УДК 534.121.1

O. N. Petrishchev[1], *D.Sc.(Engineering), professor,*
V. M. Sharapov[2], *D.Sc.(Engineering), professor,*
Zh. V. Sotula[2], *Ph. D.,*
K. V. Bazilo[2], *Ph. D.*
[1]National Technical University of Ukraine «Kyiv Polytechnic Institute»
Peremogy ave., 37, Kyiv, 03056, Ukraine
[2]Cherkasy State Technological University
Shevchenko blvd., 460, Cherkasy, Ukraine, 18006
v_sharapov@rambler.ru

PRINCIPLES OF CALCULATION OF PIEZOELECTRIC ELEMENTS WITH SURFACES PARTIAL COVERING BY ELECTRODES

A mathematical formulation of boundary problems of mathematical physics, sequential solution of which allows to describe stress-strain state and transfer characteristics of piezoelectric elements with surfaces partial covering by electrodes is made. The set of formulated boundary problems makes mathematical content of energy-power method of the analysis of physical condition of piezoelectric elements in the regime of forced oscillations under the influence of an external source of electrical energy. The proposed sequence of computational procedures can be used in the study of forced oscillations of microelectromechanical structures.

Keywords: *surfaces partial covering of piezoelectric element by electrodes, microelectromechanical structures.*

Introduction. In 1986 [1] for the first time at the pages of scientific journals the abbreviation MEMS, which replaced the long spoken phrase microelectromechanical structures, appeared. Currently leading companies – manufacturers of electronic components serially produced a rather extensive list of elements, in which various MEMS are included. These, above all, are various accelerometers, which are produced by millions of copies, resonators and implemented on the basis of their electrical signals filters, transformers and other microminiature electromechanical systems.

MEMS manufacturing technology is currently called as microsystem technology. MEMS or, what is the same, piezoelectric elements produced by means of microsystem technology have much in common with conventional piezoelectric elements, i.e. not of microscopic size, which are made of piezoelectric ceramics. For the realization of various features in MEMS polycrystal ferroelectrics are used, which are polarized by constant electric field in predetermined direction. Conventional piezoelectric elements are made of piezoelectric ceramics, which initially is a polycrystalline ferroelectric, which on the last process step of piezoceramic product manufacturing is polarized by constant electric field of given orientation. A distinctive feature between the MEMS and conventional piezoelectric elements is a method of working surfaces covering by electrodes. Conventional piezoelectric elements are usually of continuous electrodes surface. In some special cases, the electrodes are divided (cut) into areas which have no galvanic connection between them. In MEMS partial working surfaces covering by electrodes is usually used, when only a part of polarized ferroelectric surface is covered with a metal film. This method of covering by electrodes makes it possible to excite in the volume of MEMS several types of elastic vibrations. Manipulating by geometric parameters of surfaces covering by electrodes, you can manage energy oscillatory processes in MEMS, i.e. create conditions where one type of oscillatory motions will dominate over the rest in amplitude of elastic displacement vector of material particles.

Partial surfaces covering by electrodes has the effect that electric field in the volume of the ferroelectric becomes dependent on the values of the coordinates of observation point for the parameters of electric field. This phenomenon is absent in conventional piezoelectric elements. On the condition of constancy of alternating electric field intensity in the volume of piezoelectric element, created by external generator of electrical signals the methods of calculation of stress-strain state of piezoelectric plates [2] and shells [3] are based. In the case of partial surfaces covering by electrodes of piezoelectric element these methods do not work.

The latter encourages the search for new approaches to the procedure of calculating of the parameters of stress-strain state and transfer characteristics of piezoelectric elements with partial surfaces covering by electrodes. Below we will present the general scheme of computational procedures execution, which, in principle, allows you to estimate the parameters of stress-strain state and transfer characteristics of piezoelectric elements with partial surfaces covering by electrodes. This scheme can be used as a theoretical basis for mathematical modeling of MEMS.

The sequence of computational procedures in the calculation of piezoelectric elements with partial surfaces covering by electrodes

Let's consider a piezoelectric element for certainty in the form of a plate bounded by arbitrarily curved contour K (Fig. 1).

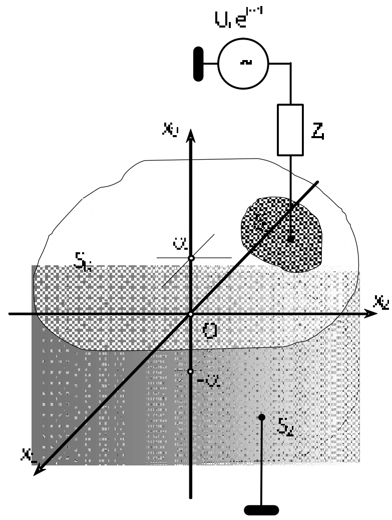

Fig. 1. Piezoelectric element with partial surfaces covering by electrodes $x_3 = \pm\alpha$

