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ON VARIANTS OF THE EXTENDED BICYCLIC SEMIGROUP

Oleg GUTIK, Kateryna MAKSYMYK

Ivan Franko National University of Lviv,
Universytetska Str., 1, 79000, Lviv,
e-mails: oleg.gutik@lnu.edu.ua, ovgutik@yahoo.com,
kate.maksymyk15@gmail.com

In the paper we describe the group $\mathbf{Aut}\,(\mathscr{C}_{\mathbb{Z}})$ of automorphisms of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and study the variants $\mathscr{C}_{\mathbb{Z}}^{m,n}$ of the extended bicycle semigroup $\mathscr{C}_{\mathbb{Z}}$, where $m,n\in\mathbb{Z}$. In particular, we prove that $\mathbf{Aut}\,(\mathscr{C}_{\mathbb{Z}})$ is isomorphic to the additive group of integers, the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and every its variant are not finitely generated, and describe the subset of idempotents $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ and Green's relations on the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$. Also we show that $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ is an ω-chain and any two variants of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isomorphic. At the end we discuss shift-continuous Hausdorff topologies on the variant $\mathscr{C}_{\mathbb{Z}}^{0,0}$. In particular, we prove that if τ is a Hausdorff shift-continuous topology on $\mathscr{C}_{\mathbb{Z}}^{0,0}$ then each of inequalities $a>0$ or $b>0$ implies that (a,b) is an isolated point of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and construct an example of a Hausdorff semigroup topology τ^* on the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$ such that all its points with $ab\leqslant 0$ and $a+b\leqslant 0$ are not isolated in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$.

Key words: Semigroup, interassociate of a semigroup, variant of a semigroup, bicyclic monoid, extended bicyclic semigroup, semitopological semigroup, topological semigroup.

1. Introduction and preliminaries.

We shall follow the terminology of [7, 11, 20, 37]. In this paper all spaces are assumed to be Hausdorff. By $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ we denote the sets of all integers, non-negative integers and positive integers, respectively.

A *semigroup* is a non-empty set with a binary associative operation.

If S is a semigroup, then we shall denote the Green relations on S by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [11]). For every $a\in S$ by $\mathbf{R}_a$, $\mathbf{L}_a$ and $\mathbf{H}_a$ we denote the $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-class in S which contains the element a, respectively. A semigroup S is called *simple* if S does not contain proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If $E(S)$ is closed under multiplication, we shall refer to $E(S)$ a as *band* (or the *band of*

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S). The semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is not properly contained in any other chain of E.

The Axiom of Choice implies the existence of maximal chains in every partially ordered set. According to [35, Definition II.5.12], a chain L is called an ω-chain if L is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$ or equivalently, if L is isomorphic to $(\mathbb{N}_0, \max)$.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p, q)$ is the semigroup with the identity 1 generated by two elements p and q subject only to the condition $pq = 1$. The bicyclic monoid $\mathscr{C}(p, q)$ is a combinatorial bisimple F-inverse semigroup (see [34]) and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known O. Andersen's result [1] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup cannot be embedded into the stable semigroups [33].

An interassociate of a semigroup $(S, \cdot)$ is a semigroup $(S, *)$ such that for all $a, b, c \in S$, $a \cdot (b * c) = (a \cdot b) * c$ and $a * (b \cdot c) = (a * b) \cdot c$. This definition of interassociativity was studied extensively in 1996 by Boyd, Gould, and Nelson in [6]. Certain classes of semigroups are known to give rise to interassociates with various properties. For example, it is very easy to show that if S is a monoid, every interassociate must satisfy the condition $a * b = a \cdot c \cdot b$ for some fixed element $c \in S$ (see [6]). This type of interassociate was called a *variant* by Hickey [28]. Variants of semigroups of binary relations have been studied by Chase [8, 9, 10]. Variants of transformation semigroups and their representations have been studied [16, 17, 18, 31, 36]. A general theory of variants has been developed by a number of authors; see especially [28, 29, 32]. For a recent study of variants of finite full transformation semigroups, and for further references and historical discussion, see [16] and also [23, Chapter 13]. Variants of semilattices were studied in [12, 22]. The articles [13, 14, 15, 16] initiated the study of general sandwich semigroups in arbitrary (locally small) categories. In addition, every interassociate of a completely simple semigroup is completely simple [6]. Finally, it is relatively easy to show that every interassociate of a group is isomorphic to the group itself.

In the paper [24] the bicyclic semigroup $\mathscr{C}(p, q)$ and its interassociates are investigated. In particular, if p and q are generators of the bicyclic semigroup $\mathscr{C}(p, q)$ and m and n are fixed nonnegative integers, the operation $a *_{m,n} b = a \cdot q^m p^n \cdot b$ is known to be an interassociate. It was shown that for distinct pairs (m, n) and (s, t), the interassociates $(\mathscr{C}(p, q), *_{m,n})$ and $(\mathscr{C}(p, q), *_{s,t})$ are not isomorphic. Also in [24] the authors generalized a result regarding homomorphisms on $\mathscr{C}(p, q)$ to homomorphisms on its interassociates. Later for fixed non-negative integers m and n the interassociate $(\mathscr{C}(p, q), *_{m,n})$ of the bicyclic monoid $\mathscr{C}(p, q)$ will be denoted by $\mathscr{C}_{m,n}$.

A (*semi*)*topological semigroup* is a topological space with a (separately) continuous semigroup operation. A topology τ on a semigroup S is called:

- *shift-continuous* if (S, τ) is a semitopological semigroup;
- *semigroup* if (S, τ) is a topological semigroup.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{C}(p,q)$ is an open subset of S [19]. Bertman and West in [5] extend this result for the case of Hausdorff semitopological semigroups. The stable and Γ-compact topological semigroups do not contain the bicyclic semigroup [2, 30]. The problem of embedding of the bicyclic monoid into compact-like topological semigroups studied in [3, 4, 27]. Also in the paper [21] it was proved that the discrete topology is the unique topology on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ such that the semigroup operation on $\mathscr{C}_{\mathbb{Z}}$ is separately continuous. Amazing dichotomy for the bicyclic monoid with adjoined zero $\mathscr{C}^0=\mathscr{C}(p,q)\sqcup\{0\}$ was proved in [25]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathscr{C}^0$ is either compact or discrete.

