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LOAD-CARRYING CAPACITY OF REINFORCED CONCRETE PLATES ACCORDING TO DIFFERENT YIELD CRITERIA

This paper is devoted to the comparision of four yield criteria used for reinforced concrete plates with the amount of reinforcement not exceeding the balanced reinforcement. The basis of the assessment was the conformity of theoretical results with experimental results of tests on some series of slabs subjected to a homogeneous moment field load.

Key words: reinforced concrete plates, yield criterion, laboratory tests, statistics analysis.

Порівняно чотири критерії локального пошкодження залізобетонних плит з рівнем армування менше граничного. Основою оцінки була точність збігу результатів теоретичних розрахунків з результатами лабораторних досліджень чотирьох серій плит, завантажених однорідним полем моментів.

Ключові слова: критерій пластичності, залізобетонні плити, лабораторні дослі-дження, статистичний аналіз.

Introduction

The effort v of the slab may be defined as:

$$v = \frac{|M(p)|}{M_n} \leq 1, \tag{1}$$

where: $M(p)$ – the vector of external bending moment which increases proportionally to the increase of the loading p, M_n – the vector of yield moment at the point of the slab.

Both vectors are collinear. The ends of all vectors of yield moment at the chosen point of slab create the surface of failure. The failure of the slab in the given point occures when the end of the vector of external bending moment reaches this surface. The value of the vector of yield moment depends of the yield criterion.

From a vast amount of criteria applied to the calculation of load-carrying capacity of slabs with orthogonal reinforcement, four of the most frequently encountered in the literature are presented and analysed below. Their choice- apart from their popularity -was influenced by the wide differences in the assumptions concerning the behaviour of a slab at failure. They are namely: Gallileo-Leibnitz criterion (GL), Lenschow-Sozen criterion (LS), Sobotka criterion (S) and Huber-Mises-Hencky criterion (HMH).

Criteria of effort

Criteria of effort descripted below were constructed taking advantage of assumptions and simplifications. The most important are:

– in the class of slabs under consideration the failure does not occur by crushing of the concrete; in connection with this, the load-carrying capacity of the compressed zone in complex state of stress is not considered;

– the load-carrying capacity of the compressed zone is defined on the basis of the compressive strength of concrete in uniaxial state of stress;

– the state at failure of the compressed bars is not analysed, but their influence on the magnitude of the lever arm of internal forces and on the load-carrying capacity of the compressed zone is taken into account;

– in case of the GL, LS and HMH criteria the area of the reinforcement in each direction is replaced by a_{ij} layer of thickness aij depending of the distance beween bars; in case of the S criterion, the reinforcement mesh at each of the surfaces of the slab is replaced by a layer of reinforcement of thickness a_j, directionally orthotropic (the load-carrying capacity in i-th direction is equal to the actual capacity in this direction); i means analysed direction of bars, j means top or bottom reinforcement;

– only in case of the S criterion the reinforcement in the layer under consideration is interacting in both directions;

– for the GL, S and HMH criteria the cracks at failure develop in the planes of principal moments; for the LS criterion the cracks at failure are deviated by an angle dependent both on the moments at failure and on the principal bending moments;

– for the GL criterion reinforcement preserves its original direction; thus reinforcement is subjected to normal stresses only, perpendicular to the cross-section; for the HMH, LS and S criteria the reinforcement is subjected also to transverse deformation, which means that tangential stresses occur in addition to normal stresses.

The condition for destruction according to Gallileo (GL) , expressed in stresses for each of the orthogonal directions of reinforcement, can be written in the form:

$$\frac{\sigma_{ij}}{f_y} = 1 \quad , \qquad \sigma_{ij} > 0\,, \tag{2}$$

where: σ_{ij} – normal stresses in bars subjected to tension, of *i*–th direction and of *j*-th layer; f_y – yield limit.