The plate is located in Cartesian coordinate system (x_1, x_2, x_3) so that the origin of the system is on the middle surface of the plate. To simplify subsequent discussion, we assume that the plate is made polarized in thickness, i.e. in the axis direction Ox_3, piezoelectric ceramics. We also assume that the polarization is made by constant electric field, the axial component of the intensity vector which had a constant value at any point in the volume of the plate. This allows to assert that material constants of piezoelectric plate (dielectric constant tensor components, piezoelectric modules and elasticity modules) do not depend on the coordinates of the point inside the volume of the plate and are given by matrices of the following form:

a) matrix of dielectric constants χ_{ij}^{ε}, that are experimentally determined in the mode of constancy (equality to zero) of elastic deformations (upper symbol ε):

$$\left|\chi_{ij}^{\varepsilon}\right| = \begin{vmatrix} \chi_{11}^{\varepsilon} & 0 & 0 \\ & \chi_{22}^{\varepsilon} & 0 \\ & & \chi_{33}^{\varepsilon} \end{vmatrix}, \quad (1)$$

where $\chi_{11}^{\varepsilon} = \chi_{22}^{\varepsilon} \neq \chi_{33}^{\varepsilon}$;

b) matrix of piezoelectric modules $e_{k\beta}$ ($k = 1,2,3$; $\beta = 1,2,3,4,5,6$ – Voigt index):

$$\left|e_{k\beta}\right| = \begin{vmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{vmatrix}, \quad (2)$$

where $e_{31} = e_{32} \neq e_{33}$; $e_{15} = e_{24} = (e_{33} - e_{31})/2$;

c) matrix of elasticity modules $c_{\beta\lambda}^{E}$, that are experimentally determined in the mode of constancy (equality to zero) of electric field intensity (upper symbol E), ((β, λ) = 1, … , 6 – Voigt index):

$$\left|c_{\beta\lambda}^{E}\right| = \begin{vmatrix} c_{11}^{E} & c_{12}^{E} & c_{13}^{E} & 0 & 0 & \\ & c_{22}^{E} & c_{23}^{E} & 0 & 0 & 0 \\ & & c_{33}^{E} & 0 & 0 & 0 \\ & & & c_{44}^{E} & 0 & 0 \\ & & & & c_{55}^{E} & 0 \\ & & & & & c_{66}^{E} \end{vmatrix}, \quad (3)$$

where $c_{11}^{E} = c_{22}^{E} \neq c_{33}^{E}$; $c_{12}^{E} = c_{13}^{E} = c_{23}^{E}$; $c_{44}^{E} = c_{55}^{E}$; $c_{66}^{E} = \left(c_{11}^{E} - c_{12}^{E}\right)/2$.

Let's suppose that on the upper ($x_3 = \alpha$) and lower ($x_3 = -\alpha$) surfaces of piezoelectric plate are arbitrarily located areas S_1 and S_2 accordingly (Fig. 1), which are coated with a thin layer of metal, i.e. covered by electrodes. Thus, in general $S_1 \neq S_2 \neq S_K$, where S_K – the surface of the plate bounded by the contour K. The electrical potential $U_0 e^{i\omega t}$ is supplied to the surface S_1 from electrical signals generator (U_0 – amplitude of electrical potential on the surface S_1 covered by electrodes; naturally, $U_0 \neq U_r$, where U_r – amplitude value of electric potential at the output of the generator; $i = \sqrt{-1}$; ω – angular frequency; t – time). Symbol Z_r in Fig. 1 denotes electric output impedance of electrical signals generator.

Denote the amplitude value of intensity vector of alternating electric field by symbol $\vec{E}^*(x_k)$. We'll determine electric polarization, which is created by alternating electric field in the volume of piezoelectric plate, by time-varying electric induction vector according to the law $e^{i\omega t}$, amplitude value of which is denoted by $\vec{D}^*(x_k)$. At the same time there is linear conformity between the components of electric induction vector and electric field intensity vector of the following form:

$$D_k^*(x_k) = \chi_{kj}^{\varepsilon} E_j^*(x_k). \tag{4}$$

In (4) and in all subsequent recordings of this type the summation over twice repeated index is assumed. In the formula (4) such index is symbol j which sequentially assumes values 1, 2, 3.

Characteristics of alternating electric field in the volume of piezoelectric element are determined by Maxwell's equations, which for the amplitude values of electric and magnetic components of electromagnetic field can be written in the following form:

$$\mathrm{rot}\vec{H}^*(x_k) = \vec{J}^*(x_k) + i\omega\vec{D}^*(x_k), \tag{5}$$

$$\mathrm{rot}\vec{E}^*(x_k) = -i\omega\vec{B}^*(x_k), \tag{6}$$

where $\vec{B}^*(x_k) = \mu_0\vec{H}^*(x_k)$; $\mu_0 = 4\pi\cdot 10^{-7}$ Гн/м – magnetic permeability of vacuum; $\vec{B}^*(x_k)$ and $\vec{H}^*(x_k)$ – amplitudes of harmonically time-varying vectors of induction and magnetic field intensity; $\vec{J}^*(x_k)$ – amplitude of conduction current surface density vector. Since piezoelectric ceramics is a pretty good dielectric we can write $\vec{J}^*(x_k) = 0$. This means that the properties of ideal dielectric are attributed to real piezoelectric ceramics.

