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A CRITERION OF AGGREGATING EXPERT ESTIMATIONS INTO CONSENSUS PAIRWISE COMPARISON MATRIX BY A GIVEN COMPARISON SCALE WITHIN THE CORRESPONDING SPACE OF POSITIVE INVERSE-SYMMETRIC MATRICES

An approach of aggregating expert estimations into consensus pairwise comparison matrix is suggested. The aggregation criterion is minimization of the weighted distance between the consensus pairwise comparison matrix and pairwise comparison matrices of experts. The matrix distance is Euclidean-based metric in the space of all positive inverse-symmetric matrices whose subset is the space of all pairwise comparison matrices. The consensus is found slightly simpler for experts with identical competences. Expert estimations are treated consistent if they do not differ badly. For checking consistency or concordance of expert estimations, two inequalities are controlled. The first inequality addresses maximal distance among weighted pairwise comparison matrices of experts. The second one addresses maximal difference among entries of these matrices. For experts with identical competences, the two inequalities are stated simpler, without weighting. The suggested approach is applicable, regardless the comparison scale, for solving hierarchical multicriteria problems by finite number of alternatives.

Keywords: expert estimations, comparison scale, pairwise comparison matrix, aggregation of expert estimations, matrix distance, consensus, competences of experts, consistency of expert estimations.

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КРИТЕРІЙ АГРЕГУВАННЯ ЕКСПЕРТНИХ ОЦІНОК В УЗАГАЛЬНЕНУ МАТРИЦЮ ПОПАРНИХ ПОРІВНЯНЬ ЗА ДАНОЮ ШКАЛОЮ ПОРІВНЯНЬ У ВІДПОВІДНОМУ ПРОСТОРІ ДОДАТНИХ ОБЕРНЕНО-СИМЕТРИЧНИХ МАТРИЦЬ

Пропонується підхід до агрегування експертних оцінок в узагальнену матрицю попарних порівнянь. Критерієм агрегування є мінімізація зваженої відстані між узагальненою матрицею попарних порівнянь і матрицями попарних порівнянь експертів. Матричною відстанню є метрика на основі евклідової відстані у просторі усіх додатних обернено-симетричних матриць, підмножиною якого є простір усіх матриць попарних порівнянь. Узагальнення знаходиться дещо простіше для експертів з однаковими компетентностями. Експертні оцінки вважаються узгодженими, якщо вони не сильно різняться. Для перевірки узгодженості експертних оцінок контролюються дві нерівності. Перша нерівність звертається до максимальної відстані між зваженими матрицями попарних порівнянь експертів. Друга нерівність звертається до максимальної різниці між елементами цих матриць. Дані дві нерівності записуються простіше, без зважування, для експертів з однаковими компетентностями. Незалежно від шкали порівнянь, запропонований підхід є застосовним для розв'язування ієрархічних багатокритеріальних задач зі скінченною кількістю альтернатив.

Ключові слова: експертні оцінки, шкала порівнянь, матриця попарних порівнянь, агрегування експертних оцінок, матрична відстань, узагальнення, компетентності експертів, узгодженість експертних оцінок.

Problem of aggregating expert estimations

Expert estimations (EE) may differ badly. Simple aggregation of such estimations, e. g., arithmetic or geometrical mean, may turn inconsistent or incorrect. Therefore, a criterion of EE consistency should be. If EE are consistent by the criterion, then the corresponding solution analysis is applicable.

As a pattern, consider pairwise comparison matrix (PCM) used in solving hierarchical multicriteria problems (HMCP) by finite number of alternatives (possible solutions or strategies). PCM is widely applied within the well-known Saaty method of analytic hierarchy process (AHP). Using the spread scale of comparisons [1, 2], the variety of PCM is finite. Obviously, the consensus PCM must be a matrix of those ones in the PCM variety. Even a trivial example shows, however, that averaging arithmetically or geometrically gives an off-the-scale matrix which is not a PCM: if 2 and 3 are values given by two experts, their arithmetic mean 2.5 is off-the-scale value, and their geometric mean $\sqrt{6}$ is off-the-scale value as well.

This is the motivation to develop an approach of EE aggregation such that the approach could be applicable regardless the scale of comparisons. In other words, this approach should work on any variety of PCM.