Multiplying the numerator and the denominator of the expression (2) by the amount of reinforcement in the *i* direction and by the lever arm of internal forces characteristic for the state of full effort we arrive at the condition of failure expressed in terms of bending moments:

$$\frac{\sigma_{ij} \cdot a_{ij} \cdot z_{ij}}{f_y \cdot a_{ij} \cdot z_{ij}} = \frac{|M_{ij}|}{M_{nij}} \quad , \qquad \frac{|M_{ij}|}{M_{nij}} = 1\ , \tag{3}$$

where: a_{ij} – the amount of reinforcement, z_{ij} – the lever arm of internal forces, M_{ij} – the bending moment in *i*–th direction, M_{nij} – absolute value of the corresponding moment at failure.
Taking advantage of the definition of the local effort of the slab, the condition that the slab be not destroyed can be written as the inequality:

$$v_{ij} = \frac{|M_{ij}|}{|M_{nij}|} \quad , \qquad v_{ij} \le 1 \quad , \qquad v = max(v_{ij})\ , \tag{4}$$

where: v_{ij} – effort in *j*–th layer of reinforcement in *i*–th direction,equale to the effort of the slab. Failure of the slab in the given point occures when bars of only one or of both diections reaches $v=1$.

According to the criterion developped by Lenschow and Sozen (LS) [1], a reinforced concrete slab shall fail at a section depending on the distribution of its load-carrying capacity and external loading. The direction of the crack at failure (Fig. 1) is is obtained from the equation:

$$\frac{d}{d\beta} \cdot \frac{M_{nrj}}{M_r} = 0\,, \tag{5}$$

where: M_{nrj} – actual magnitude of moment vector at failure with yield of *j*–th layer of reinforcement, M_r – bending moment in a plane perpendicular to the crack at failure, β – angle of inclination of the crack at failure to the direction of M_1 one of the principal bending moment.

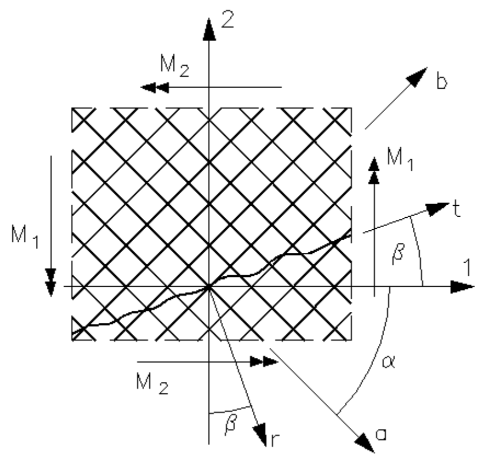


Fig. 1. Equilibrium of internal forces in the crack at failure [1]

The equation is transformed into:

$$-tg^2\beta\cdot(1-\lambda)\cdot ctg\alpha-\left[(\lambda-\varpi)\cdot ctg^2\alpha+1-\lambda\cdot\varpi\right]\cdot tg\beta+\varpi\cdot(1-\lambda)\cdot ctg\alpha=0$$
$$\lambda=\frac{M_{nyj}}{Mnxj}\qquad \varpi=\frac{M_2}{M_1}\quad, \tag{6}$$

where: λ – the degree of orthotropy of the slab, α – angle of inclination of the direction of moment M_1 to the x axis., ω – ratio of the principal bending moments M_1 to M_2, M_{nyj} and M_{nxj}- absolute value of the corresponding moment at failure.

Load-carrying capacity analysis of a slab section according to this criterion is limited to one, the most dangerous direction. It is therefore possible to assess the effort of the slab connected with the behaviour of the top or the bottom reinforcement only:

$$v_j=\frac{M_r}{M_{nrj}}\ , \tag{7}$$

where: v_j – the effort of the slab in j-th layer of reinforcement.