Calculating the divergence of the left and right parts of Maxwell equations (5), we obtain, in the case of ideal dielectric, the following result:

$$\mathrm{div}\vec{D}^*(x_k) = 0. \tag{7}$$

In [3] it is proved that in frequency range from zero to tens of megahertz the equation (6) for ideal dielectrics can be written in the following form:

$$\mathrm{rot}\vec{E}^*(x_k) \cong 0. \tag{8}$$

The condition (8) indicates a potential character of alternating electric field in piezoelectric element volume. For this reason the description of this field with the help of scalar potential $\Phi^*(x_k)e^{i\omega t}$ is possible. At the same time amplitude value $\vec{E}^*(x_k)$ of intensity vector of alternating electric field created by an external source, i.e. the generator of electrical signals, is determined in the standard way [4]:

$$\vec{E}^*(x_k) = -\mathrm{grad}\,\Phi^*(x_k). \tag{9}$$

Substituting the definition (9) into (4), and obtained result into the condition (7) of the absence of electricity free carriers, we obtain Laplace differential equation the solution of which determines the amplitudes of electric potential $\Phi^*(x_k)$:

$$\frac{\partial^2\Phi^*(x_k)}{\partial x_1^2} + \frac{\partial^2\Phi^*(x_k)}{\partial x_2^2} + \xi^2\frac{\partial^2\Phi^*(x_k)}{\partial x_3^2} = 0, \tag{10}$$

where $\xi^2 = \chi_{33}^{\varepsilon}/\chi_{11}^{\varepsilon}$ – squared coefficient of the anisotropy of piezoelectric ceramics dielectric constant.

General solution of differential equation (10) must satisfy the following conditions at the boundaries of the region of existence:

$$\Phi^*(x_k)\big|_{x_3=\alpha} = U_0\ \forall(x_1,x_2)\in S_1; \tag{11}$$

$$\left.\frac{\partial\Phi^*(x_k)}{\partial x_3}\right|_{x_3=\alpha} = 0\ \forall(x_1,x_2)\in S_K - S_1, \tag{12}$$

$$\frac{\partial\Phi^*(x_k)}{\partial n} = 0\ \forall(x_k)\in K, \tag{13}$$

$$\left.\frac{\partial\Phi^*(x_k)}{\partial x_3}\right|_{x_3=-\alpha} = 0\ \forall(x_1,x_2)\in S_K - S_2, \tag{14}$$

$$\Phi^*(x_k)\big|_{x_3=-\alpha} = 0\ \forall(x_1,x_2)\in S_2, \tag{15}$$

where symbol $\partial/\partial n$ denotes the derivative in the direction of outward unit normal to lateral surface of the plate, which relies on the contour K.

Conditions (11) and (15) are self-evident. Conditions (12)–(14) are approximate [3] and are performed more accurately when piezoelectric dielectric constant differs more from dielectric constant of vacuum $\chi_0 = 8{,}85\cdot 10^{-12}$ Ф/м. For piezoelectric ceramics PZT-type dielectric constant is $\chi_{ij}^{\varepsilon} \geq 10^3\chi_0$ and thus boundary conditions (12)–(14) can be considered practically exact. If dielectric constant of polarized ferroelectric in MEMS is less than $10\chi_0$, that, incidentally, has not yet happened, then the conditions (12)–(14) must be reformulated, taking into account the existence of stray fields.

The solution of boundary problem (10)–(15) is the first computational procedure when performing calculations of parameters of stress-strain state of piezoelectric elements that are compatible with microsystem technologies. At this stage energy levels of electric field are actually determined at any arbitrarily predetermined point in the volume of piezoelectric element. It must be emphasized that the potential $\Phi^*(x_k)$ is calculated assuming that it is completely determined by geometric parameters of piezoelectric element, and does not depend on the parameters of elastic and electric fields, that are generated in the volume of vibrating piezoelectric element.

After determining the potential $\Phi^*(x_k)$ the equation of physical condition of oscillating piezoelectric element, calculation scheme of which is shown in fig. 1, can be written in the following form:

$$\sigma_{ij} = c^E_{ijk\ell}\frac{\partial u_\ell}{\partial x_k} - e_{kij}\left(\frac{\partial \Phi}{\partial x_k} + \frac{\partial \Phi^*}{\partial x_k}\right), \quad (16)$$

$$D_m = e_{mij}\frac{\partial u_i}{\partial x_j} + \chi^\varepsilon_{mj}\left(\frac{\partial \Phi}{\partial x_j} + \frac{\partial \Phi^*}{\partial x_j}\right), \quad (17)$$

where σ_{ij} – amplitude value of tensor component of resulting mechanical stresses; u_ℓ – the amplitude of time-varying ℓ-th component of displacement vector of piezoelectric material particles according to harmonic law $e^{i\omega t}$; $\Phi \equiv \Phi(x_k)$ - scalar potential of internal electric field [5], which arises as a result of the displacement of piezoelectric ions from equilibrium positions of crystal lattice sites; $\Phi^* \equiv \Phi^*(x_k)$ - known scalar potential of alternating electric field created by an external source (generator). By its physical content the equation of physical condition (16) is a generalized Hooke's law for an elastic medium with piezoelectric effects, and equation (17) is the law of electric polarization of the dielectric with piezoelectric properties.