In the paper [26] we studied semitopological interassociates $(\mathscr{C}(p,q),*_{m,n})$ of the bicyclic monoid $\mathscr{C}(p,q)$ for arbitrary non-negative integers m and n. Some results from [5, 19, 25] obtained for the bicyclic semigroup to its interassociate $(\mathscr{C}(p,q),*_{m,n})$ were extended. In particular, we showed that for arbitrary non-negative integers m, n and every Hausdorff topology τ on $\mathscr{C}_{m,n}$ such that $(\mathscr{C}_{m,n},\tau)$ is a semitopological semigroup, is discrete. Also, we proved that if an interassociate of the bicyclic monoid $\mathscr{C}_{m,n}$ is a dense subsemigroup of a Hausdorff semitopological semigroup $(S,\cdot)$ and $I=S\setminus\mathscr{C}_{m,n}\neq\varnothing$ then I is a two-sided ideal of the semigroup S and show that for arbitrary non-negative integers m, n, any Hausdorff locally compact semitopological semigroup $\mathscr{C}^0_{m,n}$ ($\mathscr{C}^0_{m,n}=\mathscr{C}_{m,n}\sqcup\{0\}$) is either discrete or compact.

On the Cartesian product $\mathscr{C}_{\mathbb{Z}}=\mathbb{Z}\times\mathbb{Z}$ we define the semigroup operation as follows:

$$(a,b)\cdot(c,d)=\begin{cases}(a-b+c,d), & \text{if } b<c;\\ (a,d), & \text{if } b=c;\\ (a,d+b-c), & \text{if } b>c,\end{cases} \tag{1}$$

for $a,b,c,d\in\mathbb{Z}$. The set $\mathscr{C}_{\mathbb{Z}}$ with such defined operation will be called the *extended bicyclic semigroup* [38].

In the paper [21] algebraic properties of $\mathscr{C}_{\mathbb{Z}}$ were described and it was proved therein that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. Also it was shown that the semigroup $\mathscr{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology and the closure $\operatorname{cl}_T(\mathscr{C}_{\mathbb{Z}})$ of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup T was studied.

In this paper we describe the group $\mathbf{Aut}(\mathscr{C}_{\mathbb{Z}})$ of automorphisms of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and study a variant $\mathscr{C}_{\mathbb{Z}}^{m,n}=(\mathscr{C}_{\mathbb{Z}},*_{m,n})$ of the extended bicycle semigroup $\mathscr{C}_{\mathbb{Z}}$, where $m,n\in\mathbb{Z}$, which is defined by the formula

$$(a,b)*_{m,n}(c,d)=(a,b)\cdot(m,n)\cdot(c,d). \tag{2}$$

In particular, we prove that $\mathbf{Aut}(\mathscr{C}_{\mathbb{Z}})$ is isomorphic to the additive group of integers, the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and every its variant are not finitely generated, and describe the subset of idempotents $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ and Green's relations on the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$. Also we show that $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ is an ω-chain and any two variants of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isomorphic. At the end we discuss shift-continuous Hausdorff topologies on the variant $\mathscr{C}_{\mathbb{Z}}^{0,0}$. In particular, we prove that if τ is a Hausdorff shift-continuous topology on $\mathscr{C}_{\mathbb{Z}}^{0,0}$ then each of inequalities $a>0$ or $b>0$ implies that (a,b) is an isolated point of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and constructe an example a Hausdorff semigroup topology τ^* on the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$ such that all its points with the properties $ab\leqslant 0$ and $a+b\leqslant 0$ are not isolated in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$.

2. On the group of automorphisms of the extended bicyclic semigroup.

Lemma 1. *For arbitrary integer k the set*

$$\mathscr{C}_{\mathbb{Z}}^{\geqslant k}=\{(i,j)\in\mathscr{C}_{\mathbb{Z}}\colon i,j\geqslant k\}$$

with the induced from $\mathscr{C}_{\mathbb{Z}}$ semigroup operation is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$ by the mapping $h_k\colon\mathscr{C}(p,q)\to\mathscr{C}_{\mathbb{Z}}^{\geqslant k}$, $q^ip^j\mapsto(i+k,j+k)$.

Proof. Since

$$\begin{aligned}
h_k\left(q^mp^n\cdot q^ip^j\right)&=\begin{cases} h_k\left(q^{m-n+i}p^j\right), & \text{if } m\leqslant i;\\ h_k\left(q^np^{m-i+j}\right), & \text{if } m\geqslant i\end{cases}=\\
&=\begin{cases} (m-n+i+k,j+k), & \text{if } m\leqslant i;\\ (n+k,m-i+j+k), & \text{if } m\geqslant i\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
h_k\left(q^mp^n\right)\cdot h_k\left(q^ip^j\right)&=(m+k,n+k)\cdot(i+k,j+k)=\\
&=\begin{cases} (m-n+i+k,j+k), & \text{if } m\leqslant i;\\ (n+k,m-i+j+k), & \text{if } m\geqslant i.\end{cases}
\end{aligned}$$

for any $q^mp^n,q^ip^j\in\mathscr{C}(p,q)$ we have that the so defined map $h_k\colon\mathscr{C}(p,q)\to\mathscr{C}_{\mathbb{Z}}^{\geqslant k}$ is a homomorphism, and simple verifications imply that it is a bijection. □

Theorem 1. *For an arbitrary integer k the map $h_k\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ defined by by the formula*

$$h_k\left((i,j)\right)=(i+k,j+k), \tag{3}$$

is an automorphism of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and every automorphism $\mathfrak{h}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$ of $\mathscr{C}_{\mathbb{Z}}$ has form (3). Moreover, the group $\mathbf{Aut}\left(\mathscr{C}_{\mathbb{Z}}\right)$ of automorphisms of $\mathscr{C}_{\mathbb{Z}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$ and this isomorphism $\mathfrak{H}\colon\mathbb{Z}(+)\to\mathbf{Aut}\left(\mathscr{C}_{\mathbb{Z}}\right)$ is defined by the formula $\mathfrak{H}(k)=h_k$, $k\in\mathbb{Z}$.

Proof. For any $(m,n),(i,j)\in\mathscr{C}_{\mathbb{Z}}$ we have that

$$\begin{aligned}
h_k\left((m,n)\cdot(i,j)\right)&=\begin{cases} h_k\left((m-n+i,j)\right), & \text{if } m\leqslant i;\\ h_k\left((n,m-i+j)\right), & \text{if } m\geqslant i\end{cases}=\\
&=\begin{cases} (m-n+i+k,j+k), & \text{if } m\leqslant i;\\ (n+k,m-i+j+k), & \text{if } m\geqslant i\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
h_k\left((m,n)\right)\cdot h_k\left((i,j)\right)&=(m+k,n+k)\cdot(i+k,j+k)=\\
&=\begin{cases} (m-n+i+k,j+k), & \text{if } m\leqslant i;\\ (n+k,m-i+j+k), & \text{if } m\geqslant i.\end{cases}
\end{aligned}$$

Simple verifications imply that for every integer k the so defined map h_k is a bijection, and hence it is an automorphism of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$.