Z. Sobotka formulated the plasticity condition (S) [2] for homogeneous slabs with layer-and directional orthotropy, which can be applied to concrete slabs with top and bottom reinforcement at right angles. It is also valid for slabs having layer orthotropy. The equation given by Sobotka can be transformed to express efforts in terms of moment functions:

$$v=\sqrt{\left(\frac{M_x}{M_{nxj}}\right)^2+\left(\frac{M_y}{M_{nyj}}\right)^2-\left(\frac{1}{M_{nxj}^2}+\frac{1}{M_{nyj}^2}-\frac{1}{S_{nxyj}^2}\right)\cdot M_xM_y+\left(\frac{M_{xy}}{M_{nxy}}\right)^2}\le 1$$
$$S_{nxyj}=\frac{M_{nxj}\cdot M_{nyj}}{\sqrt{M_{nxj}^2+M_{nyj}^2-M_{nxj}\cdot M_{nyj}}}\quad, \tag{8}$$

where: j – the reinforcement layer: t – top or b – bottom, in which tension occurs under loading by a particular moment, M_{nxy} – limit torsional moment, equal to one-half of the volume of the solid bounded by the plane of constant inclination $\pi/4$ multiplied by the plasticity limit of steel, S_{nxy} – teoretical limit ending moment at uniform biaxial bending for the case when $M_x{\cdot}M_y>0$.

According to the criterion developped by Goszczyński (HMH) [3], a reinforced concrete slab shall fail at a section in which:

$$\sqrt{\frac{\sigma_{ij}^2}{f_y^2}+3\cdot\frac{\sigma_{xyj}^2}{f_y^2}}=1\ , \tag{9}$$

where: σ_{xyj} – tangential stresses in j-th layer of reinforcement.

The reinforcement of the slab in the vicinity of the initiated crack is under tension in both directions. Tangential forces T_{xyb} and T_{yxb} (Fig.2) are present in both directions of reinforcement on the side surface of the slab segment. From the condition of equilibrium, torsional moment M_{xy} are calculated as:

$$M_{xy}=T_{xyb}\cdot z_{xyb}=a_{xb}\cdot\sigma_{xyb}\cdot z_{xyb}=M_{yx}=T_{yxb}\cdot z_{yxb}=a_{yb}\cdot\sigma_{yxb}\cdot z_{yxb}\qquad \sigma_{xyb}\approx\sigma_{yxb}\ , \tag{10}$$

where: T_{xyb}, T_{yxb} – the resultant of tangential forces in fuzzy reinforcement of bottom layer, z_{xyb}, z_{yxb} – lever arm of T_{xyb} and T_{yxb} forces.

If the amount of reinforcement in both directions is not equal, to fullfil the condition of equality of tangential stresses redistribution ΔT of tangential forces must occure:

$$M_{xy}=\left(a_{xb}\cdot\sigma_{xyb}+\Delta T\right)\cdot z_{xyb}\ ,\qquad M_{yx}=\left(a_{yb}\cdot\sigma_{yxb}-\Delta T\right)\cdot z_{yxb}$$
$$M_{xy}=M_{yx}\quad and\quad \sigma_{xyb}=\sigma_{yxb}=\sigma_{xy}\quad, \tag{11}$$

where: M_{xy}, M_{yx} – torsional moments in x and y direction, ΔT – change in the magnitude of tangential force resulting from the redistribution of internal forces.

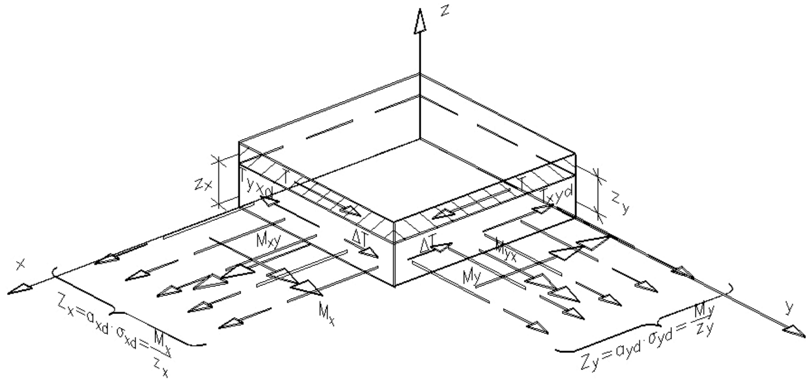


Fig. 2. Internal forces in a segment ofslab with bottom reinforcement [2]

On adding these equations we obtain:

$$2M_{xy} = a_{xd} \cdot \sigma_{xy} \cdot z_{xyb} + a_{xd} \cdot \sigma_{xy} \cdot z_{xyb} \,. \tag{12}$$

Since at the moment of failure the stresses σ_{xy} attain the mignitude $fy/\sqrt{3}$, and M_{xy} reaches the limit value M_{nxy}, the formula (12) can be transformed to the form:

$$M_{nxyb} = \frac{1}{2 \cdot \sqrt{3}} \cdot (M_{nxb} + M_{nyb}) \,. \tag{13}$$

where: M_{nxyb} – limit torsional moment for the bottom layer of reinforcement.
The limit torsional moment for the top reinforcement we obtain by analogy.