Newton's second law in differential form or, what is the same, – the motion equation of a material particle of elastically deformable solid in general case can be written as following:

$$\frac{\partial \sigma_{ji}(x_k,t)}{\partial x_j} = \rho_0\frac{\partial^2 u_i(x_k,t)}{\partial t^2}, \quad (18)$$

where ρ_0 - the density of deformable solid. Substituting the equation (16) into the definition (18), and taking into account that all of physical fields in the volume of deformable piezoelectric element change in time according to harmonic law $e^{i\omega t}$ we obtain the following equation:

$$c^E_{ijk\ell}\frac{\partial^2 u_\ell}{\partial x_k \partial x_j} - e_{kij}\frac{\partial^2 \Phi}{\partial x_k \partial x_j} + \rho_0\omega^2 u_i = f^K_i(x_k), \quad (19)$$

where $f^K_i(x_k) = e_{kij}\dfrac{\partial^2 \Phi^*}{\partial x_k \partial x_j}$ – amplitude value of the *i*-th component of volume density of Coulomb forces that are generated by an external source (generator) of electric field. Naturally, the second term on the left-hand side of the equation (19) also makes sense as the *i*-th component of volume density vector amplitude value of Coulomb forces, which arise in the volume of deformable piezoelectric by internal electric field and prevent its deformation by electric fields of external sources. For small values of piezoelectric modules, i.e. when $e_{kij} \le 1\,\text{Кл}/\text{м}^2$, the second summand in the equation, in principle, can be ignored. But for piezoelectric ceramics of PZT type, that have $e_{kij} \le 20\,\text{Кл}/\text{м}^2$ the Coulomb forces of internal electric field cannot be ignored. These forces, i.e. the second summand in the equation (19), act in accordance with the forces of elasticity (the first summand) and increase the effective rigidity of the piezoelectric. In some directions the rigidity of piezoelectric ceramics can be increased more than 50%.

The condition (7) of electricity free carriers absence in piezodielectric is true, obviously, for any representation of electric induction vector. Since $\operatorname{div}\vec{D}(x_k) = 0$, then after substitution of the equation (17) into this condition we obtain the following result:

$$\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \xi^2\frac{\partial^2 \Phi}{\partial x_3^2} + \frac{1}{\chi^\varepsilon_1}e_{mij}\frac{\partial^2 u_i}{\partial x_m \partial x_j} = 0. \quad (20)$$

Equations (19) and (20) are, in general, a system of four differential equations of second order partial derivatives. Uniqueness of the solution of this equations system is provided by boundary conditions. If piezoelectric element fluctuates in a vacuum, i.e. has no mechanical contact with other material objects, then on its lateral surfaces due to the execution of the third law of Newton the normal and tangential stresses acting on elementary areas of these surfaces should turn to zero. With regard to the calculation scheme, which is shown in fig. 1, the foregoing can be written as following:

$$c_{3jk\ell}\frac{\partial u_\ell}{\partial x_k}-e_{k3j}\left(\frac{\partial \Phi}{\partial x_k}+\frac{\partial \Phi^*}{\partial x_k}\right)\bigg|_{x_3=\pm\alpha}= , \quad (21)$$

$$=0\,\forall(x_1,x_2)\in S_K$$

$$n_i\left[c_{ijk\ell}\frac{\partial u_\ell}{\partial x_k}-e_{kij}\left(\frac{\partial \Phi}{\partial x_k}+\frac{\partial \Phi^*}{\partial x_k}\right)\right]=0\,\forall x_k\in K, \quad (22)$$

where n_i – vector components of outward unit normal to lateral cylindrical surface of piezoelectric element, whose base is curved contour K (fig. 1).

Scalar potential $\Phi(x_k)$ of internal electric field must satisfy the following boundary conditions:

$$\Phi(x_k)\big|_{x_3=\alpha}=0\,\forall(x_1,x_2)\in S_1, \quad (23)$$

$$\frac{\partial\Phi(x_k)}{\partial x_3}\bigg|_{x_3=\alpha}=0\,\forall(x_1,x_2)\in S_K-S_1, \quad (24)$$

$$\frac{\partial\Phi(x_k)}{\partial x_1}\bigg|_{x_3=\alpha}=\frac{\partial\Phi(x_k)}{\partial x_2}\bigg|_{x_3=\alpha}=0\,\forall(x_1,x_2)\in S_1, \quad (25)$$

$$n_i\frac{\partial\Phi(x_k)}{\partial x_i}=0\,\forall x_k\in K, \quad (26)$$

$$\Phi(x_k)\big|_{x_3=-\alpha}=0\,\forall(x_1,x_2)\in S_2, \quad (27)$$

$$\frac{\partial\Phi(x_k)}{\partial x_3}\bigg|_{x_3=-\alpha}=0\,\forall(x_1,x_2)\in S_K-S_2, \quad (28)$$

$$\frac{\partial\Phi(x_k)}{\partial x_1}\bigg|_{x_3=\alpha}=\frac{\partial\Phi(x_k)}{\partial x_2}\bigg|_{x_3=\alpha}=0\,\forall(x_1,x_2)\in S_1. \quad (29)$$