How the consensus PCM is obtained

In most practical cases, while an HMCP is solved by PCM involved at least two experts, a question of how the consensus PCM is obtained appears unclear. For instance, the paper [3] derives a group PCM, wherein a consensus index of a sequence of individual PCM is defined and two consensus improving methods are developed by introducing a general aggregation operator based on Abelian linearly ordered group. Based on multiplicative AHP model with lognormal errors, the paper [4] proposes a Bayesian revision method for improving a sequence of individual PCM under the assumption that the consensus exists among decision makers, which is considered an aid to aggregation of individual judgments and aggregation of the individual priorities. In the paper [5], it is declared that using the row geometric mean as the prioritization procedure, consensus is sought between the different decision makers when the modifications of their initial positions or judgments are guaranteed to be within the range of values accepted for a given inconsistency level. A general model to generate crisp priority weights of the

alternatives from possibly inconsistent and conflicting fuzzy preference relations is proposed in [6], where the model is expressed in terms of fuzzy matrix approximations, and in the aggregation process, the importance of each expert is taken into consideration according to the agreement of the group with the expert.

A lot of apparent drawbacks of those decisional tools spring from superfluous complications, assumptions needing supplementary substantiations, impossibility to achieve a complete matrix [5], invalidity of fuzzifying numerical judgments in the AHP [7, 8]. Besides, often supplementary statistics is needed [4, 6, 7, 9, 10]. Thus, the EE aggregation is an issue.

Goal and items to be accomplished to meet it

The final goal is to state an approach of the EE aggregation into a consensus PCM along with a criterion of consistency of this PCM. Note that here consistency of the consensus PCM is treated in the sense of succeeding to EE consistency, rather than concerning the principal eigenvalue (PE) and consistency index [7, 10, 11]. This is so because validity is the target in decision-making, not consistency, which can be successively improved by manipulating the judgments as the answer gets farther and farther from reality [7]. And validity is founded on that EE do not differ badly, what allows to bring them into the consensus.

For meeting the said goal, the following items are going to be made:

1. Formalize the scale of the comparison result.

2. Formalize the space (called earlier variety) of all PCM.

3. Formalize the space of all positive inverse-symmetric matrices (PISM) whose subset is the space of all PCM.

4. Introduce a metric in the PISM space.

5. Suggest how to aggregate EE in their PCM into a consensus PCM belonging to the space of all PCM.

6. Suggest a criterion which would ensure consistency of EE in their PCM and let apply the corresponding consensus PCM in solving an HMCP.

The comparison scale and spaces of PCM and PISM

By the Saaty method of AHP, the comparison result is reflected usually with the 17-pointed scale whose values are [1, 2, 7, 11, 12]

$$\left\{ \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9 \right\} \tag{1}$$

and only these ones are elements of PCM. Each value in the set (1) has its own interpretation suitable to the appropriate situation. Here the first nine natural numbers are used. Generalization of the comparison scale is that the comparison result is reflected on the scale of the first S natural numbers for $2S-1$ graduation marks

$$A(S) = \left\{ \left\{ m^{-1} \right\}_{m=2}^{S}, 1, \left\{ m \right\}_{m=2}^{S} \right\} \quad \text{by} \quad S \in \mathbb{N} \quad . \tag{2}$$

Thus, having N objects (alternatives, criteria, subcriteria, etc.) by $N \in \mathbb{N}$ and J experts by $J \in \mathbb{N}$, PCM of the j-th expert is $\mathbf{A}_j = \left[a_{ik}^{\langle j \rangle} \right]_{N \times N}$ where

$$a_{ik}^{\langle j \rangle} \in A(S) \quad \text{by} \quad a_{ik}^{\langle j \rangle} = \frac{1}{a_{ki}^{\langle j \rangle}} . \tag{3}$$

Denote by

$$\mathbf{A}_N(S) = \left\{ \mathbf{A} = \left[a_{ik} \right]_{N \times N} : a_{ik} \in A(S), \, a_{ik} = a_{ki}^{-1} \right\} \tag{4}$$

the space of all $N \times N$ PCM with the comparison scale (2). It is clear that $\mathbf{A}_j \in \mathbf{A}_N(S)$ and the space (4) is finite. However, the space (4) is a partial case of PCM variety generated by the scale (2). The space (4) contains not all PISM. But the infinite space

$$\mathbf{B}_N = \left\{ \mathbf{B} = \left[b_{ik} \right]_{N \times N} : b_{ik} > 0, \, b_{ik} = b_{ki}^{-1} \right\} \tag{5}$$

contains all $N \times N$ PCM generated by any scale of positive values. So, $\mathbf{A}_N(S) \subset \mathbf{B}_N$ and a metric to measure distance between two PCM should be introduced just right in the PISM space (5).