Taking advantage of the assumption of constant layer arm of the internal forces, we obtain a linear relation between stresses and the magnitude of the corresponding moments up till the plastic limit is reached by steel. This allows to formulate the magnitude of the effort for the definite layer and direction of the reinforcement in function of the moments:

$$v_{ij} = \sqrt{\left(\frac{M_i}{M_{nij}}\right)^2 + \left(\frac{M_{xy}}{M_{nxyj}}\right)^2} \le 1 \,. \tag{14}$$

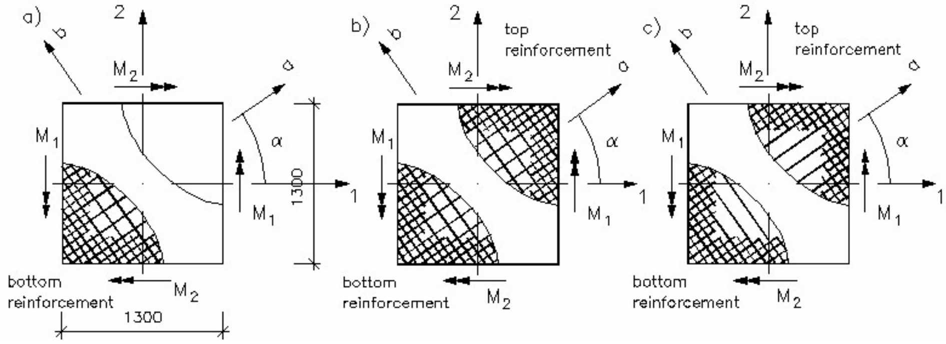
The general condition for non-destruction of the slab is as follow:

$$v = max\left(v_{ij}\right) < 1 \,. \tag{15}$$

Tests

In the assessment of the criteria experimental results of tests on four series of slabs subjected to a homogeneous moment field load were used. Criteria of selection were: detailed description of tests and slabs and differentation of kinds of reinforcement and loads.

Bauss and Tolaccia [4] tested rectangular slabs with one-sided orthogonal reinforcement and two-sided, with orthogonal reinforcement as well or in only one direction, but perpendicular one to another as it is shown in Fig.3. The load of tested slabs was a homogeneous moment field, but with the differentation of values and sense of vectors. 39 slabs were analysed – set *A* in statistical analysis. The percentage of reinforcement was 2,61% for external layer and 3,08% for internal layer.

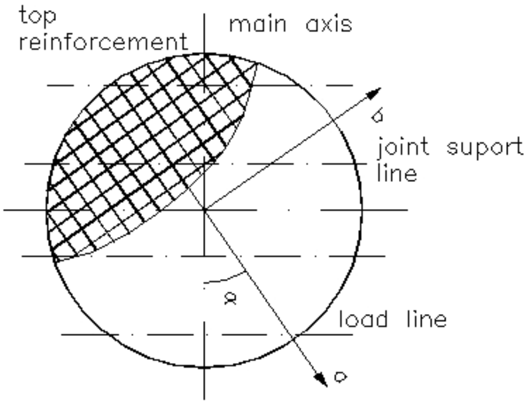


Type of reinforcement	Symbol of slab	α
a)	1/I – 6/I	0°
	1/II – 5/II	45°
	1/III – 3/III	22,5°
b)	15/II – 18/II	45°
	4/III – 8/III	22,5°
c)	7/I – 11/I	0°
	6/II – 14/II	45°
	9/III – 10/III	30°

thickness 80mm

Fig. 3. Tests of Bauss and Tolaccia [4]

