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CALCULATION OF PARAMETRES OF TENSELY DEFORMED STATE OF DISK PIEZOELEMENT WITH PARTIAL COVERING BY ELECTRODES SURFACES

In the paper general formulation of boundary problem concerning the calculation of scalar potential of axially symmetric electric field in a disk, the material of which has anisotropic dielectric constant, that is at least an order higher than dielectric constant of vacuum, is presented. Proposed scheme for solving of the problem uses the method of sequential approximations that allows to obtain analytical expressions for the calculation of coefficients in mathematical description of electric field potential.

Keywords: *scalar potential, Laplace's equation, anisotropy of dielectric constant, partial covering by electrodes of dielectric disk's end surfaces.*

Introduction. In disk piezoelectric elements with surface partial covering by electrodes we can simultaneously excite oscillations of compression-tension and transverse bending vibrations. Manipulating the geometric parameters of electrodes and their location relative to each other, you can have a significant effect on the energy of oscillatory motion particular type of material particles of piezoelectric disk volume. If transverse bending oscillations dominate then disk piezoelectric element may be used as a low frequency acoustic wave's projector. If one of electrodes is subjected to special processing to operate as light beams reflector, then disk piezoelectric element with surface partial covering by electrodes can be used as a focusing light flux deflector. In this case, it is necessary to emphasize that this piezoelectric element has the compatibility with microsystem technologies, i.e. may be made as a microelectromechanical structure (MEMS) [1].

If partial covering by electrodes is done with violation of axial symmetry of the whole structure, then this provides further opportunities to manage the parameters of stress-strain state of piezoelectric disk and transfer characteristics of electromechanical system in general.

All of the above is a good motivation for the study of parameters and characteristics of dynamic electroelastic fields in the volume of disk piezoelectric elements with surface partial covering by electrodes.

In [2] the general scheme of mathematical modeling of processes in piezoelectric elements with surface partial covering by electrodes is described. Following these calculation principles of piezoelectric elements with surface partial covering by electrodes, we can consider the first computational procedure, i.e. the calculation of spatial distribution of alternating electric field in the volume of piezoelectric element. Solving this problem allows to assess the quantity of energy that is consumed by piezoelectric element from an external source, i.e. from the generator of electrical signals. This procedure is the first stage of energy-power method [3-6] of physical state analysis of piezoelectric elements in the regime of forced oscillations under the influence of an external source of electrical energy.

Calculation of alternating electric field in the volume of disk piezoelectric element with surfaces partial covering by electrodes

Let's consider a disk (fig. 1) of thickness polarized PZT-type piezoelectric ceramics. The electrode on top surface ($z = \alpha$) of piezoceramic disk (position 1 in fig. 1) has circle shape of radius R_0 centered on the axis Oz of cylindrical coordinate system. Harmonically electric potential $U(t) = U_0 e^{i\omega t}$ varying in time according to $e^{i\omega t}$ ($i = \sqrt{-1}$; ω – angular frequency; t – time) is applied to this electrode. The electrode on bottom ($z = 0$) disk surface (position 2 in fig. 1) is formed in a ring shape centered on the axis Oz. Ring electrode is grounded, i.e. its potential is always zero.

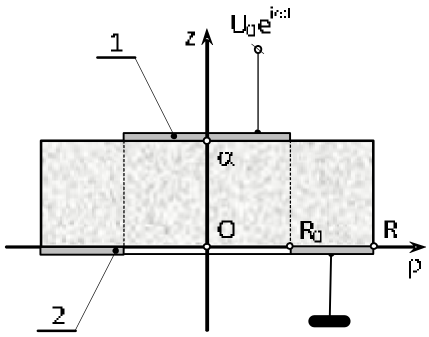

Fig. 1. Piezoelectric disk with partial covering by electrodes surfaces

Let's investigate the character of distribution of alternating electric field in the volume of piezoceramic disk assuming that outer radius R_0 of top electrode coincides with inner radius of bottom electrode.

In [2] it is shown that an alternating electric field in the volume of piezoelectric disk is determined by scalar potential $\Phi^*(\rho,\varphi,z)e^{i\omega t}$. With axial location of top and bottom electrodes the value of scalar potential amplitude depends only on radial ρ and axial z coordinates of cylindrical coordinate system. In other words, as shown in fig. 1, electrode structure creates axially symmetric alternating electric field in thickness polarized piezoceramic disk volume, scalar potential is determined by Laplace equation of the following form:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Phi^*(\rho,z)}{\partial\rho}\right] + \xi^2\frac{\partial^2\Phi^*(\rho,z)}{\partial z^2} = 0, \quad (1)$$

where $\xi^2 = \chi_{33}^{\varepsilon}/\chi_{11}^{\varepsilon}$ - squared coefficient of dielectric constant anisotropy of polarized ceramics; χ_{33}^{ε} – tensor component of dielectric constant in the direction of electric polarization; χ_{11}^{ε} – tensor component of dielectric constant in any direction on a plane that is perpendicular to the direction of disk material electric polarization.

To simplify further calculations we divide piezoceramic disk volume into two areas. First area ($0 \le \rho \le R_0$; $0 \le z \le \alpha$) will be called the inner one and electric potential within its limits will be denoted by $\Phi_{(1)}^*(\rho,z)$. Ring area ($R_0 \le \rho \le R$; $0 \le z \le \alpha$) will be called the outer area or the second area, and electric potential within its limits will be denoted by $\Phi_{(2)}^*(\rho,z)$.

The potentials on external and internal surfaces of inner and outer areas $\Phi_{(1)}^*(\rho,z)$ and $\Phi_{(2)}^*(\rho,z)$ must satisfy the following boundary conditions:

$$\Phi_{(1)}^*(\rho,\alpha) = U_0 \;\forall\rho\in[0, R_0], \quad (2)$$

$$\left.\frac{\partial\Phi_{(1)}^*(\rho,z)}{\partial z}\right|_{z=0} = 0 \;\forall\rho\in[0, R_0], \quad (3)$$

$$\Phi_{(1)}^*(R_0,z) - \Phi_{(2)}^*(R_0,z) = 0 \;\forall z\in[0, \alpha], \quad (4)$$

$$\frac{\partial\Phi_{(1)}^*(R_0,z)}{\partial z} - \frac{\partial\Phi_{(2)}^*(R_0,z)}{\partial z} = 0 \;\forall z\in[0, \alpha], \quad (5)$$

$$\left.\frac{\partial\Phi_{(1)}^*(\rho,z)}{\partial\rho}\right|_{\rho=R_0} - \left.\frac{\partial\Phi_{(2)}^*(\rho,z)}{\partial\rho}\right|_{\rho=R_0} = 0 \;\forall z\in[0, \alpha], \quad (6)$$

$$\left.\frac{\partial\Phi_{(2)}^*(\rho,z)}{\partial z}\right|_{z=\alpha} = 0 \;\forall\rho\in[R_0, R], \quad (7)$$

$$\Phi_{(2)}^*(\rho,0) = 0 \;\forall\rho\in[R_0, R], \quad (8)$$

$$\left.\frac{\partial\Phi_{(2)}^*(\rho,z)}{\partial\rho}\right|_{\rho=R} = 0 \;\forall z\in[0, \alpha]. \quad (9)$$

Conditions (3), (7) and (9) are approximate [7] and are performed the more accurate when dielectric constants χ_{33}^{ε} and χ_{11}^{ε} more differ the dielectric constant of vacuum $\chi_0 = 8{,}85\cdot10^{-12}$ Ф/м. If dielectric constants χ_{33}^{ε} and χ_{11}^{ε} more than three orders exceed the dielectric constant of piezoceramic disk surrounding, these boundary conditions can be considered as practically exact. Conditions (4)–(6) are matching conditions meaningful the solutions of equation (1) on the boundary $\rho = R_0$ of inner and outer areas.