Euclidean-based metric in the space of PISM

Denote the distance between PISM

$$\mathbf{X} = \left[x_{ik} \right]_{N \times N} \in \mathbf{B}_N \quad \text{and} \quad \mathbf{Y} = \left[y_{ik} \right]_{N \times N} \in \mathbf{B}_N \tag{6}$$

by $\rho_{\mathbf{B}_N}(\mathbf{X}, \mathbf{Y})$. These two PISM (6) can be represented as ordinary N^2-dimensional points in Euclidean arithmetic space $\mathbb{R}$. Therefore, it is natural to measure distance [13] between PISM (6) by the Euclidean-based metric:

$$\rho_{\mathbf{B}_N}(\mathbf{X}, \mathbf{Y}) = \left\| \mathbf{X} - \mathbf{Y} \right\| = \sqrt{\sum_{i=1}^{N} \sum_{k=1}^{N} \left(x_{ik} - y_{ik} \right)^2} . \tag{7}$$

Properties of PISM declared in (5) allow to deduce a formula for simpler calculation of the distance (7):

$$\rho_{\mathbf{B}_N}(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\| = \sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\left[(x_{ik} - y_{ik})^2 + (x_{ik}^{-1} - y_{ik}^{-1})^2\right]} =$$

$$= \sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\left[(x_{ik} - y_{ik})^2 + \frac{(y_{ik} - x_{ik})^2}{(x_{ik}y_{ik})^2}\right]} = \sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\frac{1 + (x_{ik}y_{ik})^2}{(x_{ik}y_{ik})^2}\cdot(x_{ik} - y_{ik})^2}. \qquad (8)$$

Excluding main diagonal, by formula (8) we need to sum up $\dfrac{N(N-1)}{2}$ terms instead of $N(N-1)$ terms by formula (7). This is twice faster.

The consensus PCM belonging to the space of all PCM

If there was just a single expert, no consensus PCM would be required because PCM of this expert would be already applicable. In the case of J experts, whose weights $\left\{\xi_j\right\}_{j=1}^{J}$ are such that

$$\xi_j \in (0;1) \ \text{ for } \ j = \overline{1, J} \ \text{ by } \ \sum_{j=1}^{J}\xi_j = 1, \qquad (9)$$

where the weight ξ_j indicates at the j-th expert's competence, the consensus PCM must be selected among $\left|\mathbf{A}_N(S)\right|$ PCM of the space $\mathbf{A}_N(S)$. Formally, this is to determine a mapping of J PCM $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ into the consensus PCM $\tilde{}\tilde{}$ so that $\tilde{}\ \mathbf{A}\ )$. Neither weighted arithmetic mean

$$\mathbf{M} = \left[m_{ik}\right]_{N \times N} \ \text{ by } \ m_{ik} = \sum_{j=1}^{J}\xi_j a_{ik}^{\langle j\rangle}$$

nor weighted geometric mean

$$\mathbf{G} = \left[g_{ik}\right]_{N \times N} \ \text{ by } \ g_{ik} = \prod_{j=1}^{J}\left(a_{ik}^{\langle j\rangle}\right)^{\xi_j}$$

is the mapping unless all matrices $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ are identical. But generally $\mathbf{M} \notin \mathbf{A}_N(S)$ and $\mathbf{G} \notin \mathbf{A}_N(S)$. Moreover, $\mathbf{M} \notin \mathbf{B}_N$ though $\mathbf{G} \in \mathbf{B}_N$.