Formulated by (19)–(29), boundary problem is the most complete and general mathematical formulation of electroelasticity boundary problem of steady harmonic vibrations of finite size piezoelectric elements.

Without going into a lengthy discourse, it can be argued that exact solution of boundary problem cannot be implemented. However, it is possible to create an algorithm for approximate solution of the problem (19)–(29). Computational procedure, which will be discussed, can be called the method of sequential approximations. In this case, the unknown components of displacement vector $u_\ell(x_k)$ and scalar potential $\Phi(x_k)$ are represented by convergent series as following:

$$u_\ell(x_k)=u_\ell^{(0)}(x_k)+\sum_{\nu=1}^{\infty}\Delta u_\ell^{(\nu)}(x_k),$$

$$\Phi(x_k)=\Phi^{(0)}(x_k)+\sum_{\nu=1}^{\infty}\Delta\Phi^{(0)}(x_k), \quad (30)$$

where $u_\ell(x_k)$ and $\Phi(x_k)$ - exact solutions of equations (19), (20) system; $u_\ell^{(0)}(x_k)$ and $\Phi^{(0)}(x_k)$ - zero approximation to exact solutions; $\Delta u_\ell^{(\nu)}(x_k)$ and $\Delta\Phi^{(0)}(x_k)$ - ν-th order corrections to zero approximations of exact $u_\ell(x_k)$ and $\Phi(x_k)$.

Zero approximation $u_\ell^{(0)}(x_k)$ to exact value of the ℓ-th component amplitude of displacement vector of piezoelectric element material particles is found by solving the following stationary boundary problem:

$$c_{ijk\ell}^{E}\frac{\partial^2 u_\ell^{(0)}}{\partial x_k\partial x_j}+\rho_0\omega^2 u_\ell^{(0)}=f_i^K, \quad (31)$$

$$\left[c_{3jk\ell}^{E}\frac{\partial u_\ell^{(0)}}{\partial x_k}-\sigma_{3j}^{K}\right]\bigg|_{x_3=\pm\alpha}=0\,\forall(x_1,x_2)\in S_K, \quad (32)$$

$$n_i\left[c_{ijk\ell}^{E}\frac{\partial u_\ell^{(0)}}{\partial x_k}-\sigma_{ij}^{K}\right]=0\,\forall x_k\in K, \quad (33)$$

where $\sigma_{3j}^K=e_{k3j}(\partial\Phi^*/\partial x_k)$, $\sigma_{ij}^K=e_{kij}(\partial\Phi^*/\partial x_k)$ – Coulomb forces surface densities of external source electric field or, which is the same, Coulomb tensions.

Thus, zero approximation to exact values of displacement vector components of piezoelectric material particles is formed by the solution of boundary problem (31)–(33). According to physical content, this boundary problem of elasticity dynamic theory is the problem of the excitation of piezoelectric element harmonic oscillations by the system of volumetric (f_i^K) and surface (σ_{ij}^K) loads. It should be emphasized that at full piezoelectric elements surfaces covering by electrodes and even in the case of separated electrodes Coulomb forces volume density is zero, and the excitation of elastic waves in such piezoelectric elements is operated by surface loads σ_{ij}^K.

After solving boundary problem (31)–(33) zero approximation to exact value of scalar potential of internal electric field is determined. The equation (20) can be written in the following form:

$$\frac{\partial^2\Phi^{(0)}}{\partial x_1^2}+\frac{\partial^2\Phi^{(0)}}{\partial x_2^2}+\xi^2\frac{\partial^2\Phi^{(0)}}{\partial x_3^2}=-\frac{\rho_{пэ}^{(0)}}{\chi_1^{\varepsilon}}. \quad (34)$$

where $\rho_{пэ}^{(0)}=e_{mij}\frac{\partial^2 u_i^{(0)}}{\partial x_m \partial x_j}$ - zero approximation to exact value of volume density of polarization electric charge in the volume of deformable piezoelectric. General solution of Poisson equation (34) means that the function $\Phi^{(0)}(x_k)$ must satisfy boundary conditions (23)–(29).