The solution of equation (1) for both external and internal areas will search at standard technology of separation of variables [8], i.e., we assume that

$$\Phi_{(k)}^*(\rho,z) = R^{(k)}(\rho)Z^{(k)}(z), \; k = 1,2, \quad (10)$$

where $R^{(k)}(\rho)$ and $Z^{(k)}(z)$ - functions depending only on radial and axial coordinates respectively.

Substituting the expected form of the solution (10) into the equation (1), we get the opportunity to write it in the following form:

$$\frac{1}{R^{(k)}(\rho)}\left[\frac{1}{\rho}\frac{\partial R^{(k)}(\rho)}{\partial \rho}+\frac{\partial^2 R^{(k)}(\rho)}{\partial \rho^2}\right]= =-\frac{\xi^2}{Z^{(k)}(z)}\frac{\partial^2 Z^{(k)}(z)}{\partial z^2}. \quad (11)$$

Equation (11) can be satisfied for arbitrary values of variables ρ and z, and only in one case, when left and right parts do not depend on ρ and z respectively, and both are equal to the same constant, that is called separation constant [8]. The choice of separation constant is largely predetermined by physical content of solved problem. For the convenience of subsequent records we denote separation constant by symbol β^2. In this case, from the equation (11) two ordinary differential equations of the following form can be obtained:

$$\frac{1}{R^{(k)}(\rho)}\left[\frac{1}{\rho}\frac{\partial R^{(k)}(\rho)}{\partial \rho}+\frac{\partial^2 R^{(k)}(\rho)}{\partial \rho^2}\right]=\beta^2, \quad (12)$$

$$-\frac{\xi^2}{Z^{(k)}(z)}\frac{\partial^2 Z^{(k)}(z)}{\partial z^2}=\beta^2. \quad (13)$$

The abundance of boundary conditions (see relations (2)–(9)) requires an appropriate set of constants in general solutions $\Phi^*_{(k)}(\rho,z)$. To provide appropriate number of constants is possible as follows.

Let's consider inner area ($k=1$).

If in equations (12) and (13) to determine the parameter β as a real number β_1, we obtain the following expression for scalar potential:

$$\Phi^*_{(11)}(\rho,z)= =I_0(\beta_1\rho)\left[A_1\cos(\lambda_1 z)+B_1\sin(\lambda_1 z)\right], \quad (14)$$

where $I_0(\beta_1\rho)$ - modified Bessel function of zero order; A_1 and B_1 - constants; $\lambda_1=\beta_1/\xi$; β_1 - determined real number.

Suppose now that the parameter β is an imaginary number $i\beta_2$, i.e. $\beta^2=-\beta_2^2$. Then from equations (12) and (13) we obtain the following expression for scalar potential calculation of inner area of disk piezoelectric element:

$$\Phi^*_{(12)}(\rho,z)= =J_0(\beta_2\rho)\left[A_2 ch(\lambda_2 z)+B_2 sh(\lambda_2 z)\right], \quad (15)$$

where $J_0(\beta_2\rho)$ - Bessel function of zero order; A_2 and B_2 - constants; $\lambda_2=\beta_2/\xi$; β_2 - determined number. It is obvious that the superposition of general solutions $\Phi^*_{(11)}(\rho,z)$ and $\Phi^*_{(12)}(\rho,z)$ is also a general solution of the equation (1), i.e.

$$\Phi^*_{(1)}(\rho,z)=\Phi^*_{(11)}(\rho,z)+\Phi^*_{(12)}(\rho,z). \quad (16)$$

From boundary condition (3) follows that

$$\lambda_1 I_0(\beta_1\rho)B_1+\lambda_2 J_0(\beta_2\rho)B_2= =0\ \forall\rho\in[0,R_0]. \quad (17)$$

If we assume $B_1=B_2=0$, the equation (17) and, respectively, boundary condition (3) will be performed automatically.

As the numbers β_1 and β_2 can be assigned arbitrarily, we will determine those according to the following conditions:

$$\cos(\lambda_1\alpha)=0, \quad (18)$$

$$J_0(\beta_2 R_0)=0. \quad (19)$$

From the conditions (18) and (19) it follows that

$$\beta_1=\beta_n=\frac{\pi}{2\alpha}(1+2n)\xi,\ n=0,1,2,..., \quad (20)$$

$$\beta_2=q_m/R_0,\ m=1,2,3,..., \quad (21)$$

where q_m - m-th root of the equation $J_0(x)=0$. The first five roots of this equation have the following numerical values: $q_1=2{,}404826$, $q_2=5{,}520078$, $q_3=8{,}653728$, $q_4=11{,}791534$ and $q_5=14{,}930918$. It's easy to see that $q_m-q_{m-1}\approx\pi$, while approximate equality holds more accurate when number m of the root is higher.

Since the eigenvalues β_k and λ_k ($k=1,2$) form infinite sets then they must comply with infinite sets of constants A_{1n} and A_{2m}. The expression for the calculation of scalar potential $\Phi^*_{(1)}(\rho,z)$ takes the following form:

$$\Phi^*_{(1)}(\rho,z)=\sum_{n=0}^{\infty}A_{1n}I_0(\beta_n\rho)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]+ +\sum_{m=1}^{\infty}A_{2m}J_0\left(\frac{q_m\rho}{R_0}\right)ch(\lambda_m z), \quad (22)$$

where $\lambda_m=q_m/(\xi R_0)$.