Let $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$ be an arbitrary automorphism of $\mathscr{C}_{\mathbb{Z}}$. Since $(0,0)$ is an idempotent of $\mathscr{C}_{\mathbb{Z}}$, $\mathfrak{h}\left((0,0)\right)$ is an idempotent of $\mathscr{C}_{\mathbb{Z}}$ as well, and hence by Proposition 2.1(i) from [21] we have that $\mathfrak{h}\left((0,0)\right) = (k,k)$ for some integer k. Since $(1,1)$ is the maximum of the subset

$$\{(n,n) \in E(\mathscr{C}_{\mathbb{Z}})\colon (n,n) \preccurlyeq (0,0)\} \setminus \{(0,0)\}$$

of the poset $(E(\mathscr{C}_{\mathbb{Z}}), \preccurlyeq)$ and $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$ is an automorphism of $\mathscr{C}_{\mathbb{Z}}$ we get that $\mathfrak{h}\left((1,1)\right) = (k+1,k+1)$ because $(k+1,k+1)$ is the maximum of the subset

$$\{(n,n) \in E(\mathscr{C}_{\mathbb{Z}})\colon (n,n) \preccurlyeq (k,k) = \mathfrak{h}\left((0,0)\right)\} \setminus \{(k,k)\}$$

of the poset $(E(\mathscr{C}_{\mathbb{Z}}), \preccurlyeq)$. Then by induction we obtain that $\mathfrak{h}\left((i,i)\right) = (i+k,i+k)$ for every positive integer i. Also, since $(-1,-1)$ is the minimum of the subset

$$\{(n,n) \in E(\mathscr{C}_{\mathbb{Z}})\colon (0,0) \preccurlyeq (n,n)\} \setminus \{(0,0)\}$$

of the poset $(E(\mathscr{C}_{\mathbb{Z}}), \preccurlyeq)$ and $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$ is an automorphism of $\mathscr{C}_{\mathbb{Z}}$ we obtain that $\mathfrak{h}\left((-1,-1)\right) = (k-1,k-1)$ because $(k-1,k-1)$ is the minimum of the subset

$$\{(n,n) \in E(\mathscr{C}_{\mathbb{Z}})\colon (k,k) = \mathfrak{h}\left((0,0)\right) \preccurlyeq (n,n)\} \setminus \{(k,k)\}$$

of the poset $(E(\mathscr{C}_{\mathbb{Z}}), \preccurlyeq)$. Then by induction we get that $\mathfrak{h}\left((-i,-i)\right) = (-i+k,-i+k)$ for every positive integer i.

Since $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$ is an automorphism of $\mathscr{C}_{\mathbb{Z}}$, $\mathscr{C}_{\mathbb{Z}}$ is an inverse semigroup and by Proposition 2.1(iv) of [21] every $\mathscr{H}$-class in $\mathscr{C}_{\mathbb{Z}}$ is a singleton, the equalities

$$\mathbf{L}_{(i,j)} = \mathbf{L}_{(j,j)}, \qquad \mathbf{R}_{(i,j)} = \mathbf{R}_{(i,i)} \qquad \text{and} \qquad \mathbf{H}_{(i,j)} = \mathbf{L}_{(i,j)} \cap \mathbf{R}_{(i,j)}$$

imply that

$$\mathbf{L}_{\mathfrak{h}((i,j))} = \mathbf{L}_{\mathfrak{h}((j,j))}, \qquad \mathbf{R}_{\mathfrak{h}((i,j))} = \mathbf{R}_{\mathfrak{h}((i,i))} \qquad \text{and} \qquad \mathbf{H}_{\mathfrak{h}((i,j))} = \mathbf{L}_{\mathfrak{h}((i,j))} \cap \mathbf{R}_{\mathfrak{h}((i,j))},$$

and hence we have that

$$\{\mathfrak{h}((i,j))\} = \mathbf{H}_{\mathfrak{h}((i,j))} = \mathbf{L}_{\mathfrak{h}((i,j))} \cap \mathbf{R}_{\mathfrak{h}((i,j))} = \mathbf{L}_{(i+k,j+k)} \cap \mathbf{R}_{(i+k,j+k)} = \{(i+k,j+k)\},$$

for all integers i and j. This completes the proof of the first statement of the theorem.

For arbitrary integers k_1 and k_2 we have that

$$\begin{aligned}
(h_{k_1} \circ h_{k_2})(i,j) &= h_{k_1}\left(h_{k_2}\left((i,j)\right)\right) = \\
&= h_{k_1}\left((i+k_2, j+k_2)\right) = \\
&= (i+k_2+k_1, j+k_2+k_1) = \\
&= h_{k_1+k_2}(i,j),
\end{aligned}$$

$h_0\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$, $(i,j) \mapsto (i,j)$ is the identity automorphism of $\mathscr{C}_{\mathbb{Z}}$ and $h_{-k_1}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$, $(i,j) \mapsto (i-k_1, j-k_1)$ is the converse map to the map $h_{k_1}\colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}$. This completes the proof of the second statement of the theorem. □

Serhii Bardyla asked the following question on the Seminar on S-act Theory and Spectral Spaces at Lviv University.

Question. *Are the semigroups $\mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_{\mathbb{Z}}^{m,n}$, $m,n \in \mathbb{Z}$, finitely generated?*

Later we shall give a negative answer to this question.

Lemma 2. *For every finite subset $F = \{(i_1, j_1), \ldots, (i_n, j_n)\}$ of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ there exists a subsemigroup S of $\mathscr{C}_{\mathbb{Z}}$ such that S is isomorphic to the bicyclic semigroup and S contains the semigroup $\langle F\rangle$ which is generated by the set F. Moreover $\langle F\rangle$ is a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{\geqslant k}$, where $k = \min\{i_1, j_1, \ldots, i_n, j_n\}$.*

Proof. By Lemma 1, $\mathscr{C}_{\mathbb{Z}}^{\geqslant k}$ is an inverse subsemigroup of $\mathscr{C}_{\mathbb{Z}}$ for any integer k and formula (1) implies that $\langle F\rangle$ is a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{\geqslant k}$ for $k = \min\{i_1, j_1, \ldots, i_n, j_n\}$ □

The following theorem is a consequence of Lemma 2.

Theorem 2. *The extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ is not finitely generated as an inverse semigroup.*

Corollary 1. *The extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ is not finitely generated as a semigroup.*

3. Algebraic properties of the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$.

Since a semigroup S is simple if and only if $SsS = S$ for every $s \in S$ we have that $S(csc)S = S$ for all $s, c \in S$. Since the extended bicycle semigroup $\mathscr{C}_{\mathbb{Z}}$ is simple, the above arguments imply the following property of the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$.

