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A LOCAL ADEQUATE RING IS A RING OF ADEQUATE RANGE 1

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We introduce the notion of a local adequate ring and construct an example of a local adequate ring, which is not adequate. Moreover, we prove that a ring of adequate range 1 is an elementary divisor ring.

Key words: an adequate ring, a ring of adequate range 1, a local adequate ring, an elementary divisor ring.

The class of adequate rings was introduced by O. Helmer as a class of elementary divisor rings where no conditions of finiteness of all possible chains of ideals and non-associative atomic divisors of elements are imposed [1]. The ring of analytic functions may be one of the best illustrations of these properties of the adequate rings [1]. The adequate rings with zero divisors in the Jacobson radical were considered by I. Kaplansky [2]. L. Gilman and M. Henriksen showed that a commutative regular ring is adequate [3]. At the same time the structure of these rings was not investigated enough. Among the existing properties we pay attention to M. Henriksen's result that every non-zero principal ideal of an adequate ring is contained in a unique maximal ideal [4]. Moreover, in the case of semi-local Bezout ring this property is necessary and sufficient for an adequate ring [5]. For commutative Bezout domains with Nether spectrum this result was presented in [6]. An example of a commutative elementary divisor domain which is not adequate, is considered in [4]. Taking into consideration these results, B. Zabavsky in [7] described a class of generalized adequate rings which is close to adequate, except the condition that every non-zero principal ideal of an adequate ring is contained in a singular maximal.

Among results on adequate rings we would distinguish the results obtained by B. Zabavsky and S. Bilyavska [8]. The authors proved that the commutative Bezout

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domain is adequate if and only if any of its non-trivial finite homomorphic images is a semi-regular ring. They also prove that the stable range of an adequate ring equals 2 and the stable range of an adequate ring with nonzero Jacobson radical equals 1 (see [9]).

As noted above any commutative regular ring is adequate [3]. A. Osba, M. Henriksen and O. Alkman in [11] and B. Zabavsky in [12] considered the local regular rings as generalizations of regular rings.

The following questions arose: *are local adequate rings adequate?* Otherwise, is there a connection between local adequate rings and adequate elementary divisor rings? What is the stable range of these rings? In this paper, we can answers these questions.

All rings in this paper are commutative with identity, $1 \neq 0$.

Let us remind some necessary definitions. A ring R is called *a Bezout ring* if every its finitely generated ideal is principal. An element $a \in R$ is called *adequate* if for every nonzero element $b \in R$ the element a can be represented as a product $a = rs$, where $rR + bR = R$ and for any non-invertible divisor s' of s we have $s'R + bR \neq R$. A ring is called *adequate* if every its nonzero element is adequate [1].

A von Neumann regular ring is a ring R such that for every nonzero $a \in R$ there exists $x \in R$ such that $axa = a$. A ring R is called *a local regular ring*, if for every element $a \in R$ at least one of the elements a or $1 - a$ is regular [11].

Let us introduce the notion of a local adequate ring. A ring R is called *a local adequate ring*, if for every element $a \in R$ at least one of the elements a or $1 - a$ is adequate.

Obviously, any adequate ring is local adequate. An example of local adequate ring which is not adequate, is any local regular ring which is not regular [10], [11]. Another example is Henriksen's one [4] namely

$$R = \left\{ z_0 + a_1 x + a_2 x^2 + \ldots | z_0 \in \mathbf{Z}, a_i \in \mathbf{Q} \right\}.$$

It must be noticed that in this ring for every elements $f(x), g(x) \in R$ such that $f(x)R + g(x)R = R$ possible two cases are:

- both $f(x)$ and $g(x)$ are adequate;
- at least one of these element is adequate (another is invertible).

The case when $f(x)$ and $g(x)$ are not adequate is not impossible because $f(x) \in J(R)$ and $g(x) \in J(R)$.

It is known that every prime ideal of a local regular ring is contained in a unique maximal ideal. However, the ring in Henriksen's example is not the ring, where every nonzero prime ideal is contained in a unique maximal ideal. Clearly, it is not a local ring.

Proposition 1. *A commutative Bezout domain R is local adequate if and only if for every $a, b \in R$ such that $aR + bR = R$, one of the elements a or b is adequate.*

Proof. The necessity is obvious. Let us prove the sufficiency.

Let R be a local adequate ring and $aR + bR = R$ for $a, b \in R$. Then there exist $u, v \in R$ such that $au + bv = 1$. Given the restriction imposed on the ring R, au or bv is adequate. Since the set of adequate elements is saturated multiplicatively closed [12], a or b is adequate as being a divisor of au or bv. □

An element a of a ring R is called *clean* if $a = u + e$, where u is an invertible element of R and e is an idempotent ($e^2 = e$). A ring is called *clean* if every its element is clean. An element a of a ring R is called *neat* if the quotient ring R/aR is clean [13].

According to [8], if a is an adequate element of a commutative Bézout domain R, then R/aR is a semiregular ring. Since semiregular rings are exchange rings and commutative semiregular rings are clean, we obtain the following result.

Proposition 2. *Any adequate element of a commutative Bézout domain is neat.*

A ring R is called *a ring of neat range 1* if $aR+bR = R$, $a, b \in R$ imply the existence of $t \in R$ such that $a + bt$ is a neat element of R. Similarly, a ring R is called *a ring of adequate range 1* if $aR + bR = R$, $a, b \in R$, imply the existence of $t \in R$ such that $a + bt$ is an adequate element of R.

Theorem 1. *A local adequate ring is a ring of adequate range 1.*

Proof. Let R be a local adequate ring and $aR + bR = R$, $a, b \in R$. If a is an adequate element of R then obviously $a + b\cdot 0$ is an adequate element of R. If a is not an adequate element of R let us consider the element $a+b$. Since $aR+(a+b)R = R$ by Proposition 2, $a + b$ is an adequate element of R. □

In [13] B. Zabavsky proved the following result.

Theorem 2. *A commutative Bézout domain is an elementary divisor domain if and only if it is a ring of neat range 1.*

Now we are able to prove the main result of this article.

Theorem 3. *Any commutative Bézout domain of adequate range 1 is an elementary divisor ring.*

Proof. Let R be a commutative Bézout ring of adequate range 1, which means that $aR + bR = R$ ($a, b \in R$) implies the existence of $t \in R$ such as $a + bt = \alpha$ is an adequate element of R. Since the adequate elements are neat, $R/\alpha R$ is a clean ring. By Theorem 2, R is an elementary divisor ring. □

Theorem 4. *A local adequate commutative Bézout domain is an elementary divisor ring.*

Proof. By Theorem 1, any local adequate ring R is a ring of adequate range 1. By Theorem 3, R is an elementary divisor ring. □

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ЛОКАЛЬНЕ АДЕКВАТНЕ КІЛЬЦЕ Є КІЛЬЦЕМ АДЕКВАТНОГО РАНГУ 1

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Введено означення локально адекватного кільця та доведено, що таке кільце не є адекватним кільцем. Крім того, доведено, що кільце адекватного рангу 1 є кільцем елементарних дільників.

Ключові слова: адекватне кільце, кільце адекватного рангу 1, локально адекватне кільце, кільце елементарних дільників.