When $z=\alpha$, boundary condition (3) must be performed. Substituting $z=\alpha$ to the calculation formula (22), we obtain

$$\sum_{m=1}^{\infty}A_{2m}J_0\left(\frac{q_m\rho}{R_0}\right)ch(\lambda_m\alpha)=U_0. \quad (23)$$

Bessel functions $J_0(q_m\rho/R_0)$ in the interval $0\le\rho\le R_0$ form a system of orthogonal functions, i.e. there exists the integral [9] of the following form:

$$\int_0^{R_0} \rho J_0\left(q_m\rho/R_0\right) J_0\left(q_p\rho/R_0\right) d\rho = \\ = \begin{cases} 0 \,\forall m \neq p, \\ R_0^2 J_1^2\left(q_m\right)/2 \text{ при } m=p, \end{cases} \qquad (24)$$

where $J_1(q_m)$ - Bessel function of the first order. Using orthogonality property (24), from the equation (23) we obtain the following values of the coefficients A_{2m}:

$$A_{2m} = U_0 \tilde{A}_{2m}(q_m), \qquad (25)$$

where dimensionless factor $\tilde{A}_{2m}(q_m)$ is determined by the following expression:

$$\tilde{A}_{2m}(q_m) = \frac{2}{q_m J_1(q_m) \operatorname{ch}\left(\frac{q_m \alpha}{\xi R_0}\right)}. \qquad (26)$$

Table 1 shows the results of calculations of dimensionless weight factors $\tilde{A}_{2m}(q_m)$ for the first twenty roots of the equation $J_0(x) = 0$. While doing calculations the anisotropy coefficient $\xi = \sqrt{\chi_{33}^{\varepsilon}/\chi_{11}^{\varepsilon}}$ of dielectric constant of polarized piezoceramics is taken as unity, i.e. accepted that $\chi_{33}^{\varepsilon}/\chi_{11}^{\varepsilon} = 1$. In fact, for the type of piezoelectric ceramics ЦТС-19 the coefficient $\xi = 0{,}97 \div 0{,}99$. The third column of the table 1 shows the values of weight coefficient $\tilde{A}_{2m}^0 = 1/[q_m J_1(q_m)]$ that do not depend on geometrical parameter α/R_0. Dimensionless weight factor $\tilde{A}_{2m}(q_m)$ was calculated for different values of the parameter α/R_0 that is specified in the headers of the columns of table 1.

Table 1

The numerical values of dimensionless weight factors $\tilde{A}_{2m}(q_m)$ for the first twenty roots q_m of the equation $J_0(x) = 0$

m	q_m	$\tilde{A}_{2m}^0$	$\tilde{A}_{2m}(q_m)$			
			$\alpha/R_0 = 0{,}1$	$\alpha/R_0 = 0{,}2$	$\alpha/R_0 = 0{,}4$	$\alpha/R_0 = 0{,}8$
1	2,404826(00)	1,601975(00)	1,556743(00)	1,433008(00)	1,068376(00)	4,581400(-01)
2	5,520078(00)	-1,064799(00)	-9,208959(-01)	-6,361205(-01)	-2,312844(-01)	-2,572544(-02)
3	8,653728(00)	8,513992(-01)	6,088411(-01)	2,924759(-01)	5,338622(-02)	1,677065(-03)
4	1,179153(01)	-7,296452(-01)	-4,100099(-01)	-1,367963(-01)	-1,305292(-02)	-1,167732(-04)
5	1,493092(01)	6,485236(-01)	2,774129(-01)	6,530816(-02)	3,305117(-03)	8,422161(-06)
6	1,807106(01)	-5,895428(-01)	-1,884451(-01)	-3,173932(-02)	-8,556177(-04)	-6,208900(-07)
7	2,121164(01)	5,441802(-01)	1,286367(-01(	1,564097(-02)	2,248713(-04)	4,646171(-08)
8	2,435247(01)	-5,078936(-01)	-8,828160(-02)	-7,790196(-03)	-5,975099(-05)	-3,514693(-09)
9	2,749348(01)	4,780125(-01)	6,090726(-02)	3,912089(-03)	1,600895(-05)	2,680750(-10)
10	3,063461(01)	-4,528506(-01)	-4,222733(-02)	-1,977401(-03)	-4,317264(-06)	-2,057938(-11)
11	3,377582(01)	4,312839(-01)	2,940508(-02)	1,004760(-03)	1,170395(-06)	1,588077(-12)
12	3,691710(01)	-4,125307(-01)	-2,055546(-02)	-5,127524(-04)	-3,186614(-07)	-1,230758(-13)
13	4,005843(01)	3,960282(-01)	1,441773(-02)	2,626185(-04)	8,707525(-08)	9,572677(-15)
14	4,319979(01)	-3,813595(-01)	-1,014249(-02)	-1,349206(-04)	-2,386668(-08)	-7,468262(-16)
15	4,634119(01)	3,682084(-01)	7,153367(-03)	6,949911(-05)	6,558958(-09)	5,841790(-17)
16	4,948261(01)	-3,563301(-01)	-5,056596(-03)	-3,588208(-05)	-1,806645(-09)	-4,579974(-18)
17	5,262405(01)	3,455318(-01)	3,581565(-03)	1,856312(-05)	4,986362(-10)	3,597905(-19)
18	5,576551(01)	-3,356591(-01)	-2,541305(-03)	-9,620500(-06)	-1,378691(-10)	-2,831427(-20)
19	5,890698(01)	3,265869(-01)	1,806041(-03)	4,993823(-06)	3,818014(-11)	2,231754(-21)
20	6,204847(01)	-3,182126(-01)	-1,285331(-03)	-2,595887(-06)	-1,058825(-11)	-1,761575(-22)

Note. Record 1,856312(-05) is equivalent to $1{,}856312 \times 10^{-5}$.

Thus, scalar potential $\Phi_{(1)}^{*}(\rho, z)$ in inner area $0 \le \rho \le R_0$ of disk piezoelectric element is determined by the following expression:

$$\Phi_{(1)}^{*}(\rho, z) = \sum_{n=0}^{\infty} A_{1n} I_0\left[\frac{\pi\rho}{2\alpha}(1+2n)\xi\right] \cos\left[\frac{\pi z}{2\alpha}(1+2n)\right] + \\ + U_0 \sum_{m=1}^{\infty} \tilde{A}_{2m}(q_m) J_0\left(\frac{q_m \rho}{R_0}\right) \operatorname{ch}\left(\frac{q_m z}{\xi R_0}\right). \qquad (27)$$

General solutions of the equations (12) and (13) for outer area $R_0 \le \rho \le R$ of disk piezoelectric element can be written in the following form:

a) separation constant – real number β_1:

$$R_{(1)}^{(2)}(\rho) = C_1 I_0(\beta_1\rho) + D_1 K_0(\beta_1\rho), \quad (28)$$

$$Z_{(1)}^{(2)}(z) = E_1 \cos(\lambda_1 z) + F_1 \sin(\lambda_1 z), \quad (29)$$

b) separation constant – imaginary number $i\beta_2$:

$$R_{(2)}^{(2)}(\rho) = C_2 J_0(\beta_2\rho) + D_2 N_0(\beta_2\rho), \quad (30)$$

$$Z_{(2)}^{(2)}(z) = E_2 \mathrm{ch}(\lambda_2 z) + F_2 \mathrm{sh}(\lambda_2 z), \quad (31)$$

where C_k, D_k, E_k and F_k ($k = 1,2$) – constants; $K_0(\beta_1\rho)$ – Macdonald function of zero order; $N_0(\beta_2\rho)$ - Neumann function of zero order.