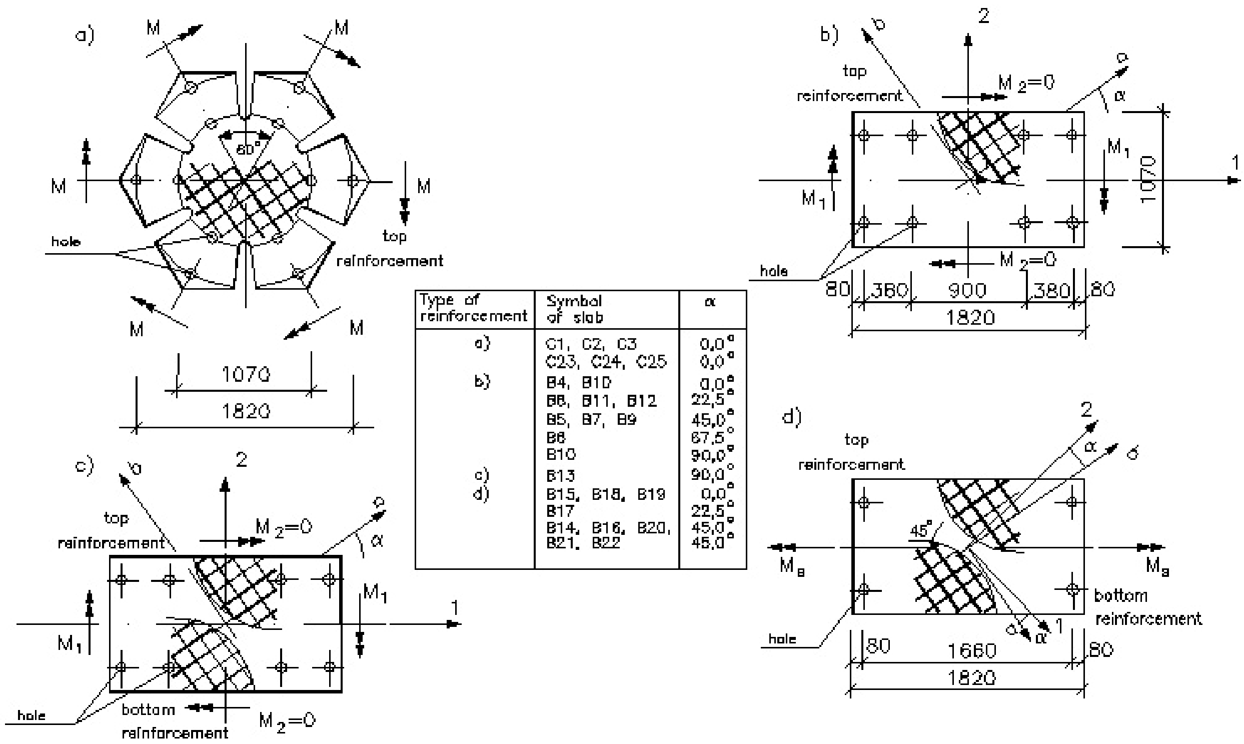
Lenkei [5] tested slabs of circle shape, with one-sided orthogonal reinforcement (Fig.4) and the percentage of reinforcement 0,90% for external layer and 1,06% for internal layer. All tested slabs (set *B*) in the numer of 63 were bent only in one direction, but with the differentation of the direction of bending moment in relations to the direction of reinforcement.



Symbol of slab	α
1,2,3/a, 16,17/b 31,32/c, 40/d	0,0°
4,5,6/a, 18,19,20,21/b 33,34/c, 41/d	22,5°
7,8,9/a, 22,23,24,25/b 35,36/c, 42,43/d	45,0°
10,11,12/a, 26,27,28/b 37,38/c, 44/d,	67,5°
13,14,15/a, 29,30/b 39/c, 45/d	67,5°

Fig. 4. Tests of Lenkei [5]

Lenschow and Sozen [6] investigated slabs in two shapes as shown in Fig.5.