Proposition 1. *$\mathscr{C}_{\mathbb{Z}}^{m,n}$ is a simple semigroup for all $m, n \in \mathbb{Z}$.*

Proposition 2. *Let m and n be arbitrary integers. Then an element (a, b) of the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$ is an idempotent if and only if $(a, b) = (n + i, m + i)$ for some $i \in \mathbb{N}_0$.*

Proof. ($\Leftarrow$) Suppose that $a = n + i$ and $b = m + i$ for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}
(a, b) *_{m,n} (a, b) &= (n + i, m + i) \cdot (m, n) \cdot (n + i, m + i) = \\
&= (n + i, m + i - m + n) \cdot (n + i, m + i) = \\
&= (n + i, i + n) \cdot (n + i, m + i) = \\
&= (n + i, m + i) = \\
&= (a, b).
\end{aligned}$$

($\Rightarrow$) Formulae (1), (2), and items (ix) and (x) of Proposition 2.1 [21] imply that for any element (a, b) of the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$ we have that

$$\begin{aligned}
(a, b) *_{m,n} \mathscr{C}_{\mathbb{Z}}^{m,n} &= (a, b) \cdot (m, n) \cdot \mathscr{C}_{\mathbb{Z}} = \\
&= \begin{cases} (a, b - m + n) \cdot \mathscr{C}_{\mathbb{Z}}, & \text{if } b \geqslant m; \\ (a - b + m, n) \cdot \mathscr{C}_{\mathbb{Z}}, & \text{if } b < m \end{cases} = \\
&= \begin{cases} \{(x, y) \in \mathscr{C}_{\mathbb{Z}}^{m,n} \colon x \geqslant a\}, & \text{if } b \geqslant m; \\ \{(x, y) \in \mathscr{C}_{\mathbb{Z}}^{m,n} \colon x \geqslant a - b + m\}, & \text{if } b < m \end{cases}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\mathscr{C}_{\mathbb{Z}}^{m,n} *_{m,n} (a, b) &= \mathscr{C}_{\mathbb{Z}} \cdot (m, n) \cdot (a, b) = \\
&= \begin{cases} \mathscr{C}_{\mathbb{Z}} \cdot (a - n + m, b), & \text{if } a \geqslant n; \\ \mathscr{C}_{\mathbb{Z}} \cdot (m, n - a + b), & \text{if } a < n \end{cases} = \\
&= \begin{cases} \{(x, y) \in \mathscr{C}_{\mathbb{Z}}^{m,n} \colon y \geqslant b\}, & \text{if } a \geqslant n; \\ \{(x, y) \in \mathscr{C}_{\mathbb{Z}}^{m,n} \colon y \geqslant n - a + b\}, & \text{if } a < n. \end{cases}
\end{aligned} \tag{5}$$

Since

$$(a,b) = (a,b) *_{m,n} (a,b) \subseteq (a,b) *_{m,n} \mathscr{C}_{\mathbb{Z}}^{m,n} \cap \mathscr{C}_{\mathbb{Z}}^{m,n} *_{m,n} (a,b),$$

formulae (4), (5) imply that $b \geqslant m$ and $a \geqslant n$. Then

$$\begin{aligned}(a,b) *_{m,n} (a,b) &= (a,b)\cdot(m,n)\cdot(a,b) = \\ &= (a, b-m+n)\cdot(a,b) = \\ &= \begin{cases} (2a-b-n+m, b), & \text{if } a \geqslant b-m+n; \\ (a, 2b-a-m+n), & \text{if } a < b-m+n \end{cases}\end{aligned}$$

and hence the equality $(a,b) *_{m,n} (a,b) = (a,b)$ implies that $a-b = n-m$. Since a and b are integers such that $b \geqslant m$ and $a \geqslant n$ the equality $a-b=n-m$ implies that $(a,b) = (n+i, m+i)$ for some non-negative integer i. □

Since $E(\mathscr{C}_{\mathbb{Z}}^{m,n}) = \{(n+i, m+i) : i \in \mathbb{N}_0\}$ we denote the idempotent $(n+i, m+i)$ of $\mathscr{C}_{\mathbb{Z}}^{m,n}$ by e_i for arbitrary $i \in \mathbb{N}_0$.

Lemma 3. *Let m and n be arbitrary integers. Then $e_i \preccurlyeq e_j$ in $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ if and only if $j \leqslant i$ and hence $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ is an ω-chain.*

Proof. If $e_i \preccurlyeq e_j$ in $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ then

$$\begin{aligned}e_i *_{m,n} e_j &= (n+i, m+i)\cdot(m,n)\cdot(n+j, m+j) = \\ &= (n+i, n+i)\cdot(n+j, m+j) = \\ &= (n+i, m+i) = \\ &= e_i\end{aligned}$$

and

$$\begin{aligned}e_j *_{m,n} e_i &= (n+j, m+j)\cdot(m,n)\cdot(n+i, m+i) = \\ &= (n+j, n+j)\cdot(n+i, m+i) = \\ &= (n+i, m+i) = \\ &= e_i\end{aligned}$$

imply that $j \leqslant i$. The converse statement follows from the semigroup operation of $\mathscr{C}_{\mathbb{Z}}^{m,n}$.

An isomorphism $\varphi \colon E(\mathscr{C}_{\mathbb{Z}}^{m,n}) \to (\mathbb{N}_0, \max)$ we define by the formula $\varphi(e_i) = i$, $i \in \mathbb{N}_0$. □

The following proposition describes Green's relations on the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$.

Proposition 3. *Let m and n be arbitrary integers, (a,b) and (c,d) be elements of $\mathscr{C}_{\mathbb{Z}}^{m,n}$. Then the following statements hold.*

(1) $(a,b)\mathscr{R}(c,d) \iff (a=c) \wedge ((b=d) \vee (b,d \geqslant m))$.
(2) $(a,b)\mathscr{L}(c,d) \iff (b=d) \wedge ((a=c) \vee (a,c \geqslant n))$.
(3) $(a,b)\mathscr{H}(c,d) \iff (a,b) = (c,d)$.
(4) $(a,b)\mathscr{D}(c,d) \iff (a,b) = (c,d) \vee (a,c \geqslant n) \vee (b,d \geqslant m)$.
(5) $(a,b)\mathscr{J}(c,d)$ *for all* $(a,b),(c,d) \in \mathscr{C}_{\mathbb{Z}}^{m,n}$.

Proof. By formula (4) we get that

$$\{(a,b)\} \cup (a,b) *_{m,n} \mathscr{C}_{\mathbb{Z}}^{m,n} = \begin{cases} \{(x,y) \in \mathscr{C}_{\mathbb{Z}}^{m,n} : x \geqslant a\}, & \text{if } b \geqslant m; \\ \{(a,b)\} \cup \{(x,y) \in \mathscr{C}_{\mathbb{Z}}^{m,n} : x \geqslant a-b+m\}, & \text{if } b < m. \end{cases}$$

The above formula implies statement (1). The proof of statement (2) is similar.

Statement (3) follows from (1) and (2).

For the proof of assertion (4) we consider the following three cases.