If the q-th PCM in the space $\mathbf{A}_N(S)$ is

$$\mathbf{C}_q = \left[c_{ik}^{\langle q\rangle}\right]_{N \times N} \in \mathbf{A}_N(S) \subset \mathbf{B}_N \ \text{ by } \ q = \overline{1, \left|\mathbf{A}_N(S)\right|} \qquad (10)$$

then the weighted distance between this PCM and J PCM $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ is

$$\rho\left(\mathbf{C}_q, \left\{\mathbf{A}_j\right\}_{j=1}^{J}\right) = \sum_{j=1}^{J}\xi_j \cdot \rho_{\mathbf{B}_N}(\mathbf{A}_j, \mathbf{C}_q) = \sum_{j=1}^{J}\xi_j \cdot \sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\frac{1 + \left(a_{ik}^{\langle j\rangle}c_{ik}^{\langle q\rangle}\right)^2}{\left(a_{ik}^{\langle j\rangle}c_{ik}^{\langle q\rangle}\right)^2}\cdot\left(a_{ik}^{\langle j\rangle} - c_{ik}^{\langle q\rangle}\right)^2}. \qquad (11)$$

Then

$$\tilde{} \min_{\substack{\mathbf{C}_q \in \mathbf{A}_N(S) \subset \mathbf{B}_N \\ q = \overline{1, \left|\mathbf{A}_N(S)\right|}}}\rho\left(\mathbf{C}_q, \left\{\mathbf{A}_j\right\}_{j=1}^{J}\right) \qquad (12)$$

is the required consensus PCM. Note that if expert procedures involve experts with identical competences then the consensus PCM (12) is found slightly simpler:

$$\tilde{} \min_{\substack{\mathbf{C}_q \in \mathbf{A}_N(S) \subset \mathbf{B}_N \\ q = \overline{1, \left|\mathbf{A}_N(S)\right|}}}\left(\sum_{j=1}^{J}\sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\frac{1 + \left(a_{ik}^{\langle j\rangle}c_{ik}^{\langle q\rangle}\right)^2}{\left(a_{ik}^{\langle j\rangle}c_{ik}^{\langle q\rangle}\right)^2}\cdot\left(a_{ik}^{\langle j\rangle} - c_{ik}^{\langle q\rangle}\right)^2}\right). \qquad (13)$$

Either the minimization problem (12) by (11) or the minimization problem (13) is solved trivially (Figure 1) owing to that they deal with the finite set $\mathbf{A}_N(S)$ of possible solutions.
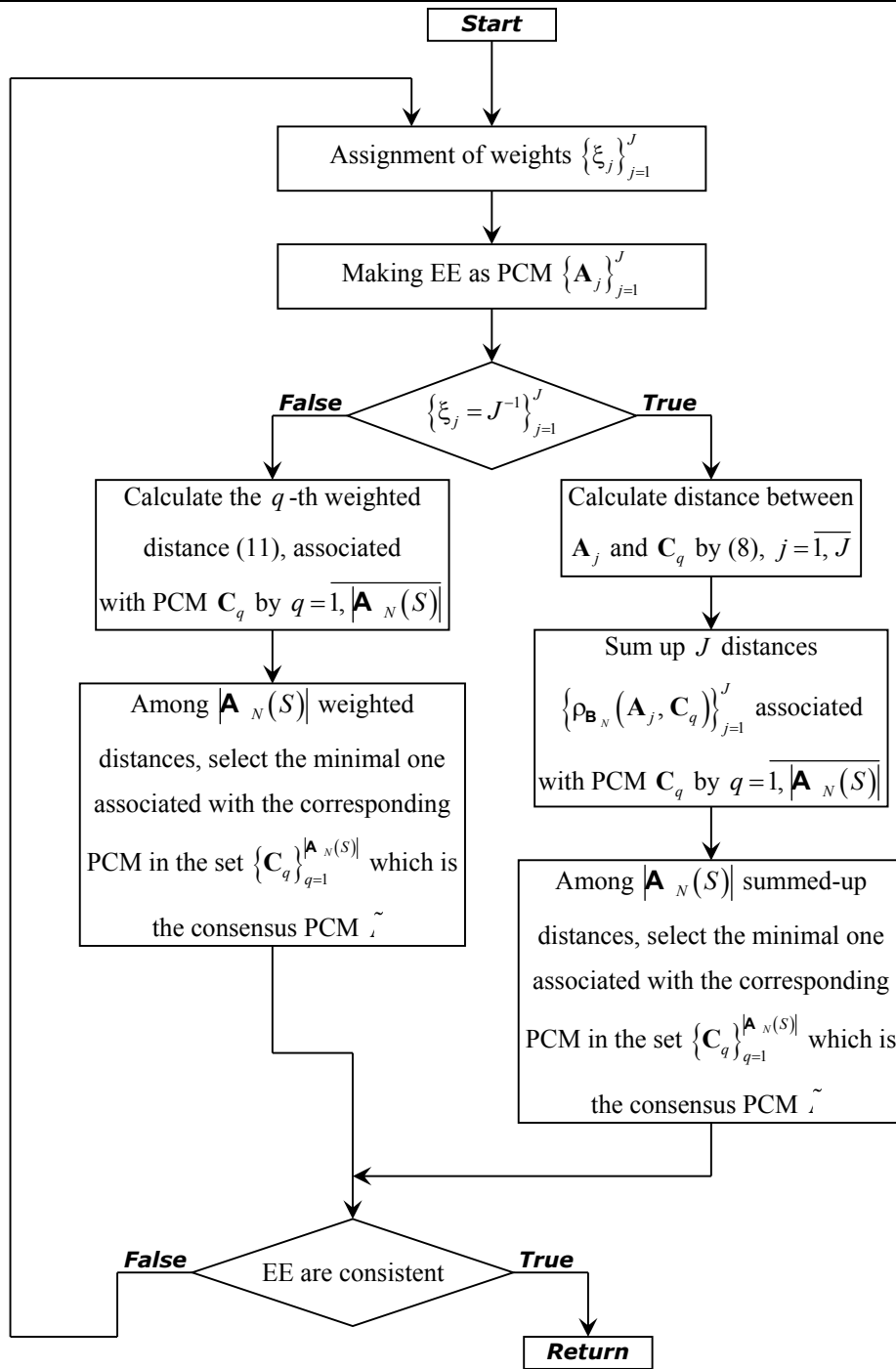
Figure 1. A routine for finding the consensus PCM $\tilde{\imath}$