Type of reinforcement	Symbol of slab	α
a)	C1, C2, C3	0,0°
	C23, C24, C25	0,0°
b)	B4, B10	0,0°
	B8, B11, B12	22,5°
	B5, B7, B9	45,0°
	B6	67,5°
	B10	90,0°
c)	B13	90,0°
d)	B15, B18, B19	0,0°
	B17	22,5°
	B14, B16, B20,	45,0°
	B21, B22	45,0°

Fig. 5. Tests of Lenschow and Sozen [6]

Slabs in the number of 20 were used to the analysis (set *C*). The percentage of reinforcement in *x* direction of top and bottom layer was about 1,49%, 1,72% and 2,77%, in *y* direction of top and bottom layer was about 0,9%, 0,72% and 3,19% in various combinations. The load of slabs was a homogeneous moment field, but with the differentation of values and sense of vectors.

Kozakow [7] tested rectangular slabs with orthogonal reinforcement in top and bottom layer (Fig. 6) – set *D*. The percentage of reinforcement was 1,5% in *x* direction of bottom layer and *y* direction of top layer, and 1,76% in *x* direction of top layer and *y* direction of bottom layer. The load of tested slabs was a homogeneous moment field, in which vectors of main moments were of a different sign.

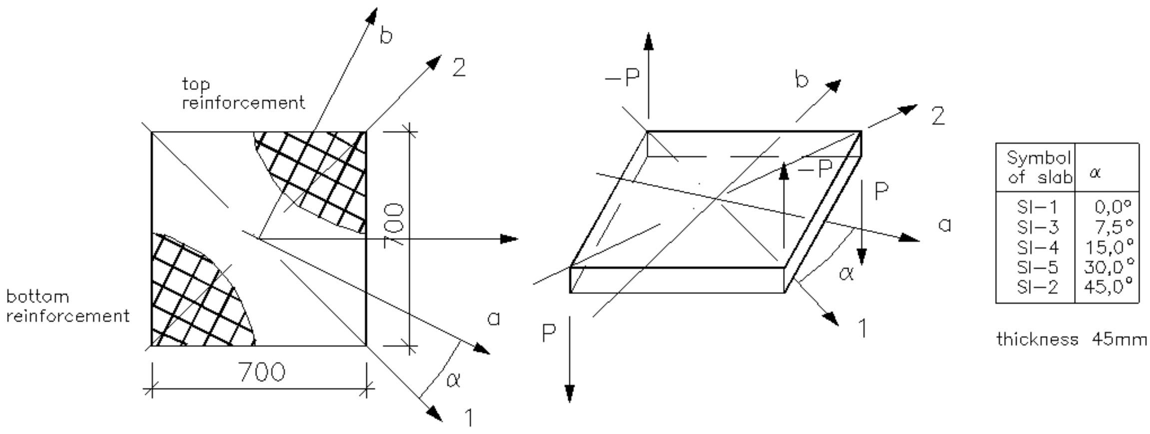


Fig. 6. Tests of Kozakow [7]

Statistical analysis of effort

The real magnitudes of moment vector at failure display random differences in relation to the theoretical results. The subject of further analysis shall be a variable *k*: the relative magnitude M_r of the real vector at slab failure in comparison with the theoretical magnitude M_t calculated in accordance with all descripted criteria.

In the analysis of the S criterion, test results of 21 slabs belonging to three sets were used, as only this number of test elements was reinforced orthogonally, top and bottom.

Bar charts in Fig. 7 show the distribution of random variables *k*, which depend on the criterion of failure adopted. The parameters of these distributions, i.e. mean values and variances of random variables *k* for single sets and for global sets are presented in Table I.