(*i*) If $a < n$ and $b < m$ then by statements (1) and (2) we have that

$$(x, y)\mathscr{R}(a, b) \iff (x, y) = (a, b)$$

and

$$(x, y)\mathscr{L}(a, b) \iff (x, y) = (a, b),$$

for $(x, y) \in \mathscr{C}_{\mathbb{Z}}^{m,n}$, and hence in this case we get that

$$(c, d)\mathscr{R}(a, b) \iff (c, d) = (a, b).$$

(*ii*) If $a \geqslant n$ then by statements (1) and (2) we obtain that the $\mathscr{L}$-class $\mathbf{L}_{(a,b)}$ of the element (a, b) intersects the $\mathscr{R}$-class $\mathbf{R}_{(x,y)}$ of any element (x, y) with $y \geqslant m$.

(*iii*) If $b \geqslant m$ then by statements (1) and (2) we obtain that the $\mathscr{R}$-class $\mathbf{R}_{(a,b)}$ of the element (a, b) intersects the $\mathscr{L}$-class $\mathbf{L}_{(x,y)}$ of any element (x, y) with $x \geqslant n$.

Thus, in the case of (*ii*) or (*iii*) we have that

$$(a, b)\mathscr{D}(c, d) \iff (a, c \geqslant n) \vee (b, d \geqslant m)$$

which with case (*i*) implies statement (4).

Proposition 1 implies statement (5). □

Lemma 4. *For arbitrary idempotents (i, i) and (j, j) of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ variants $\mathscr{C}_{\mathbb{Z}}^{i,i}$ and $\mathscr{C}_{\mathbb{Z}}^{j,j}$ are isomorphic.*

Proof. By Theorem 1 for every positive integer k the map

$$h_k \colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}, \qquad (i, j) \mapsto (i + k, j + k)$$

is an automorphism of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$. This implies that the map h_k determines the isomorphism $\mathfrak{h}_k \colon \mathscr{C}_{\mathbb{Z}}^{0,0} \to \mathscr{C}_{\mathbb{Z}}^{k,k}$ of variants $\mathscr{C}_{\mathbb{Z}}^{0,0}$ and $\mathscr{C}_{\mathbb{Z}}^{k,k}$ for every integer k. Indeed, we put $\mathfrak{h}_k\left((a, b)\right) = h_k\left((a, b)\right)$ for each $(a, b) \in \mathscr{C}_{\mathbb{Z}}^{0,0}$. Then

$$\begin{aligned}
\mathfrak{h}_k\left((a, b) *_{(0,0)} (c, d)\right) &= h_k\left((a, b) *_{(0,0)} (c, d)\right) = \\
&= h_k\left((a, b) \cdot (0, 0) \cdot (c, d)\right) = \\
&= h_k\left((a, b)\right) \cdot h_k\left((0, 0)\right) \cdot h_k\left((c, d)\right) = \\
&= (a + k, b + k) \cdot (k, k) \cdot (c + k, d + k) = \\
&= (a + k, b + k) *_{(k,k)} (c + k, d + k) = \\
&= h_k\left((a, b)\right) *_{(k,k)} h_k\left((c, d)\right) = \\
&= \mathfrak{h}_k\left((a, b)\right) *_{(k,k)} \mathfrak{h}_k\left((c, d)\right),
\end{aligned}$$

for arbitrary $(a, b), (c, d) \in \mathscr{C}_{\mathbb{Z}}^{0,0}$. Since for any positive integer k the map

$$h_k \colon \mathscr{C}_{\mathbb{Z}} \to \mathscr{C}_{\mathbb{Z}}, \qquad (i, j) \mapsto (i + k, j + k)$$

as a self-mapping of the set $\mathscr{C}_{\mathbb{Z}}$ is bijective, we conclude that $\mathfrak{h}_k \colon \mathscr{C}_{\mathbb{Z}}^{0,0} \to \mathscr{C}_{\mathbb{Z}}^{k,k}$ is an isomorphism of variants $\mathscr{C}_{\mathbb{Z}}^{0,0}$ and $\mathscr{C}_{\mathbb{Z}}^{k,k}$. This completes the proof of the lemma. □

Lemma 5. *For an arbitrary integer r and an arbitrary positive integer p the variants $\mathscr{C}_{\mathbb{Z}}^{r,r}$ and $\mathscr{C}_{\mathbb{Z}}^{r+p,r}$ of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isomorphic.*

Proof. Fix an arbitrary integer r and an arbitrary positive integer p. We define a map $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}}^{r,r} \to \mathscr{C}_{\mathbb{Z}}^{r+p,r}$ by the formula

$$\mathfrak{h}\left((r+i,r+j)\right) = (r+i,r+j+p).$$

Then for arbitrary $(r+i,r+j),(r+k,r+l)\in \mathscr{C}_{\mathbb{Z}}^{r,r}$ we have that

$$\begin{aligned}
&\mathfrak{h}\left((r+i,r+j) *_{r,r} (r+k,r+l)\right) = \mathfrak{h}\left((r+i,r+j)\cdot(r,r)\cdot(r+k,r+l)\right) = \\
&= \begin{cases} \mathfrak{h}\left((r+i-j,r)\cdot(r+k,r+l)\right), & \text{if } r+j\leqslant r; \\ \mathfrak{h}\left((r+i,r+j)\cdot(r+k,r+l)\right), & \text{if } r+j> r \end{cases} = \\
&= \begin{cases} \mathfrak{h}\left((r+i-j,r+l-k)\right), & \text{if } r+j\leqslant r \text{ and } r+k\leqslant r; \\ \mathfrak{h}\left((r+i-j+k,r+l)\right), & \text{if } r+j\leqslant r \text{ and } r+k> r; \\ \mathfrak{h}\left((r+i,r+j-k+l)\right), & \text{if } r+j> r \text{ and } r+k\leqslant r+j; \\ \mathfrak{h}\left((r+i-j+k,r+l)\right), & \text{if } r+j> r \text{ and } r+k> r+j \end{cases} = \\
&= \begin{cases} \mathfrak{h}\left((r+i-j,r+l-k)\right), & \text{if } j\leqslant 0 \text{ and } k\leqslant 0; \\ \mathfrak{h}\left((r+i-j+k,r+l)\right), & \text{if } j\leqslant 0 \text{ and } k> 0; \\ \mathfrak{h}\left((r+i,r+j-k+l)\right), & \text{if } j> 0 \text{ and } k\leqslant j; \\ \mathfrak{h}\left((r+i-j+k,r+l)\right), & \text{if } j> 0 \text{ and } k> j \end{cases} = \\
&= \begin{cases} (r+i-j,r+l-k+p), & \text{if } j\leqslant 0 \text{ and } k\leqslant 0; \\ (r+i-j+k,r+l+p), & \text{if } j\leqslant 0 \text{ and } k> 0; \\ (r+i,r+j-k+l+p), & \text{if } j> 0 \text{ and } k\leqslant j; \\ (r+i-j+k,r+l+p), & \text{if } j> 0 \text{ and } k> j \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{h}\left(r+i,r+j\right) *_{r+p,r} \mathfrak{h}\left((r+k,r+l)\right) = \\
&= (r+i,r+j+p)\cdot(r+p,r)\cdot(r+k,r+l+p) = \\
&= \begin{cases} (r+i-j,r)\cdot(r+k,r+l+p), & \text{if } r+j+p\leqslant r+p; \\ (r+i,r+j)\cdot(r+k,r+l+p), & \text{if } r+j+p> r+p \end{cases} = \\
&= \begin{cases} (r+i-j,r+l-k+p), & \text{if } r+j+p\leqslant r+p \text{ and } r+k\leqslant r; \\ (r+i-j+k,r+l+p), & \text{if } r+j+p\leqslant r+p \text{ and } r+k> r; \\ (r+i,r+j-k+l+p), & \text{if } r+j+p> r+p \text{ and } r+k\leqslant r+j; \\ (r+i-j+k,r+l+p), & \text{if } r+j+p> r+p \text{ and } r+k> r+j \end{cases} = \\
&= \begin{cases} (r+i-j,r+l-k+p), & \text{if } j\leqslant 0 \text{ and } k\leqslant 0; \\ (r+i-j+k,r+l+p), & \text{if } j\leqslant 0 \text{ and } k> 0; \\ (r+i,r+j-k+l+p), & \text{if } j> 0 \text{ and } k\leqslant j; \\ (r+i-j+k,r+l+p), & \text{if } j> 0 \text{ and } k> j, \end{cases}
\end{aligned}$$