It is important that the routine in Figure 1 describes the succession of steps for finding the consensus PCM $\tilde{\imath}$ before consistency of EE is checked. Though EE consistency was expected to be checked straight off after weights $\left\{\xi_j\right\}_{j=1}^{J}$ and PCM $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ are known, matrix $\tilde{\imath}$ may come indispensable for reasoning upon EE consistency. That is why the EE consistency checking module has been put in the end of the routine, after the consensus PCM $\tilde{\imath}$ is selected in the course of solving a trivial minimization problem.

Consistency of EE

The consensus PCM $\tilde{\imath}$, when found, can be applied in solving an HMCP only if there is a concordance (consistency) of EE. Consistency is a requirement of that EE would not be badly diverse. This will prevent premature conclusions on relationships among objects which are compared.

Basically, for making a conclusion on EE consistency, we should check how bad the expert matrices $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ are scattered. For this, we calculate the maximal distance in the space $\mathbf{A}_N(S)$. The maximal distance between two PCM is reached when every pair of their entries of the same location is maximally opposite. Therefore,

all the pairs (except the main diagonal) must include entries $\{S, S^{-1}\}$ and

$$\max_{\substack{q=1,|\mathbf{A}_N(S)|-1 \\ r=q+1,|\mathbf{A}_N(S)|}} \rho_{\mathbf{B}_N}\left(\mathbf{C}_q, \mathbf{C}_r\right) = \max_{\substack{q=1,|\mathbf{A}_N(S)|-1 \\ r=q+1,|\mathbf{A}_N(S)|}} \sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} \frac{1+\left(c_{ik}^{\langle q\rangle} c_{ik}^{\langle r\rangle}\right)^2}{\left(c_{ik}^{\langle q\rangle} c_{ik}^{\langle r\rangle}\right)^2}\cdot\left(c_{ik}^{\langle q\rangle}-c_{ik}^{\langle r\rangle}\right)^2} =$$

$$=\sqrt{\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\frac{1+\left(S\cdot S^{-1}\right)^2}{\left(S\cdot S^{-1}\right)^2}\cdot\left(S-S^{-1}\right)^2} = \sqrt{\frac{\left(S^2-1\right)^2}{S^2}\cdot 2\cdot\frac{N(N-1)}{2}} = \frac{S^2-1}{S}\sqrt{N(N-1)}. \tag{14}$$

Furthermore, matrices $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ may appear scattered not much, but if a pair of entries of the same location includes too different comparisons, that is an evidence of inconsistency. Obviously, that the maximally different comparisons constitute the same pair $\{S, S^{-1}\}$ and $\mathbb{R}$-distance between them is $S-S^{-1}=\dfrac{S^2-1}{S}$. The minimally different comparisons always constitute, without loss of generality, a pair

$$\left\{c_{ik}^{\langle q\rangle}, c_{ik}^{\langle r\rangle}\right\} = \left\{c_{ik}^{\langle q\rangle}, c_{ik}^{\langle q\rangle}+1\right\}$$

and $\mathbb{R}$-distance between them is 1.