To make scalar potential $\Phi_{(2)}^{*}(\rho,z)$ satisfying the condition (9) it is necessary and sufficient that the functions $R_{(k)}^{(2)}(\rho)$ ($k = 1,2$) satisfy the conditions of the following form:

$$\left.\frac{\partial R_{(k)}^{(2)}(\rho)}{\partial \rho}\right|_{\rho = R} = 0. \quad (32)$$

The condition (32) is equivalent to the following two equations:

$$\beta_1\left[C_1 I_1(\beta_1 R) - D_1 K_1(\beta_1 R)\right] = 0,$$

$$-\beta_2\left[C_2 J_1(\beta_2 R) + D_2 N_1(\beta_2 R)\right] = 0,$$

where $K_1(\beta_1 R)$ and $N_1(\beta_2 R)$ - Macdonald and Neumann functions of the first order. From last equalities it follows that

$$D_1 = C_1\frac{I_1(\beta_1 R)}{K_1(\beta_1 R)}, D_2 = -C_2\frac{J_1(\beta_2 R)}{N_1(\beta_2 R)}. \quad (33)$$

Scalar potential $\Phi_{(2)}^{*}(\rho,z)$ will be determined as follows:

$$\Phi_{(2)}^{*}(\rho,z) = \Phi_{(21)}^{*}(\rho,z) + \Phi_{(22)}^{*}(\rho,z), \quad (34)$$

where

$$\Phi_{(21)}^{*}(\rho,z) = \left[I_0(\beta_1\rho) + \frac{I_1(\beta_1 R)}{K_1(\beta_1 R)}K_0(\beta_1\rho)\right]\times \quad (35)$$
$$\times\left[E\cos(\lambda_1 z) + F\sin(\lambda_1 z)\right]$$

$$\Phi_{(22)}^{*}(\rho,z) = \left[J_0(\beta_2\rho) - \frac{J_1(\beta_2 R)}{N_1(\beta_2 R)}N_0(\beta_2\rho)\right]\times; \quad (36)$$
$$\times\left[M\mathrm{ch}(\lambda_2 z) + N\mathrm{sh}(\lambda_2 z)\right]$$

$E = E_1 C_1$; $F = F_1 C_1$; $M = E_2 C_2$ and $N = F_2 C_2$.

In order to simplify subsequent computations, we determine separation constants β_1 and β^2 as follows:

$$\beta_1 = \beta_\ell = \frac{\pi}{2\alpha}(1 + 2\ell)\xi, \ell = 0,1,2,\ldots, \quad (37)$$

$$\beta_2 = \beta_p = \zeta_p / R_0, p = 1,2,3,\ldots, \quad (38)$$

where ζ_p - root of transcendental equation number p,

$$J_0(x)N_1(kx) - J_1(kx)N_0(x) = 0, \quad (39)$$

where $k = R/R_0$ – geometrical parameter of disk piezoelectric element.

Table 2 shows numerical values of the first twenty roots of transcendental equation (39), which is calculated at the values of the ratio $R_0/R = 0,8$; $0,6$; $0,4$ and $0,2$, that correspond to the values $k = 1,25$; $1,67$; $2,5$ and 5. It is easy to note that numerical values of the roots increase without bound when $k \to 1$.

Table 2

The first twenty roots of transcendental equation $J_0(x)N_1(kx) - J_1(kx)N_0(x) = 0$

p	ζ_p			
	$k = 1,25$	$k = 1,6(6)$	$k = 2,50$	$k = 5,00$
1	1,875902(01)	2,120370(00)	8,660582(-01)	2,823584(-01)
2	3,136174(01)	6,993940(00)	3,083539(00)	1,139215(00)
3	4,394362(01)	1,173633(01)	5,201067(00)	1,939183(00)
4	5,651859(01)	1,646150(01)	7,305411(00)	2,731207(00)
5	6,909044(01)	2,118098(01)	9,405344(00)	3,520405(00)
6	8,166059(01)	2,589788(01)	1,150327(01)	4,308266(00)
7	9,422974(01)	3,061338(01)	1,360011(01)	5,095389(00)
8	1,067982(02)	3,532806(01)	1,569630(01)	5,882061(00)
9	1,193663(02)	4,004220(01)	1,779206(01)	6,668439(00)
10	1,319340(02)	4,475597(01)	1,988754(01)	7,454613(00)
11	1,445015(02)	4,946947(01)	2,198281(01)	8,240642(00)
12	1,570688(02)	5,418279(01)	2,407793(01)	9,026562(00)

Continuation of the tbl. 2

13	1,822030(02)	5,889595(01)	2,617294(01)	9,812399(00)
14	1,947700(02)	6,360900(01)	2,826785(01)	1,059817(01)
15	2,073337(02)	6,832196(01)	3,036269(01)	1,138389(01)
16	2,199038(02)	7,303484(01)	3,245748(01)	1,216957(01)
17	2,324705(02)	7,774767(01)	3,455221(01)	1,295522(01)
18	2,450373(02)	8,246044(01)	3,664691(01)	1,374084(01)
19	2,576040(02)	8,717317(01)	3,874158(01)	1,452643(01)
20	2,701707(02)	9,188587(01)	4,083622(01)	1,531200(01)

Sets of separation constants correspond to sets of constants, equivalent by power E_ℓ, F_ℓ, M_p and N_p. In this case expressions (35) and (36) can be written as follows:

$$\Phi^*_{(21)}(\rho,z)=\sum_{\ell=0}^{\infty}W_0(\beta_\ell\rho)\left\{E_\ell\cos\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]+F_\ell\sin\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]\right\}, \quad (40)$$

$$\Phi^*_{(22)}(\rho,z)=\sum_{p=1}^{\infty}\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)\left[M_p\mathrm{ch}\left(\frac{\zeta_p z}{\xi R_0}\right)+N_p\mathrm{sh}\left(\frac{\zeta_p z}{\xi R_0}\right)\right], \quad (41)$$

where $W_0(\beta_\ell\rho)=I_0(\beta_\ell\rho)+\frac{I_1(\beta_\ell R)}{K_1(\beta_\ell R)}K_0(\beta_\ell\rho)$; $\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)=J_0\left(\frac{\zeta_p\rho}{R_0}\right)-\frac{J_1(k\zeta_p)}{N_1(k\zeta_p)}N_0\left(\frac{\zeta_p\rho}{R_0}\right)$; $k=\frac{R}{R_0}$.

Expressions (27), (40) and (41) contain five sets of constants (A_{1n}, E_ℓ, F_ℓ, M_p and N_p), and must satisfy matching conditions of the solutions (4), (5) and (6) on the boundary $\rho=R_0$ of inner and outer areas and conditions (7) and (8) on top and bottom boundary of outer area of disk piezoelectric element. The construction of expressions (27), (40) and (41) provides automatic execution of boundary conditions (2), (3) and (9).