Table I

Statistical parameters of random variables k

Tests	Statistical parameters	Criteria			
		GL	HMH	LS	S
Baus – Tolaccia (A)	k_m	0,65513	1,01146	0,90659	0,97367
	s_{n-1}^2	0,10645	0,03399	0,00844	0,06525
Lenkei (B)	k_m	0,78895	0,98830	0,97065	-
	s_{n-1}^2	0,02240	0,00533	0,00551	-
Lenschow – Sozen (C)	k_m	0,78235	1,09725	1,00865	1,34286
	s_{n-1}^2	0,13467	0,05535	0,00271	0,10553
Kozakow (D)	k_m	0,54900	0,94900	0,74500	1,16120
	s_{n-1}^2	0,16137	0,00173	0,03461	0,20084
Kozakow and Baus – Tolaccia (A+D)	k_m	0,64307	1,00436	0,88823	1,04640
	s_{n-1}^2	0,11024	0,03060	0,01337	0,11065
Global (A+B+C+D)	k_m	0,73737	1,01102	0,94808	1,14138
	s_{n-1}^2	0,07371	0,02287	0,00977	0,12489

In view of the small size of *D* set, an attempt was made to join it with *A*, *B* or *C* set for each criteria, provided that the probability of such a pair of sets originating from a common general population is sufficiently high. In order to determine suitable probabilities, the *H* rank sum test (Greń [8]) was applied. The minimum level of significance was assumed to be 0,05. In cases of GL and S criteria of effort, the *D* set can be joined to any other, as the levels of significance here are higher than 0,05. Using HMH hypothesis it is possible to join the *D* set to an *A* or *B* set; for the LS criterion no joining is possible without violation of the accepted minimum level of significance. For GL, HMH and S criteria, the highest aggregate level of significance for the non-dismissal of the hypothesis in question was obtained in the first variant. This was decisive for the joining of *D* and *A* sets.