If experts have identical competences then, after the consensus PCM (13), we check if the inequality

$$\max_{\substack{j=1,J-1 \\ l=j+1,J}} \rho_{\mathbf{B}_N}\left(\mathbf{A}_j, \mathbf{A}_l\right) \leqslant \frac{S^2-1}{S}\sqrt{N(N-1)} \quad \text{by} \quad \gamma_N(S)\in(0;1) \tag{15}$$

is true. Along with this, the inequality

$$\max_{\substack{j=1,J-1 \\ l=j+1,J}}\left\{\max_{\substack{i=1,N-1 \\ k=i+1,N}}\max\left\{\left|a_{ik}^{\langle j\rangle}-a_{ik}^{\langle l\rangle}\right|, \left|\frac{1}{a_{ik}^{\langle j\rangle}}-\frac{1}{a_{ik}^{\langle l\rangle}}\right|\right\}\right\} \leqslant \quad \text{by} \quad \mu_N(S)\in\left[1;\frac{S^2-1}{S}\right) \tag{16}$$

is checked also. Values $\gamma_N(S)$ and $\mu_N(S)$ are parameters of consistency of EE made by identical competence experts. The lesser these parameters are taken the stronger requirement for EE consistency is.

If experts have non-identical competences, the consistency conditions (15) and (16) are not relevant. Consequently, instead of comparing PCM $\mathbf{A}_j$ and $\mathbf{A}_l$ the ξ_j-weighted PCM $\mathbf{A}_j$ must be compared to the ξ_l-weighted PCM $\mathbf{A}_l$. Denote the ξ_j-weighted PCM by $\mathbf{A}_j^{\langle\xi_j\rangle}=\left[a_{ik}^{\langle j,\xi_j\rangle}\right]_{N\times N}$. Apparently, the greater ξ_j the smaller change of the non-main-diagonal entry $a_{ik}^{\langle j\rangle}$ should be. Ultimately,

$$\lim_{\xi_j\to 1}a_{ik}^{\langle j,\xi_j\rangle} = a_{ik}^{\langle j\rangle} \quad \text{by} \quad i\neq k. \tag{17}$$

And vice versa, the smaller ξ_j the greater change of the non-main-diagonal entry $a_{ik}^{\langle j\rangle}$ should be, but here the inaccessible ultimate value $\xi_j=0$ must correspond to the unitary PCM implying that, by zero competence EE, objects are indistinguishable:

$$\lim_{\xi_j\to 0}a_{ik}^{\langle j,\xi_j\rangle} = 1 \quad \text{by} \quad i\neq k. \tag{18}$$

Basing upon conditions (17) and (18), define the mapping of PCM $\mathbf{A}_j$ into the ξ_j-weighted PCM as

$$\mathbf{A}_j^{\langle\xi_j\rangle}=\mathbf{Z}_{N,S}\left(\mathbf{A}_j,\xi_j\right) \quad \text{by} \quad \mathbf{A}_j^{\langle\xi_j\rangle}\in\mathbf{B}_N \quad \text{and} \quad a_{ik}^{\langle j,\xi_j\rangle}=1+\xi_j\left(a_{ik}^{\langle j\rangle}-1\right) \quad \text{for} \quad k>i \quad \forall\, k=\overline{i+1,N}. \tag{19}$$