Write down the conditions (4)–(8) explicitly.

$$\sum_{n=0}^{\infty}A_{1n}I_0(\beta_n R_0)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]=\sum_{\ell=0}^{\infty}W_0(\beta_\ell R_0)\left\{E_\ell\cos\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]+F_\ell\sin\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]\right\}, \quad (42)$$

$$-\frac{\pi}{2\alpha}\sum_{n=0}^{\infty}(1+2n)A_{1n}I_0(\beta_n R_0)\sin\left[\frac{\pi z}{2\alpha}(1+2n)\right]=$$

$$=\frac{\pi}{2\alpha}\sum_{\ell=0}^{\infty}(1+2\ell)W_0(\beta_\ell R_0)\left\{-E_\ell\sin\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]+F_\ell\cos\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]\right\}, \quad (43)$$

$$\frac{\pi\xi}{2\alpha}\sum_{n=0}^{\infty}(1+2n)A_{1n}I_1(\beta_n R_0)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]-\frac{2U_0}{R_0}\sum_{m=1}^{\infty}\frac{\mathrm{ch}[q_m z/(\xi R_0)]}{\mathrm{ch}[q_m\alpha/(\xi R_0)]}=$$

$$=\frac{\pi\xi}{2\alpha}\sum_{\ell=0}^{\infty}(1+2\ell)W_1(\beta_\ell R_0)\left\{E_\ell\cos\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]+F_\ell\sin\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]\right\}-$$

$$-\frac{1}{R_0}\sum_{p=1}^{\infty}\zeta_p\Omega_1(\zeta_p)\left[M_p\mathrm{ch}\left(\frac{\zeta_p z}{\xi R_0}\right)+N_p\mathrm{sh}\left(\frac{\zeta_p z}{\xi R_0}\right)\right], \quad (44)$$

$$\frac{\pi}{2\alpha}\sum_{\ell=0}^{\infty}(1+2\ell)W_0(\beta_\ell\rho)E_\ell+\frac{1}{\xi R_0}\sum_{p=1}^{\infty}\zeta_p\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)\left[M_p\mathrm{sh}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)+N_p\mathrm{ch}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)\right]=0, \quad (45)$$

$$\sum_{\ell=0}^{\infty}W_0(\beta_\ell\rho)E_\ell+\sum_{p=1}^{\infty}\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)M_p=0, \quad (46)$$

where $W_0(\beta_\ell R_0) = I_0(\beta_\ell R_0) + \frac{I_1(\beta_\ell R)}{K_1(\beta_\ell R)} K_0(\beta_\ell R_0)$; $W_1(\beta_\ell R_0) = I_1(\beta_\ell R_0) - \frac{I_1(\beta_\ell R)}{K_1(\beta_\ell R)} K_1(\beta_\ell R_0)$;

$$\Omega_1(\zeta_p) = J_1(\zeta_p) - \frac{J_1(k\zeta_p)}{N_1(k\zeta_p)} N_1(\zeta_p).$$

Inhomogeneous system of five algebraic equations (42)–(46) contains five sets of unknown constants, i.e. values A_{1n}, E_ℓ, F_ℓ, M_p and N_p. This, in principle, ensures the uniqueness of the solution of algebraic equations system (42)–(46).

Since trigonometric functions $\cos[\pi z(1+2n)/(2\alpha)]$ and $\sin[\pi z(1+2n)/(2\alpha)]$ in the interval $0 \le z \le \alpha$ form a system of orthogonal functions, i.e. there are integrals

$$\int_0^\alpha \cos\left[\frac{\pi z}{2\alpha}(1+2k)\right]\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz = \begin{cases} 0 \ \forall\ k \ne n, \\ \alpha/2 \text{ при } k = n, \end{cases}$$

$$\int_0^\alpha \sin\left[\frac{\pi z}{2\alpha}(1+2k)\right]\sin\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz = \begin{cases} 0 \ \forall\ k \ne n, \\ \alpha/2 \text{ при } k = n, \end{cases}$$

then equations (42) and (43) are easily reduced to the following form:

$$\frac{\alpha}{2} A_{1n} I_0\left[\frac{\pi R_0}{2\alpha}(1+2n)\xi\right] = \frac{\alpha}{2} W_0(\beta_n R_0) E_n + \frac{\alpha}{2\pi}\sum_{\ell=0}^{\infty} W_0(\beta_\ell R_0) I_{\ell n}^{sc} F_\ell, \quad (47)$$

$$-\frac{\pi}{4}(1+2n) A_{1n} I_0\left[\frac{\pi R_0}{2\alpha}(1+2n)\xi\right] = -\frac{\pi}{4}(1+2n) W_0(\beta_n R_0) E_n + \frac{1}{4}\sum_{\ell=0}^{\infty}(1+2\ell) W_0(\beta_\ell R_0) I_{\ell n}^{cs} F_\ell, \quad (48)$$

where

$$I_{\ell n}^{sc} = \frac{2\pi}{\alpha}\int_0^\alpha \sin\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz = I_{\ell n}^{cs} = \frac{2\pi}{\alpha}\int_0^\alpha \cos\left[\frac{\pi z}{2\alpha}(1+2\ell)\right]\sin\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz = \frac{(-1)^{\ell-n}(1+2\ell)-(1+2n)}{(1+\ell+n)(\ell-n)} \forall n \ne \ell;$$

$$I_{\ell n}^{sc} = I_{\ell n}^{cs} = \frac{2\pi}{\alpha}\int_0^\alpha \sin\left[\frac{\pi z}{2\alpha}(1+2n)\right]\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz = \frac{2}{1+2n}.$$

From equations (47) and (48) it follows

$$A_{1n} = \frac{W_0(\beta_n R_0)}{I_0(\beta_n R_0)} E_n, \ F_\ell = 0. \quad (49)$$

Taking into account obtained results, it is expedient to write the equations (44)–(46) in the form:

$$\frac{\pi\xi}{2\alpha}\sum_{n=0}^{\infty} Q_n(\beta_n R_0) E_n \cos\left[\frac{\pi z}{2\alpha}(1+2n)\right] + \frac{1}{R_0}\sum_{p=1}^{\infty}\zeta_p \Omega_1(\zeta_p)\left[M_p \mathrm{ch}\left(\frac{\zeta_p z}{\xi R_0}\right) + N_p \mathrm{sh}\left(\frac{\zeta_p z}{\xi R_0}\right)\right] = \frac{2U_0}{R_0}\sum_{m=1}^{\infty}\frac{\mathrm{ch}[q_m z/(\xi R_0)]}{\mathrm{ch}[q_m \alpha/(\xi R_0)]}, \quad (50)$$

$$\frac{\pi}{2\alpha}\sum_{n=0}^{\infty}(1+2n) W_0(\beta_n \rho) E_n + \frac{1}{\xi R_0}\sum_{p=1}^{\infty}\zeta_p \Omega_0\left(\frac{\zeta_p \rho}{R_0}\right)\left[M_p \mathrm{sh}\left(\frac{\zeta_p \alpha}{\xi R_0}\right) + N_p \mathrm{ch}\left(\frac{\zeta_p \alpha}{\xi R_0}\right)\right] = 0, \quad (51)$$

$$\sum_{n=0}^{\infty} W_0(\beta_n \rho) E_n + \sum_{p=1}^{\infty}\Omega_0\left(\frac{\zeta_p \rho}{R_0}\right) M_p = 0, \quad (52)$$

where $Q_n(\beta_n R_0) = (1+2n)\left[\frac{I_1(\beta_n R_0)}{I_0(\beta_n R_0)} W_0(\beta_n R_0) - W_1(\beta_n R_0)\right] = \frac{2\alpha}{\pi R_0 \xi}\frac{I_1(\beta_n R)}{I_0(\beta_n R_0) K_1(\beta_n R)}$.