After the determination of zero approximation $\Phi^{(0)}(x_k)$ the correction $\Delta u_\ell^{(1)}$ is calculated. For this purpose, into the equation (19) values $\vec{u}(x_k)\approx\vec{u}^{(0)}(x_k)+\Delta\vec{u}^{(0)}(x_k)$ and $\Phi(x_k)\approx\Phi^{(0)}(x_k)$ are inserted. The equation (19) takes the form

$$c_{ijk\ell}^{E}\frac{\partial^2\Delta u_\ell^{(1)}}{\partial x_k \partial x_j}+\rho_0\omega^2\Delta u_\ell^{(1)}=f_i^{(0)}, \quad (35)$$

where $\Delta u_\ell^{(1)}$ and $\Delta u_i^{(1)}$ - first-order corrections to zero approximations $u_\ell^{(0)}$ and $u_i^{(0)}$ of components exact values $u_\ell(x_k)$ and $u_i(x_k)$ of displacement vector of piezoelectric material particles; $f_i^{(0)}=e_{kij}\frac{\partial^2\Phi^{(0)}}{\partial x_k \partial x_j}$ - zero approximation to the *i*-th component exact value of volume density vector of Coulomb forces, which are formed by internal electric field in the volume of deformable piezoelectric.

Uniqueness of the system (35) solution is ensured by boundary conditions that can be written as following:

$$\left.\left[c_{3jk\ell}^{E}\frac{\partial\Delta u_\ell^{(1)}}{\partial x_k}-\sigma_{3j}^{(0)}\right]\right|_{x_3=\pm\alpha}=0\,\forall(x_1,x_2)\in S_K, \quad (36)$$

$$n_i\left[c_{ijk\ell}^{E}\frac{\partial\Delta u_\ell^{(1)}}{\partial x_k}-\sigma_{ij}^{(0)}\right]=0\,\forall x_k\in K, \quad (37)$$

where $\sigma_{3j}^{(0)}=e_{k3j}\left(\partial\Phi^{(0)}/\partial x_k\right)$, $\sigma_{ij}^{(0)}=e_{kij}\left(\partial\Phi^{(0)}/\partial x_k\right)$ - zero approximations to exact value of surface densities of Coulomb forces, which are formed by internal electric field in the volume of vibrating piezoelectric element.

It should be emphasized that boundary problem (35)–(37) can be solved by the same manner in which boundary problem (31)–(33) has been solved. This means that after the construction of analytical expression for $\vec{u}^{(0)}(x_k)$, i.e. for zero approximation to exact value of displacement vector of piezoelectric element material particles, it isn't needed to re-construct general solutions for first-order correction $\Delta\vec{u}^{(1)}(x_k)$ and for all subsequent orders corrections $\Delta\vec{u}^{(\nu)}(x_k)$. To obtain numerical values of corrections $\Delta\vec{u}^{(\nu)}(x_k)$ it is necessary only to substitute into general solutions of boundary problem (31)–(33) the corresponding values of Coulomb forces volume and surface densities, which are formed by internal electric field in the volume of piezoelectric vibrating element.

After receiving the first approximation $\vec{u}^{(1)}(x_k)=\vec{u}^{(0)}(x_k)+\Delta\vec{u}^{(1)}(x_k)$ to exact value of displacement vector of material particles, you can implement the assessment of correction $\Delta\Phi^{(1)}(x_k)$ to exact value $\Phi(x_k)$ of scalar potential of internal electric field. Substituting $\vec{u}^{(1)}(x_k)=\vec{u}^{(0)}(x_k)+\Delta\vec{u}^{(1)}(x_k)$ into the equation (34) and setting at the same time that $\Phi(x_k)\approx\Phi^{(0)}(x_k)+\Delta\Phi^{(1)}(x_k)$ we get Poisson equation to determine the correction $\Delta\Phi^{(1)}(x_k)$:

$$\frac{\partial^2\Delta\Phi^{(1)}}{\partial x_1^2}+\frac{\partial^2\Delta\Phi^{(1)}}{\partial x_2^2}+\xi^2\frac{\partial^2\Delta\Phi^{(1)}}{\partial x_3^2}=-\frac{\Delta\rho_{пэ}^{(1)}}{\chi_1^{\varepsilon}}, \quad (38)$$

where $\Delta\rho_{пэ}^{(1)}=e_{mij}\frac{\partial^2\Delta u_i^{(1)}}{\partial x_m \partial x_j}$ - first-order correction to exact value zero approximation of polarization charge volume density in the volume of deformable piezoelectric. Uniqueness of the equation (38) solution is ensured by boundary conditions (23)–(29).

Again, as in the determination of the correction $\Delta\vec{u}^{(1)}(x_k)$, obvious conclusion can be made. To receive the correction $\Delta\Phi^{(1)}(x_k)$ as, obviously, all subsequent corrections $\Delta\Phi^{(\nu)}(x_k)$, there is no need for a new solution of (38) and subsequent satisfaction of boundary conditions (23)–(29). To determine the correction $\Delta\Phi^{(1)}(x_k)$ it is necessary and sufficient to substitute the value $\Delta\rho_{пэ}^{(1)}$ into general solution of the equation (34) instead of volume density $\rho_{пэ}^{(0)}$. It is obvious that the correction is of ν-th order, i.e. the value $\Delta\Phi^{(\nu)}(x_k)$ is determined by general solution of the equation (34), in the right-hand part of which is written the correction $\Delta\rho_{пэ}^{(\nu)}(x_k)$.