because $\mathfrak{h}\left((r,r)\right) = (r,r+p)$, and hence $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}}^{r,r} \to \mathscr{C}_{\mathbb{Z}}^{r+p,r}$ is a homomorphism. Also, the definition of the map $\mathfrak{h}$ implies that it is a bijection, and thus $\mathfrak{h}$ is an isomorphism. □

Lemma 6. *For an arbitrary integer r and an arbitrary positive integer p the variants $\mathscr{C}_{\mathbb{Z}}^{r,r}$ and $\mathscr{C}_{\mathbb{Z}}^{r,r+p}$ of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isomorphic.*

Proof. We define a map $\mathfrak{h}\colon \mathscr{C}_{\mathbb{Z}}^{r,r} \to \mathscr{C}_{\mathbb{Z}}^{r,r+p}$ by the formula

$$\mathfrak{h}\left((r+i,r+j)\right) = (r+i,r+j+p).$$

The proof that so defined map $\mathfrak{h}$ is an isomorphism, is similar as in Lemma 5. □

Lemmas 4, 5 and 6 imply the following theorem.

Theorem 3. *Any two variants of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isomorphic.*

Theorem 4. *The variant $\mathscr{C}_{\mathbb{Z}}^{0,0}$ of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ is not finitely generated.*

Proof. Formulae (4) and (5) imply

$$\{(a,b)\}\cup(a,b)*_{0,0}\mathscr{C}_{\mathbb{Z}}^{0,0}=\begin{cases}\left\{(x,y)\in\mathscr{C}_{\mathbb{Z}}^{0,0}\colon x\geqslant a\right\}, & \text{if } b\geqslant 0;\\ \{(a,b)\}\cup\left\{(x,y)\in\mathscr{C}_{\mathbb{Z}}^{0,0}\colon x\geqslant a-b\right\}, & \text{if } b<0\end{cases}$$

and

$$\{(a,b)\}\cup\mathscr{C}_{\mathbb{Z}}^{0,0}*_{0,0}(a,b)=\begin{cases}\left\{(x,y)\in\mathscr{C}_{\mathbb{Z}}^{0,0}\colon y\geqslant b\right\}, & \text{if } a\geqslant 0;\\ \{(a,b)\}\cup\left\{(x,y)\in\mathscr{C}_{\mathbb{Z}}^{0,0}\colon y\geqslant b-a\right\}, & \text{if } a<0.\end{cases}$$

Hence for every finite subset F of the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$ we have that the set

$$\left\{(x,y)\in\mathscr{C}_{\mathbb{Z}}^{0,0}\colon x,y<0\right\}\setminus\langle F\rangle$$

is infinite, where $\langle F\rangle$ is a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{0,0}$ generated by the set F, which implies the statement of the theorem. ☐

Theorems 3 and 4 imply the following corollary.

Corollary 2. *For any integers m and n the variant $\mathscr{C}_{\mathbb{Z}}^{m,n}$ of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ is not finitely generated.*

4. Shift-continuous topologies on the variant $\mathscr{C}_{\mathbb{Z}}^{0,0}$.

Simple calculations and formula (1) imply the following lemma.

Lemma 7. *If $(a,b)\cdot(c,d)=(i,j)$ in $\mathscr{C}_{\mathbb{Z}}$ then $a-b+c-d=i-j$.*

Lemma 7 implies the following proposition.

Proposition 4. *Let m and n be arbitrary integers. If $(a,b)*_{m,n}(c,d)=(i,j)$ in $\mathscr{C}_{\mathbb{Z}}^{m,n}$ then*

$$a-b+m-n+c-d=i-j.$$

Corollary 3. *If $(a,b)*_{0,0}(c,d)=(i,j)$ in $\mathscr{C}_{\mathbb{Z}}^{0,0}$ then $a-b+c-d=i-j$.*

Later, for every $(a,b)\in\mathscr{C}_{\mathbb{Z}}^{0,0}$ by $\lambda_{(a,b)}$ and $\rho_{(a,b)}$ we denote left and right shift (translation) on the element (a,b) in the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$, respectively, i.e.,

$$\lambda_{(a,b)}\colon\mathscr{C}_{\mathbb{Z}}^{0,0}\to\mathscr{C}_{\mathbb{Z}}^{0,0},\qquad (x,y)\mapsto(a,b)*_{0,0}(x,y)$$

and

$$\rho_{(a,b)}\colon\mathscr{C}_{\mathbb{Z}}^{0,0}\to\mathscr{C}_{\mathbb{Z}}^{0,0},\qquad (x,y)\mapsto(x,y)*_{0,0}(a,b).$$

Proposition 5. *Let τ be a Hausdorff shift-continuous topology on the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$. Then the following assertions hold:*

(i) (a,b) is an isolated point in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ for any positive integers a and b;

(*ii*) *for any integers a and b the set $\{(a-i,b-i)\colon i\in\mathbb{N}_0\}$ is open in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$;*
(*iii*) *(a,b) is an isolated point in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ for any positive integer a and any integer b;*
(*iv*) *(a,b) is an isolated point in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ for any integer a and any positive integer b.*

