основним засобом дослідження є теорема Хеймана. Наведено деякі аналітичні у бікрузі розв'язки таких систем рівнянь з частинними похідними.

Ключові слова: аналітична функція, бікруг, функція двох змінних, обмежений **L**-індекс за сукупністю змінних, система лінійних РЧП вищих порядків, аналітична теорія диференціальних РЧП, аналітичний розв'язок, лінійне диференціальне рівняння вищих порядків.