It is clear that properties of the mapping (19) satisfy two conditions (17) and (18) of weighting a PCM. Then, after the consensus PCM (12) is found by (11), we check if the inequality

$$\max_{\substack{j=1,J-1 \\ l=j+1,J}} \rho_{\mathbf{B}_N}\left(\mathbf{A}_j^{\langle\xi_j\rangle}, \mathbf{A}_l^{\langle\xi_l\rangle}\right) \leqslant \frac{S^2-1}{S}\sqrt{N(N-1)} \quad \text{by} \quad \gamma_N(S)\in(0;1) \tag{20}$$

is true. Along with this, the inequality

$$\max_{\substack{j=1,J-1 \\ l=j+1,J}}\left\{\max_{\substack{i=1,N-1 \\ k=i+1,N}}\max\left\{\left|a_{ik}^{\langle j,\xi_j\rangle}-a_{ik}^{\langle j,\xi_l\rangle}\right|, \left|\frac{1}{a_{ik}^{\langle j,\xi_j\rangle}}-\frac{1}{a_{ik}^{\langle j,\xi_l\rangle}}\right|\right\}\right\} \leqslant \quad \text{by} \quad \mu_N(S)\in\left(0;\frac{S^2-1}{S}\right) \tag{21}$$

is checked. Parameters of EE consistency $\gamma_N(S)$ and $\mu_N(S)$ for (20) and (21) are adjusted in the same way as for (15) and (16), although their order may be different (Figure 2).
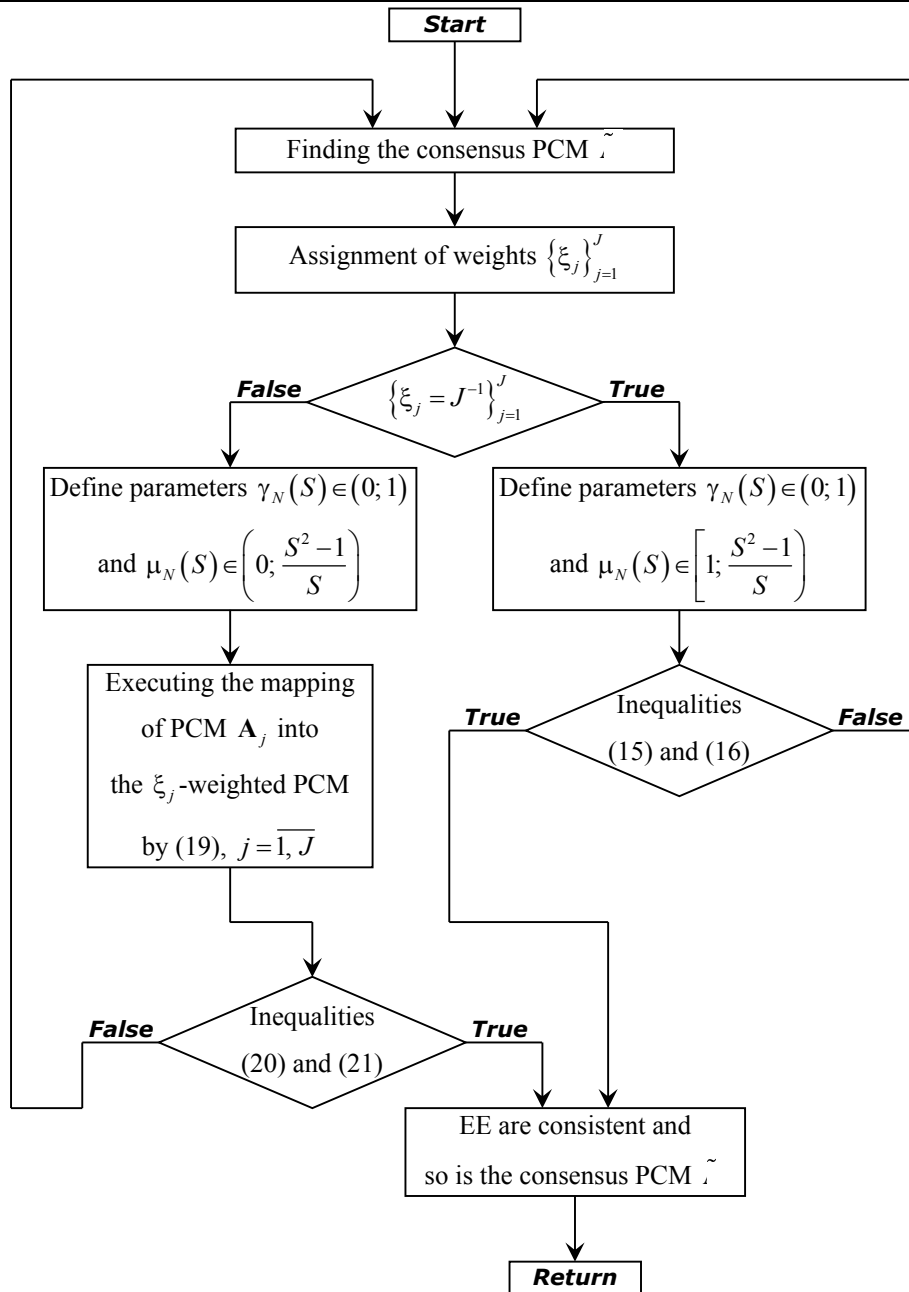
Figure 2. A routine for checking consistency of EE in their PCM $\left\{ \mathbf{A}_j \right\}_{j=1}^{J}$