The system of equations (50)–(52) has no exact solution. However, we can construct a computational procedure, which provides good enough approximations to the exact solution of this algebraic equations system. In computational mathematics, this procedure is called the method of sequential approximations [10]. In this case the unknown constants E_n, M_p and N_p are represented by the following series:

$$E_n = E_n^{(0)} + \sum_{\nu=1}^{\infty} \Delta E_n^{(\nu)}, \quad M_p = M_p^{(0)} + \sum_{\nu=1}^{\infty} \Delta M_p^{(\nu)}, \quad N_p = N_p^{(0)} + \sum_{\nu=1}^{\infty} \Delta N_p^{(\nu)}, \qquad (53)$$

where E_n, M_p and N_p - exact solutions of equations system (50)–(52); $E_n^{(0)}$, $M_p^{(0)}$ and $N_p^{(0)}$ - zero, i.e. very rough approximations to exact solutions of equations system (50)–(52); $\Delta E_n^{(\nu)}$, $\Delta M_p^{(\nu)}$ and $\Delta N_p^{(\nu)}$ - corrections of ν order to the approximation of $\nu-1$ order to exact solutions of equations system (50)–(52).

Let's look for the rudest, i.e. zero approximations to exact values of the constants E_n, M_p and N_p, in a modified system of equations (50)–(52), that can be written as follows:

$$\frac{\pi\xi}{2\alpha}\sum_{n=0}^{\infty} Q_n(\beta_n R_0) E_n^{(0)} \cos\left[\frac{\pi z}{2\alpha}(1+2n)\right] = \frac{2U_0}{R_0}\sum_{m=1}^{\infty}\frac{\mathrm{ch}\left[q_m z/(\xi R_0)\right]}{\mathrm{ch}\left[q_m \alpha/(\xi R_0)\right]}, \qquad (54)$$

$$\frac{\pi}{2\alpha}\sum_{n=0}^{\infty}(1+2n) W_0(\beta_n\rho) E_n^{(0)} + \frac{1}{\xi R_0}\sum_{p=1}^{\infty}\zeta_p \Omega_0\left(\frac{\zeta_p \rho}{R_0}\right)\left[M_p^{(0)}\mathrm{sh}\left(\frac{\zeta_p\alpha}{\xi R_0}\right) + N_p^{(0)}\mathrm{ch}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)\right] = 0, \qquad (55)$$

$$\sum_{n=0}^{\infty} W_0(\beta_n\rho) E_n^{(0)} + \sum_{p=1}^{\infty}\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right) M_p^{(0)} = 0. \qquad (56)$$

Using orthogonality property of trigonometric functions $\cos[\pi z(1+2n)/(2\alpha)]$ in the interval $0 \le z \le \alpha$, the equation (54) can be easily reduced to the following form:

$$\frac{\pi\xi}{4} Q_n(\beta_n R_0) E_n^{(0)} = \frac{2U_0}{R_0}\sum_{m=1}^{\infty}\frac{J_{mn}}{\mathrm{ch}[q_m\alpha/(\xi R_0)]},$$

where

$$J_{mn} = \int_0^{\alpha}\mathrm{ch}\left(\frac{q_m z}{\xi R_0}\right)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz =$$

$$= -\xi R_0 \pi\left(\frac{\xi R_0}{2\alpha}\right)\frac{(1+2n)}{\left\{q_m^2 + \left[\frac{\pi(1+2n)\xi R_0}{2\alpha}\right]\right\}}\mathrm{ch}\left(\frac{q_m\alpha}{\xi R_0}\right).$$

From the last equation it follows that

$$E_n^{(0)} = -\frac{4R_0\xi}{\alpha} U_0 e_n(\beta_n R_0), \qquad (57)$$

where

$$e_n(\beta_n R_0) = \frac{1}{Q_n(\beta_n R_0)}\sum_{m=1}^{\infty}\frac{(1+2n)}{\left[q_m^2 + \left(\frac{\pi\xi R_0}{2\alpha}\right)^2 (1+2n)^2\right]}.$$

The function $\Omega_0(\zeta_p\rho/R_0)$ from equations (55) and (56) in the interval $R_0 \le \rho \le R$ forms a system of orthogonal functions, i.e. there exists the integral of the following form:

$$B_{pq} = \int_{R_0}^{R}\rho\,\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)\Omega_0\left(\frac{\zeta_q\rho}{R_0}\right)d\rho = \begin{cases} 0 \ \forall\ p \ne q, \\ B_0 \text{ при } p = q. \end{cases} \qquad (58)$$

Direct calculations show that

$$B_0 = \frac{2R_0^2}{\pi^2}\Xi_p(k,\zeta_p), \qquad (59)$$

where:

$$\Xi_p(k,\zeta_p) = \frac{1}{\zeta_p^2} + \frac{\pi^2}{2}\frac{J_1(k\zeta_p)}{N_1(k\zeta_p)}\times$$

$$\times\left\{J_0(\zeta_p)N_0(\zeta_p) + J_1(\zeta_p)N_1(\zeta_p) - - k^2\left[J_0(k\zeta_p)N_0(k\zeta_p) + J_1(k\zeta_p)N_1(k\zeta_p)\right]\right\};$$

$k = R/R_0$ - geometric parameter of piezoelectric disk with surface partial covering by electrodes.

Using the orthogonality of functions $\Omega_0(\zeta_p\rho/R_0)$, i.e. properties of the integral (58), equations (55) and (56) are reduced to the following form:

$$-U_0\frac{\xi R_0\pi}{\alpha^2}\sum_{n=0}^{\infty}(1+2n)\frac{Q_{np}(\beta_n,\zeta_p)}{\left[\zeta_p^2 + (\beta_n R_0)^2\right]} e_n(\beta_n R_0) + \frac{\zeta_p}{\xi R_0\pi^2}\Xi_p(k,\zeta_p)\left[M_p^{(0)}\mathrm{sh}\left(\frac{\zeta_p\alpha}{\xi R_0}\right) + N_p^{(0)}\mathrm{ch}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)\right] = 0, \qquad (60)$$

$$-U_0\frac{2\xi R_0\pi^2}{\alpha}\sum_{n=0}^{\infty}\frac{Q_{np}(\beta_n,\zeta_p)}{\left[\zeta_p^2+(\beta_n R_0)^2\right]}e_n(\beta_n R_0)+\Xi_p(k,\zeta_p)M_p^{(0)}=0, \quad (61)$$

where
$$Q_{np}(\beta_n,\zeta_p)=-\zeta_p N_1(\zeta_p)\left\{\left[I_0(\beta_n R_0)+\frac{I_1(\beta_n R)}{K_1(\beta_n R)}K_0(\beta_n R_0)\right]\frac{J_1(\zeta_p)}{N_1(\zeta_p)}-\right.$$
$$\left.-\left[I_0(\beta_n R_0)+\frac{I_1(\beta_n R)}{K_1(\beta_n R)}K_0(\beta_n R_0)\right]\frac{J_1(k\zeta_p)}{N_1(k\zeta_p)}\right\}.$$