Proof. (*i*) Fix an arbitrary point (a,b) in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ such that $a>0$ and $b>0$. Since by Lemma 1 the set $\mathscr{C}_{\mathbb{Z}}^{\geqslant 0}=\{(i,j)\in\mathscr{C}_{\mathbb{Z}}\colon i,j\geqslant 0\}$ with the induced semigroup from $\mathscr{C}_{\mathbb{Z}}$ operation is isomorphic to the bicyclic semigroup $\mathscr{C}(p,q)$ and by Proposition 1 of [5] every shift-continuous Hausdorff topology on the bicyclic semigroup $\mathscr{C}(p,q)$ is discrete, there exists an open neighbourhood $U_{(a,b)}$ of the point (a,b) in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ such that $U_{(a,b)}\cap\mathscr{C}_{\mathbb{Z}}^{\geqslant 0}=\{(a,b)\}$. Since

$$(i,i)*_{0,0}(x,y)=(i,i)(0,0)(x,y)=(i,i)(x,y)=\begin{cases}(i,i-x+y), & \text{if } x\leqslant i;\\ (x,y), & \text{if } x>i\end{cases}$$

for any non-negative integer i, we have that $\{(s,l+s-k)\colon s\leqslant k, s\in\mathbb{Z}\}$ is the set of solutions of the equation $(k,l)=(k,k)*_{0,0}(x,y)$ for all non-negative integers k and l. Then the separate continuity of the semigroup operation in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$, Hausdorffness of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and above arguments imply that the set

$$\{(s,b+s-a)\colon s<a, s\in\mathbb{Z}\}=\lambda^{-1}_{(a-1,a-1)}(\{(a-1,b-1)\})$$

is closed in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and the set

$$\{(s,b+s-a)\colon s\leqslant a, s\in\mathbb{Z}\}=\lambda^{-1}_{(a,a)}(U_{(a,b)})$$

is open in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$, which implies that (a,b) is an isolated point in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$.

(*ii*) The proof of item (*i*) implies that the set

$$\{(s,b+s-a)\colon s\leqslant a, s\in\mathbb{Z}\}=\lambda^{-1}_{(a,a)}(U_{(a,b)})$$

is open in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ for any positive integers a and b, because there exists an open neighbourhood $U_{(a,b)}$ of the point (a,b) in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ such that $U_{(a,b)}\cap\mathscr{C}_{\mathbb{Z}}^{\geqslant 0}=\{(a,b)\}$. If we put $i=a-s$ then

$$\{(a-i,b-i)\colon i\in\mathbb{N}_0\}=\lambda^{-1}_{(a,a)}(U_{(a,b)})$$

is an open subset of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$. It is obvious that for arbitrary integers a and b there exists a positive integer $k_{(a,b)}$ such that $a+k_{(a,b)}>0$ and $b+k_{(a,b)}>0$. Hausdorffness of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ implies that every point is a closed subset of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and hence the set

$$\{(a-i,b-i)\colon i\in\mathbb{N}_0\}=\lambda^{-1}_{(a,a)}(U_{(a,b)})\setminus\left\{(a+1,b+1),\ldots,(a+k_{(a,b)},b+k_{(a,b)})\right\}$$

is open in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$, which implies the required statement.

(*iii*) Since

$$(i,i)*_{0,0}(x,y)=(i,i)(0,0)(x,y)=(i,i)(x,y)=\begin{cases}(i,i-x+y), & \text{if } x\leqslant i;\\ (x,y), & \text{if } x>i\end{cases}$$

for any non-negative integer i, we have that $\{(s,l+s-k)\colon s\leqslant k, s\in\mathbb{Z}\}$ is the set of solutions of the equation $(k,l)=(k,k)*_{0,0}(x,y)$ for every non-negative integer k and every integer l. Then the separate continuity of the semigroup operation in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and Hausdorffness of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ imply that the set

$$\{(s,l+s-k)\colon s\leqslant k, s\in\mathbb{Z}\}=\lambda^{-1}_{(k,k)}(\{(k,l)\})$$

is closed in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ for every non-negative integer k and every integer l. Fix an arbitrary positive integer a and an arbitrary integer b. Then the above arguments and assertion (ii) imply that

$$\{(a,b)\}=\{(a-i,b-i)\colon i\in\mathbb{N}_0\}\setminus\lambda^{-1}_{(a-1,a-1)}(\{(a-1,b-1)\})$$

is an open subset of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$.

(iv) Since

$$(x,y)*_{0,0}(i,i)=(x,y)(0,0)(i,i)=(x,y)(i,i)=\begin{cases}(i-y+x,i), & \text{if } y\leqslant i;\\ (x,y), & \text{if } y>i\end{cases}$$

for any non-negative integer i, we have that $\{(l+s-k,s)\colon s\leqslant k, s\in\mathbb{Z}\}$ is the set of solutions of the equation $(l,k)=(x,y)*_{0,0}(k,k)$ for every non-negative integer k and every integer l. Then the separate continuity of the semigroup operation in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and Hausdorffness of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ imply that the set

$$\{(l+s-k,s)\colon s\leqslant k, s\in\mathbb{Z}\}=\rho^{-1}_{(k,k)}(\{(l,k)\})$$

is closed in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ for every non-negative integer k and every integer l. Fix an arbitrary integer a and an arbitrary positive integer b. Then the above arguments and assertion (ii) imply that

$$\{(a,b)\}=\{(a-i,b-i)\colon i\in\mathbb{N}_0\}\setminus\rho^{-1}_{(b-1,b-1)}(\{(a-1,b-1)\})$$

is an open subset of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$. □

We summarize the results of Proposition 5 in the following theorem.

Theorem 5. *Let τ be a Hausdorff shift-continuous topology on the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$. Then each of the inequalities $a>0$ or $b>0$ implies that (a,b) is an isolated point of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$.*

The following example shows that the statement of Theorem 5 is complete and it cannot be extended on any point (a,b) with the properties $a\leqslant 0$ and $b\leqslant 0$.

Example 1. We define the topology τ^* on $\mathscr{C}_{\mathbb{Z}}^{0,0}$ in the following way. Put

- (i) (a,b) is an isolated point of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ if and only if at least one of the following conditions holds $a>0$ or $b>0$;
- (ii) if $ab=0$ and $a+b\leqslant 0$ we let $A_{(a,b)}=\{(a-i,b-i)\colon i\in\mathbb{N}_0\}$ be an arbitrary Hausdorff space and $A_{(a,b)}$ be an open-and-closed subset of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$.

It is obvious that $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ is a Hausdorff space.