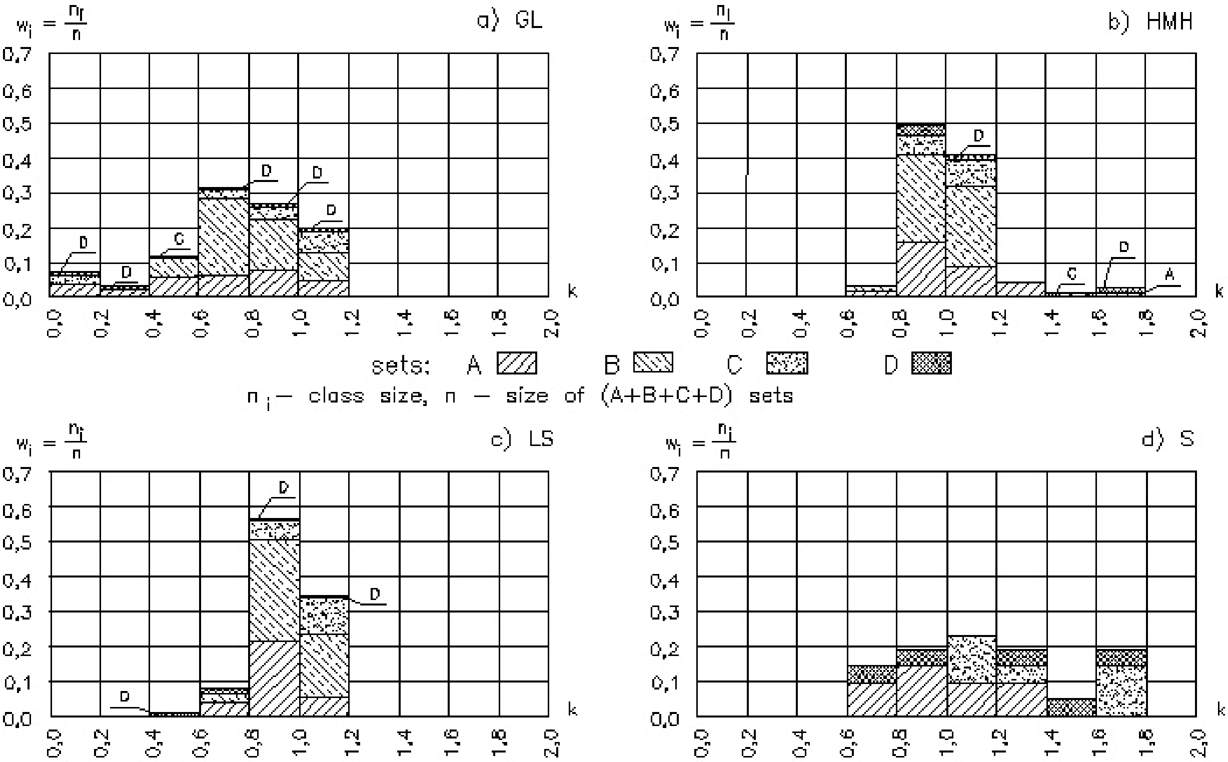


Fig. 7. Aproximation of the random k

The set of experimental tests presented is sufficiently large and varied to enable the verification of individual criteria to be made with relation to their universality. Thus, a new statistical hypothesis that the results obtained for all set of tests belong to the same general population, was investigated. The investigation was also based on the H test. Obtained test results showed that only HMH and S criteria are sufficiently universal. The non-dimissal of the hypothesis that the origin of all sets obtained applying the HMH and S criteria stems from the common general population allows further analyses to be made, based on the global set of results, including all test series.

To prove the correctness of any criterion, it is necessary to check its exactness of the assessment of the load-carrying capacity. Thus, the following statistical hypothesis was investigated: the mean value of the random variable k equals 1.0. In order to verify the statistical hypothesis in question, a test using U statistics (Greń [8]), independent of the type of distribution, was applied. This test could not have been applied to single sets obtained using Sobotka criterion, because of their small size. For that reason the hypothesis was verified for sets *A*, *B*, *C*, *A*+*D* and *A*+*B*+*C*+*D*.

The results obtained showed, that only in the case of HMH criterion the statistical hypothesis referring to individual sets was not rejected at the level of α_{min}=0.05, (0,696 for set *A*, 0,200 for set *B*, 0,058 for set *C*, 0,868 for set *A*+*D*) and for sets containing all the data (set *A*+*B*+*C*+*D*) – even at the level of α_{min}=0.41. With reference to the set of all the results, S criterion gives also a fairly correct assessment of the mean load-carrying capacity, however significantly worse than that given by HMH criterion. Despite this, it is difficult to recommend the application of the S criterion, as some tests of slabs exist, for which the assessment of the mean load-carrying capacity is unsatisfactory. GL criterion does not produce a correct assessment of the mean value, and the results of application of LS criterion depend on the type of reinforcement in the slabs; in some cases it gives a sufficiently good assessment of the load-carrying capacity, in others – the results obtained differ appreciably from test results.

An analysis of the bar charts indicates high skewness of distributions. The extension of a set of results beyond the mean value can be caused by the properties of random variable, and by reaching the strain-hardening zone beyond the plastic limit by steel reinforcement, not revealed during the tests. It shall be a reason of the difficulties in finding the theoretical limit distribution for random variable *k*, indispensable for safety analysis. It led to an attempt to approximate the part of the distribution received for HMH criterion below the median by normal distribution.

The check on the hypothesis of conformity of the distribution of the random variable under consideration to the normal distribution was based on the χ^2 test. As test results were depending on the number of intervals into which the set of random variable values was divided, the confidence level obtained varied within 0,056 and 0,119 limits and in all cases was greater than the level generally accepted as minimum equal 0,05. Estimation of the parameters of one-half of the distribution gave as the result the following paremeters: mean value k_m=0.997 and coefficient of variation n=0.09729.

Conclusions

The statistical investigations presented here included a set of results, a part of which was estimated. The inaccuracies resulting from this should not affect significantly the conclusions, as the proportion of these estimated results was small (less than 20%) and the methods used to assess the unknown quantities should not be large.

GL criterion proved useless in the analysis of the load carrying capacity of slabs. Main deficiency of the criterion is neglecting the influence of the torsional moment. LS and S criteria can be successfully applied, but only within a limited range. At present the limits of their applicability are unknown. The limits of application S criterion, given by the Author [3] with reference to slabs with top-and-bottom orthogonal renforcement – are not sufficient, but important. It can be stated that the most universal and most accurate is the HMH criterion amongst all assessed criteria. Its greater precision and universality in assessing local load-carrying capacity of slabs allows to recommend it as at least equivalent to other criteria used in engineering practice. An additional advantage is that this method can be used in cases of slabs with incomplete orthogonal reinforcement.

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