From the equation (61) it follows that
$$M_p^{(0)}=U_0\frac{2\xi R_0\pi^2}{\alpha}m_p^{(0)}(k,\zeta_p), \quad (62)$$
where
$$m_p^{(0)}(k,\zeta_p)=\frac{1}{\Xi_p(k,\zeta_p)}\sum_{n=0}^{\infty}\frac{Q_{np}(\beta_n,\zeta_p)}{\left[\zeta_p^2+(\beta_n R_0)^2\right]}e_n(\beta_n R_0).$$

Substituting determined by (62) zero approximation to exact value of the constant M_p into equation (60), we can find the value $N_p^{(0)}$:
$$N_p^{(0)}=U_0\frac{R_0}{\alpha}n_p^{(0)}(k,\zeta_p), \quad (63)$$
where

$$n_p^{(0)}(k,\zeta_p)=\frac{\xi^2R_0\pi^3}{\zeta_p\alpha\,\mathrm{ch}\left[\zeta_p\alpha/(\xi R_0)\right]\Xi_p(k,\zeta_p)}\sum_{n=0}^{\infty}\frac{Q_{np}(\beta_n,\zeta_p)}{\left[\zeta_p^2+(\beta_n R_0)^2\right]}e_n(\beta_n R_0)-2\xi\pi^2 m_p^{(0)}(k,\zeta_p)\,\mathrm{th}\left(\frac{\zeta_p\alpha}{\xi R_0}\right).$$

To determine the correction $\Delta E_n^{(1)}$ we should substitute into equation (50) approximate values of the coefficients $E_n\approx E_n^{(0)}+\Delta E_n^{(1)}$, $M_p\approx M_p^{(0)}$ and $N_p\approx N_p^{(0)}$. After that, taking into account the equation (54), we can obtain the following result:

$$\frac{\xi\pi}{2}\sum_{n=0}^{\infty}\Delta E_n^{(1)}Q_n(\beta_n R_0)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]+$$
$$+U_0\sum_{p=1}^{\infty}\zeta_p\Omega_1(\zeta_p)\left[2\xi\pi^2 m_p^{(0)}(k,\zeta_p)\mathrm{ch}\left(\frac{\zeta_p z}{\xi R_0}\right)+n_p^{(0)}(k,\zeta_p)\mathrm{sh}\left(\frac{\zeta_p z}{\xi R_0}\right)\right]=0. \quad (64)$$

Using the orthogonality of functions $\cos\left[\pi z(1+2n)/(2\alpha)\right]$ in the interval $0\le z\le\alpha$, the equation (64) can be reduced to the following form:

$$\frac{\xi\pi}{4}\Delta E_n^{(1)}Q_n(\beta_n R_0)=-2U_0\sum_{p=1}^{\infty}\zeta_p\Omega_1(\zeta_p)\left[-2\xi\pi^2 m_p^{(0)}(k,\zeta_p)I_1^*(p,n)+n_p^{(0)}(k,\zeta_p)I_2^*(p,n)\right], \quad (65)$$

where $I_1^*(p,n)=-\dfrac{1}{2\alpha}\displaystyle\int_0^{\alpha}\mathrm{ch}\left(\frac{\zeta_p z}{\xi R_0}\right)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz=\dfrac{\pi(1+2n)}{\left(\dfrac{2\zeta_p\alpha}{\xi R_0}\right)^2+\left[\pi(1+2n)\right]^2}\mathrm{ch}\left(\dfrac{\zeta_p\alpha}{\xi R_0}\right)$;

$$I_2^*(p,n)=\frac{1}{2\alpha}\int_0^{\alpha}\mathrm{sh}\left(\frac{\zeta_p z}{\xi R_0}\right)\cos\left[\frac{\pi z}{2\alpha}(1+2n)\right]dz=\frac{1}{\left(\dfrac{2\zeta_p\alpha}{\xi R_0}\right)^2+\left[\pi(1+2n)\right]^2}\left[\frac{2\zeta_p\alpha}{\xi R_0}-\pi(1+2n)\mathrm{sh}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)\right].$$

From the equation (65) we obtain an expression for the calculation of corrections $\Delta E_n^{(1)}$:
$$\Delta E_n^{(1)}=-U_0\Delta e_n^{(1)}(k,\beta_n), \quad (66)$$

where $$\Delta e_n^{(1)}(k,\beta_n) = \frac{8}{\xi\pi Q_n(\beta_n R_0)} \sum_{p=1}^{\infty} \zeta_p \Omega_1(\zeta_p) \left[-2\xi\pi^2 m_p^{(0)}(k,\zeta_p) I_1^*(p,n) + n_p^{(0)}(k,\zeta_p) I_2^*(p,n) \right]. \quad (67)$$

At the organization of functions calculation $\Delta e_n^{(1)}(k,\beta_n)$ it is necessary to consider analytic properties of numerical sequences $\Omega_1(\zeta_p)$. Table 3 shows numerical values $\Omega_1(\zeta_p)$ for the first twenty roots ζ_p of the equation (39) that were identified for the values $k = R/R_0 = 1{,}25$; $1{,}67$; $2{,}50$ and $5{,}00$.

From the data in the table it follows that numbers $\Omega_1(\zeta_p)$ for different values of the parameter k form different sequences that converge to various limits. When $R_0/R = 0{,}8$ ($k = 1{,}25$) numbers $\Omega_1(\zeta_p)$ form only one numerical sequence, and the summation in (67) may be stopped on any number p, the value of which is determined by accepted error of the calculations. When $R_0/R = 0{,}6$ ($k = 1{,}6(6)$) four sequences can be seen. Thus it is necessary to summarize even numbers of sequentially arranged four numbers $\Omega_1(\zeta_p)$. When $R_0/R = 0{,}4$ ($k = 1{,}25$) three sequences of numbers can be seen, and it is necessary to summarize even numbers of triple numbers $\Omega_1(\zeta_p)$. If the value $R_0/R = 0{,}2$ ($k = 5{,}00$) again four sequences can be seen, and it is necessary again to summarize even numbers of four numbers $\Omega_1(\zeta_p)$. From all this there should be the only conclusion – before the calculations for every parameter $k = R/R_0$ you must perform a study of numerical sequences $\Omega_1(\zeta_p)$. This will improve the precision of further calculations and increase the reliability of results of accounting.