Approximate calculation of the components of displacement vector and scalar potential of internal electric field forms the content of the second computational procedure for the processes of mathematical modeling, which are developed in the volume of piezoelectric element with partial surfaces covering by electrodes.

The results that are obtained after the first and second computing procedures, i.e. formulas for calculating of electric field potential $\Phi^*(x_k)$ created in the volume of piezoelectric element by electrical signals generator, for scalar potential $\Phi(x_k)$ of internal electric field and displacement vector $\vec{u}(x_k)$ of material particles, linearly depend on electric potential U_0 that exists on the surface S_1 covered by electrodes. From the above it follows that these relations can be calculated in the following form:

$$\Phi^*(x_k) = U_0\tilde{\Phi}^*(x_k,\Pi),$$

$$\Phi(x_k) = U_0\tilde{\Phi}(x_k,\omega,\Pi), \qquad (39)$$

$$u_\ell(x_k) = U_0\frac{e_{33}}{c_{33}^E}\tilde{u}_\ell(x_k,\omega,\Pi), \qquad (40)$$

where $\tilde{\Phi}^*(x_k,\Pi)$, $\tilde{\Phi}(x_k,\omega,\Pi)$ and $\tilde{u}_\ell(x_k,\omega,\Pi)$ - dimensionless functions of the observation point coordinates and the set of physical, mechanical and geometrical (symbol Π in argument list) parameters of piezoelectric element; the construction of the right part of (40) is due to dimension values included in it.

Since the potential $U_0 \neq U_г$, i.e. the potential U_0, is actually the uncertain quantity, so the third and final computational procedures aim to determine potential amplitude value on piezoelectric element surface covered by electrodes.

Obviously, the influence of piezoelectric element on the amplitude of electric current in the conductor, which connects it to electric signals generator, can be described using electrical impedance $Z_{эл}(\omega)$ of piezoelectric element. Electrical impedance $Z_{эл}(\omega)$ must satisfy the basic terms and definitions of theoretical electrical engineering and, therefore, Ohm's law for the section of electrical circuit, i.e.

$$Z_{эл}(\omega) = \frac{U_0}{I}, \qquad (41)$$

where I – the amplitude of electric current in the conductors which are connected to piezoelectric element surface sections covered by electrodes (fig. 1). Current or amplitude value of I is directly proportional to the velocity variation in time of electric charge Q on piezoelectric element surface covered by electrodes. For harmonic changing in time according to $e^{i\omega t}$ the amplitude I is determined as follows:

$$I = -i\omega Q_1 = -i\omega Q_2, \qquad (42)$$

where Q_1 and Q_2 - amplitude values of electric charges on the surfaces S_1 and S_2 (fig. 1).

Amplitude value of electric charge Q_1 is determined as follows:

$$Q_1 = \iint_{S_1} D_3(x_1,x_2,\alpha)dS_1 = \iint_{S_1}\left[e_{3ij}\frac{\partial u_i}{\partial x_j} + \chi_{3j}^{\varepsilon}\left(\frac{\partial\Phi}{\partial x_j} + \frac{\partial\Phi^*}{\partial x_j}\right)\right]\Bigg|_{x_3=\alpha} dS_1 =$$

$$= U_0\iint_{S_1}\left[e_{3ij}\frac{e_{33}}{c_{33}^E}\frac{\partial\tilde{u}_i(x_k,\omega,\Pi)}{\partial x_j} + \chi_{3j}^{\varepsilon}\left(\frac{\partial\tilde{\Phi}(x_k,\omega,\Pi)}{\partial x_j} + \frac{\partial\tilde{\Phi}^*(x_k,\Pi)}{\partial x_j}\right)\right]\Bigg|_{x_3=\alpha} dS_1 = U_0 C^{\varepsilon}(\omega,\Pi), \qquad (43)$$

where $C^{\varepsilon}(\omega,\Pi)$ - dynamic electric capacitance of piezoceramic element.

Thus, $I = -i\omega U_0 C^{\varepsilon}(\omega,\Pi)$ and electrical impedance of piezoelectric element is

$$Z_{эл}(\omega) = -\frac{1}{i\omega C^{\varepsilon}(\omega,\Pi)}. \qquad (44)$$

From the connection diagram of electrical signals generator (fig. 1) it follows that

$$U_0 = \frac{U_г Z_{эл}(\omega)}{Z_г + Z_{эл}(\omega)} = \frac{U_г}{1 - i\omega Z_г C^{\varepsilon}(\omega,\Pi)}, \qquad (45)$$

where $C^{\varepsilon}(\omega,\Pi)$ is determined by a double integral in (43).

The calculation of electric potential U_0 is the content of the third computational procedure in mathematical modeling of piezoelectric elements with partial surfaces covering by electrodes.