The routine in Figure 2 describes that module which checks whether EE in their PCM $\left\{ \mathbf{A}_j \right\}_{j=1}^{J}$ are consistent or not. The checking module is the bottom diamond in the routine in Figure 1. And so when the routine in Figure 1 runs, it uses the routine in Figure 2 as its subroutine.

Discussion

Consistency of EE is believed to come first before consistency related to final ranking comes. The reason is that when EE are not concordant the final ranking based on the consensus PCM is perceived just like an average, although this average may seem to reflect a fallacious "trustworthy" relationship among objects which are compared. If "trustworthy" then the consensus PCM will be PE-consistent (but inconsistent following EE inconsistency). EE inconsistency, or bad differences among expert judgments, is an evidence of that the objects are not studied well yet. And PE-consistent ranking of such objects is senseless.

There is an opinion that, however, EE consistency relates to PE-consistency. This is explained with that too "close" EE are probably going to be inherited by good PE-consistency. Nonetheless, good PE-consistency does not always mean EE consistency.

It is worthy to note that instead of Euclidean-based metric in the space of PISM, stated as (7) and calculated as (8), some other metrics could have been used [13]. They are Manhattan, Cosine, Dice, and Jaccard distance functions whose application produces significant differences in the measurement of consensus [13]. Besides, these

four metrics are not more natural than the Euclidean-based metric (7), and so their usage is a matter of another issue.

In assessing the EE consistency, when experts have non-identical competences, the weighted distances $\left\{\xi_j \cdot \rho_{\mathbf{B}_N}\left(\mathbf{A}_j, \mathbf{U}\right)\right\}_{j=1}^{J}$ or $\max\limits_{j=1,J} \rho_{\mathbf{B}_N}\left(\mathbf{A}_j^{\langle\xi_j\rangle}, \mathbf{U}\right)$ could have been used by the unitary $N \times N$ matrix $\mathbf{U}$. Matrix $\mathbf{U}$ being the unitary PCM is the inaccessible ultimate case implying indistinguishability of the studied objects. Nevertheless, the proposed mapping of PCM $\mathbf{A}_j$ into the ξ_j-weighted PCM $\mathbf{A}_j^{\langle\xi_j\rangle}$ satisfies two limit conditions (17) and (18), what lets use the metric (7) and engage inequalities (20) and (21) with inequalities (15) and (16). This eventually completes the approach of aggregating EE and checking their consistency.

Conclusion

The suggested criterion of aggregating EE into the consensus PCM is independent of the comparison scale. It ensures consistency of EE in their PCM by plain requirements which regard competences of experts. The stated approach of the EE aggregation is realized as the routine in Figure 1 with its subroutine in Figure 2, where the EE consistency checking module could have been put at the start, straight off after weights $\left\{\xi_j\right\}_{j=1}^{J}$ and PCM $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ are known, and before the consensus PCM $\tilde{\imath}$ is found. And matrix $\tilde{\imath}$ is not needful for checking consistency of EE in their PCM $\left\{\mathbf{A}_j\right\}_{j=1}^{J}$ but, not to lose generality, the consensus PCM is included into the subroutine.

Once EE is revealed to be consistent (i. e., without bad differences among expert judgments), the corresponding consensus PCM and the objects' ranking should be applied in solving an HMCP. Certainly, consistency of EE depends strongly on competences of experts. If expert procedures recur then competences change, and accurate tracking of those competences expressed as weights $\left\{\xi_j\right\}_{j=1}^{J}$ by (9) is required. Hence, the next must be an approach to re-evaluate the experts' competences while a group of objects is studied sequentially.

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