Table 3

Numerical values of the function $\Omega_1(\zeta_p) = J_1(\zeta_p) - J_1(k\zeta_p) N_1(\zeta_p) / N_1(k\zeta_p)$ for the first twenty roots of transcendental equation
$J_0(x)N_1(kx) - J_1(kx)N_0(x) = 0$

p	ζ_p			
	$k = 1{,}25$	$k = 1{,}6(6)$	$k = 2{,}50$	$k = 5{,}00$
1	-2,385497(-01)	5,782499(-01)	-2,990785(01)	-2,656170(00)
2	-1,907371(-01)	-3,276138(00)	5,918629(-01)	2,957784(00)
3	-1,635730(-01)	-2,333630(-01)	-3,694197(-01)	6,529087(-01)
4	-1,454753(-01)	4,988860(00)	1,355009(00)	5,143017(-01)
5	-1,323057(-01)	1,734729(-01)	3,567331(-01)	1,001103(00)
6	-1,221691(-01)	-6,251137(00)	-2,455062(-01)	-1,115489(00)
7	-1,140549(-01)	-1,442484(-01)	9,135880(-01)	-3,891323(-01)
8	-1,073692(-01)	7,298745(00)	1,794554(-01)	-3,525693(-01)
9	-1,017365(-01)	1,261117(-01)	-1,968125(-01)	-7,610190(-01)
10	-9,690658(-02)	-8,213966(00)	7,338983(-01)	8,102446(-01)
11	-9,270518(-02)	-1,134541(-01)	2,373926(-01)	3,038393(-01)
12	-8,90068(-02)	9,037038(00)	-1,689510(-01)	2,854409(-01)
13	-8,276970(-02)	1,039756(-01)	6,303821(-01)	6,387533(-01)
14	-8,010568(-02)	-9,791199(00)	2,099708(-01)	-6,673874(-01)
15	-7,768339(-02)	-9,653513(-02)	-1,503352(-01)	-2,577519(-01)
16	-7,546835(-02)	1,049131(01)	5,610522(-01)	-2,462304(-01)
17	-7,343259(-02)	9,049322(-02)	1,902830(-01)	-5,612772(-01)
18	-7,155317(-02)	-1,114755(01)	-1,367707(-01)	5,804990(-01)
19	-6,981104(-02)	-8,546027(-02)	5,104803(-01)	2,278097(-01)
20	-6,819027(-02)	1,176725(01)	1,782648(-01)	2,197378(-01)

To determine the corrections of $\Delta M_p^{(1)}$ and $\Delta N_p^{(1)}$ we should substitute in equations (51) and (52) approximate values of the constants $E_n \approx E_n^{(0)} + \Delta E_n^{(1)}$, $M_p \approx M_p^{(0)} + \Delta M_p^{(1)}$ and $N_p \approx N_p^{(0)} + \Delta N_p^{(1)}$. Taking into account previously used (55) and (56), we obtain the following results:

$$-\frac{\pi U_0}{2\alpha}\sum_{n=0}^{\infty}(1+2n)\Delta e_n^{(1)}(k,\beta_n)W_0(\beta_n\rho)+\frac{1}{\xi R_0}\sum_{p=1}^{\infty}\zeta_p\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)\left[\Delta M_p^{(1)}\,\mathrm{sh}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)+\Delta N_p^{(1)}\,\mathrm{ch}\left(\frac{\zeta_p\alpha}{\xi R_0}\right)\right]=0\,, \quad (68)$$

$$-U_0\sum_{n=0}^{\infty}\Delta e_n^{(1)}(k,\beta_n)W_0(\beta_n\rho)+\sum_{p=1}^{\infty}\zeta_p\Omega_0\left(\frac{\zeta_p\rho}{R_0}\right)\Delta M_p^{(1)}=0\,. \quad (69)$$

It is obvious that the corrections $\Delta M_p^{(1)}$ and $\Delta N_p^{(1)}$ are determined from the system of equations (68), (69) at predetermined potential U_0 in a unique way. After determining the corrections $\Delta M_p^{(1)}$ and $\Delta N_p^{(1)}$ from the equation (50) the correction $\Delta E_n^{(2)}$ is determined. For this purpose, into equation (50) the following approximate values of the coefficients $E_n \approx E_n^{(0)} + \Delta E_n^{(1)} + \Delta E_n^{(2)}$, $M_p \approx M_p^{(0)} + \Delta M_p^{(1)}$ and $N_p \approx N_p^{(0)} + \Delta N_p^{(1)}$ are inserted. Then, from equations (51) and (52) corrections $\Delta M_p^{(2)}$ and $\Delta N_p^{(2)}$ are determined, and then – the correction $\Delta E_n^{(3)}$. Thereafter the cycle of calculations is repeated again until it reaches a given precision of calculations.

After determining the constants E_n, M_p and N_p with given precision the constants A_{1n} are determined:

$$A_{1n} = \frac{W_0(\beta_n R_0)}{I_0(\beta_n R_0)}\left[E_n^{(0)} + \sum_{\nu=0}^{\infty}\Delta E_n^{(\nu)}\right]. \quad (70)$$

Thus, in mathematical description of scalar potential of electric field in inner and outer areas of piezoelectric element only constants A_{2m} can be precisely determined. Constants A_{1n}, E_n, M_p and N_p are determined approximately. The error of approximate determination of these constants is monitored during runtime of calculations.

Conclusions:

1. General formulation of boundary problem of scalar potential calculation of axially symmetric electric field in the disk, the material of which has an anisotropic dielectric constant that is at least an order higher than vacuum dielectric constant, is given.

2. A scheme for solving the problem in the case, when the electrodes placed on end surfaces of the disk have the shape of coaxially located circle and ring, and ring outer radius is equal to the radius of dielectric disk, while the radius of the circle coincides with inner radius of the ring, is offered.

3. Proposed scheme for solving the problem uses the method of sequential approximations that allows to obtain analytical expressions for coefficients in mathematical description of electric field potential in the volume of the disk with anisotropic dielectric constant.

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РАСЧЕТ ПАРАМЕТРОВ НАПРЯЖЕННО ДЕФОРМИРОВАННОГО СОСТОЯНИЯ ДИСКОВОГО ПЬЕЗОЭЛЕМЕНТА С ЧАСТИЧНЫМ ЭЛЕКТРОДИРОВАНИЕМ ПОВЕРХНОСТИ

В статье дана общая формулировка граничной задачи о расчете скалярного потенциала осесимметричного электрического поля в диске, материал которого имеет анизотропную диэлектрическую проницаемость, которая минимум на порядок превосходит диэлектрическую проницаемость вакуума. Предложенная схема решения задачи использует метод последовательных приближений, что позволяет получить аналитические выражения для расчета коэффициентов в математическом описании потенциала электрического поля.

Ключевые слова: *скалярный потенциал, уравнение Лапласа, анизотропия диэлектрической проницаемости, частичное электродирование торцевых поверхностей диэлектрического диска.*