Proposition 6. $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ *is a topological semigroup.*

Proof. Since (a,b) is an isolated point of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ in the case when $a>0$ or $b>0$, it is complete to show that the semigroup operation of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ is continuous in the following three cases:

(1) $(a,b)*_{0,0}(c,d)$, when $a\leqslant 0$, $b\leqslant 0$, $c\leqslant 0$ and $d\leqslant 0$;
(2) $(a,b)*_{0,0}(c,d)$, when $a\leqslant 0$, $b\leqslant 0$, and $c>0$ or $d>0$;
(3) $(a,b)*_{0,0}(c,d)$, when $c\leqslant 0$ and $d\leqslant 0$, and $a>0$ or $b>0$.

In case (1) we have that

$$(a,b)*_{0,0}(c,d)=(a,b)(0,0)(c,d)=(a-b,0)(c,d)=(a-b,d-c).$$

Also, in this case since

$$\begin{aligned}(a-i,b-i)*_{0,0}(c-j,d-j)&=(a-i,b-i)(0,0)(c-j,d-j)=\\&=(a-i-b+i,0)(c-j,d-j)=\\&=(a-b,0)(c-j,d-j)=\\&=(a-b,d-j-c+j)=\\&=(a-b,d-c)\end{aligned}$$

for any $i,j\in\mathbb{N}_0$, we obtain that $A_{(a,b)}*_{0,0}A_{(c,d)}=\{(a-b,d-c)\}$, and hence in case (1) the semigroup operation in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ is continuous.

Suppose that case (2) holds. Then we have that

$$\begin{aligned}(a,b)*_{0,0}(c,d)&=(a,b)(0,0)(c,d)=\\&=(a-b,0)(c,d)=\\&=\begin{cases}(a-b,d-c), & \text{if } c\leqslant 0;\\(c-a+b,d), & \text{if } c>0.\end{cases}\end{aligned}$$

In this case, since

$$\begin{aligned}(a-i,b-i)*_{0,0}(c,d)&=(a-i,b-i)(0,0)(c,d)=\\&=(a-i-b+i,0)(c,d)=\\&=(a-b,0)(c,d)=\\&=\begin{cases}(a-b,d-c), & \text{if } c\leqslant 0;\\(c-a+b,d), & \text{if } c>0,\end{cases}\end{aligned}$$

for every $i\in\mathbb{N}_0$ we get that

$$A_{(a,b)}*_{0,0}\{(c,d)\}=\begin{cases}\{(a-b,d-c)\}, & \text{if } c\leqslant 0;\\\{(c-a+b,d)\}, & \text{if } c>0,\end{cases}$$

which implies that the semigroup operation in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$ is continuous in case (2).

Suppose that case (3) holds. Then we have that

$$\begin{aligned}(a,b)*_{0,0}(c,d)&=(a,b)(0,0)(c,d)=\\&=(a,b)(0,d-c)=\\&=\begin{cases}(a-b,d-c), & \text{if } b\leqslant 0;\\(a,b-c+d), & \text{if } b>0.\end{cases}\end{aligned}$$

In this case, since

$$\begin{aligned}(a,b) *_{0,0} (c-j,d-j) &= (a,b)(0,0)(c-j,d-j) = \\ &= (a,b)(0,d-j-c+j) = \\ &= (a,b)(0,d-c) = \\ &= \begin{cases} (a-b,d-c), & \text{if } b \leqslant 0; \\ (a,b-c+d), & \text{if } b > 0, \end{cases}\end{aligned}$$

for every $j \in \mathbb{N}_0$ we obtain that

$$\{(a,b)\} *_{0,0} A_{(c,d)} = \begin{cases} \{(a-b,d-c)\}, & \text{if } b \leqslant 0; \\ \{(a,b-c+d)\}, & \text{if } b > 0, \end{cases}$$

and hence in case (3) the semigroup operation in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0}, \tau^*\right)$ is continuous. □

Remark 1. A topological semigroup S is called Γ-compact if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \ldots\}$ is a compactum in S (see [30]). Since by Lemma 1 the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$ contains the bicyclic semigroup as a subsemigroup the results obtained in [2], [3], [4], [30] imply that *if a Hausdorff topological semigroup S satisfies one of the following conditions:*

(*i*) *S is compact;*
(*ii*) *S is Γ-compact;*
(*iii*) *the square $S \times S$ is countably compact;*
(*iv*) *the square $S \times S$ is a Tychonoff pseudocompact space,*

then S does not contain an algebraic copy of the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$.

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ПРО ВАРІАНТИ РОЗШИРЕНОЇ БІЦИКЛІЧНОЇ НАПІВГРУПИ

Олег ГУТІК, Катерина МАКСИМИК

Львівський національний університет імені Івана Франка,
вул. Університетська 1, Львів, 79000,
e-mails: oleg.gutik@lnu.edu.ua, ovgutik@yahoo.com,
kate.maksymyk15@gmail.com

Описуємо групу $\mathbf{Aut}\,(\mathscr{C}_{\mathbb{Z}})$ автоморфізмів розширеної біциклічної напівгрупи $\mathscr{C}_{\mathbb{Z}}$ і вивчаємо варіанти $\mathscr{C}_{\mathbb{Z}}^{m,n}$ розширеної біциклічної напівгрупи $\mathscr{C}_{\mathbb{Z}}$, де $m, n \in \mathbb{Z}$. Зокрема, ми довели, що група $\mathbf{Aut}\,(\mathscr{C}_{\mathbb{Z}})$ ізоморфна адитивній групі цілих чисел, розширена біциклічна напівгрупа $\mathscr{C}_{\mathbb{Z}}$ і кожен її варіант не є скінченно породженими, описали підмножину ідемпотентів $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ і відношення Гріна на напівгрупі $\mathscr{C}_{\mathbb{Z}}^{m,n}$. Також довели, що $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ є ω-ланцюгом і довільні два варіанти розширеної біциклічної напівгрупи $\mathscr{C}_{\mathbb{Z}}$ є ізоморфними. На завершення дослідили трансляційно неперервні гаусдорфові топології на варіанті $\mathscr{C}_{\mathbb{Z}}^{0,0}$. Зокрема, довели, якщо τ — гаусдорфова трансляційно неперервна топологія на $\mathscr{C}_{\mathbb{Z}}^{0,0}$, то з кожної з нерівностей $a > 0$ або $b > 0$ випливає, шо (a, b) є ізольованою точкою в топологічному просторі $\left(\mathscr{C}_{\mathbb{Z}}^{0,0}, \tau\right)$ та побудовано приклад гаусдорфової напівгрупової топології τ^* на напівгрупі $\mathscr{C}_{\mathbb{Z}}^{0,0}$ такої, що інші точки, які задовольняють умови $ab \leqslant 0$ і $a + b \leqslant 0$ не є ізольованими в просторі $\left(\mathscr{C}_{\mathbb{Z}}^{0,0}, \tau^*\right)$.

Ключові слова: напівгрупа, інтерасоціативність напівгрупи, варіант напівгрупи, біциклічний моноїд, розширена біциклічна напівгрупа, напівтопологічна напівгрупа, топологічна напівгрупа.