After completing the third and last computational procedure we can write an expression for calculating the displacement of material particles of piezoelectric element in the next, final, form

$$u_\ell(x_k) = \frac{e_{33}U_r}{c_{33}^E\left[1 - i\omega Z_r C^\varepsilon(\omega,\Pi)\right]}\tilde{u}_\ell(x_k,\omega,\Pi). \quad (46)$$

The equation (46) is a mathematical model of dynamic stress-strain state of piezoelectric element with partial surfaces covering by electrodes and is a key relation for quantitative estimates of transfer characteristics of piezoelectric element in all variants of its functional use. Frequency dependent function $C^\varepsilon(\omega,\Pi)$ is the theoretical basis for the equivalent circuits' construction in the sense in which they were proposed in 1925–1928 by Walter G. Cady to radio engineers, who were involved in the calculation and design of high-frequency electrical signal generators with quartz resonator in the generation frequency stabilization circuit [6].

Conclusions:

1. For the first time the sequence of computational procedures is offered, which allows when calculating transfer characteristics to use the full range of physical, mechanical and geometrical parameters of piezoelectric element and the final value of output electric impedance of electrical signals generator.

2. For the first time the method is offered, which allows to perform the real situation adequate assessment of connectivity effect of elastic and electric fields in the case of their arbitrary distribution in the volume of vibrating piezoelectric element.

3. Formulated complete set of boundary problems of technical electrodynamics and elasticity dynamic theory are the mathematical content of energy-power method [7–10] of physical state analysis of piezoelectric elements in the regime of forced oscillations under the influence of external source of electrical energy.

4. Proposed sequence of computational procedures can be recommended as a theoretical basis for characteristics calculating of piezoelectric elements with partial surfaces covering by electrodes and microelectromechanical structures (MEMS).

References

1. Varadan, V., Vinoy, K. and Jose, K. (2004) RF MEMS and their applications. Moscow: Technosphera, 528 p. [in Russian]
2. Anan'eva, A. A. (1958) On the calculation of the piston piezoelectric transducer without internal losses. *Acoustic journal*, (3), Vol. 4, pp. 223–232. [in Russian]
3. Grinchenko, V. T., Ulitko, A. F. and Shulga, N. A. (1989) Mechanics of related fields in elements of constructions. Vol. 5. Electroelasticity. Kyiv: Nauk. dumka, 280 p. [in Russian]
4. Tamm, I. E. (1976) Fundamentals of the electricity theory. Moscow: Nauka, 616 p. [in Russian]
5. Petrishchev, O. N. (2012) Harmonic oscillations of piezoceramic elements. Part 1. Harmonic oscillations of piezoceramic elements in a vacuum and the method of resonance – antiresonance. Kyiv: Avers, 300 p. [in Russian]
6. Cady, W. (1949) Piezoelectricity and its practical application. Moscow: Publishing house of foreign literature, 720 p. [in Russian]

Sharapov, V. (2011) Piezoceramic sensors. Heidelberg, Dordrecht, London, New York: Springer Verlag, 498 p.

Sharapov, V. M., Musienko M. P. and Sharapova E.V. (2006) Piezoelectric sensors. Moscow: Technosphera, 632 p. [in Russian]

Sharapov, V., Sotula, Zh. and Kunitskaya, L. (2013) Piezoelectric electroacoustic transducers. Heidelberg, Dordrecht, London, New York: Springer Verlag, 240 p.

Sharapov, V. M., Minaev, I. G., Sotula, Zh. V. and Kunitskaya, L. G. (2013) Electroacoustic transducers. Moscow: Technosphera, 280 p. [in Russian]

Стаття надійшла до редакції 30.07.2014 р.

О. Н. Петрищев[1], *д.т.н., профессор*,
В. М. Шарапов[2], *д.т.н., профессор*,
Ж. В. Сотула[2], *к.т.н.*,
К. В. Базило[2], *к.т.н.*
[1]Национальный технический университет Украины «Киевский политехнический институт»
проспект Победы, 37, г. Киев, 03056, Украина
[2]Черкасский государственный технологический университет
б-р Шевченко, 460, г. Черкассы, 18006, Украина
e-mail: v_sharapov@rambler.ru

ПРИНЦИПЫ РАСЧЕТА ПЬЕЗОЭЛЕКТРИЧЕСКИХ ЭЛЕМЕНТОВ С ЧАСТИЧНО ЭЛЕКТРОДИРОВАННЫМИ ПОВЕРХНОСТЯМИ

Выполнена математическая постановка граничных задач математической физики, последовательное решение которых позволяет описать напряженно-деформированное состояние и передаточные характеристики пьезоэлектрических элементов с частичным электродированием поверхностей. Совокупность сформулированных граничных задач составляет математическое содержание энергосилового метода анализа физического состояния пьезоэлектрических элементов в режиме вынужденных колебаний под действием внешнего источника электрической энергии. Предложенная последовательность вычислительных процедур может быть использована при исследовании вынужденных колебаний микроэлектромеханических структур.

Ключевые слова: *частичное электродирование поверхностей пьезоэлемента, микроэлектромеханические структуры.*