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MAXIMIZATION OF COLLECTIVE UTILITY AND MINIMIZATION OF PAYOFF PARITY LOSSES FOR ORDERING AND SCALING EFFICIENT NASH EQUILIBRIA IN TRIMATRIX GAMES WITH ASYMMETRIC PAYOFFS

A method of refining efficient Nash equilibria in trimatrix games with asymmetric payoffs is proposed. The refinement, which here is in a wide sense, is executed via scalarizing maximization of collective utility and minimization of payoff parity losses and subsequently finding distances for every equilibrium to the three unreachable minima and one unreachable maximum. The proposed method aligns equilibria in order and also scales them by the distances' ratios.
Keywords: Nash equilibria, trimatrix games, collective utility, payoff parity loss, metaequilibrium.

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МАКСИМІЗАЦІЯ КОЛЕКТИВНОЇ КОРИСНОСТІ ТА МІНІМІЗАЦІЯ ВТРАТ ПАРИТЕТУ ВИГРАШІВ ДЛЯ УПОРЯДКУВАННЯ ТА МАСШТАБУВАННЯ ЕФЕКТИВНИХ РІВНОВАГ НЕША У ТРИМАТРИЧНИХ ІГРАХ З АСИМЕТРИЧНИМИ ВИГРАШАМИ

Пропонується метод відсіву ефективних рівноваг Неша у триматричних іграх з асиметричними виграшами. Відсів, який тут розуміється у широкому смислі, виконується шляхом скаляризації максимізації колективної корисності та мінімізації втрат паритету виграшів з подальшим знаходженням відстаней для кожної рівноваги до трьох недосяжних мінімумів та одного недосяжного максимуму. Запропонований метод розташовує рівноваги за порядком, а також масштабує їх за співвідношеннями відстаней.
Ключові слова: рівноваги Неша, триматричні ігри, колективна корисність, втрата паритету виграшів, метарівновага.

Introduction

Bimatrix games (2MGs) are good models for describing a bilateral interaction with possibilities of a decision choice [1, 2]. Nash equilibria (NEa), if any, are always an object of a refinement for selecting the best NE, although the refinement does not necessarily become easier for a lesser number of NEa [3, 4]. In article [5], a couple of collective utility (CU) and minimum payoff parity loss (PPL) rules is used as for refining NEa, as well as for ranking them. Such a refinement is executed over efficient NEa (ENEa) as they are obtained really easily. The suggested in article [5] approach exploits a two-criteria problem for finding the so-then-called metaequilibrium, wherein CU is maximized and PPL is minimized. This problem is solved via scalarization with weighing the criteria. It is obvious that a similar technique may be exploited for refining ENEa in trimatrix games (3MGs), which model a wider class of interactions than 2MGs. Apart from economical and bioecological domains, 3MGs are used for solving resources allocation problems in computer networks and distributed systems (e. g., see [1, 3] and [4]).

Goal of the article and tasks to be fulfilled

The goal of the article is to further the approach to the ENEa refinement suggested in article [5] for 3MGs. For achieving the article's goal, the following five tasks are to be fulfilled:

1) to circumscribe properties of payoff matrices (PMs), which are to be considered only for the refinement;
2) to circumscribe properties of ENEa, which are only possible to be refined for the 3MGs considered here;
3) to re-state the principle of reasonability introduced in article [5], by which players search for a metaequilibrium non-cooperatively;
4) to state the corresponding multicriteria problem for finding the metaequilibrium in 3MGs along with providing weights for the criteria to scalarize the problem;
5) to show how the relative ranking helps in not just ordering the refined ENEa (referring to the refinement in a wide sense), but also in scaling them.

Properties of PMs in the 3MGs whose ENEa undergo the refinement

The first, second, and third players in a 3MG have their respective PMs

$$\mathbf{A}=(a_{mnk})_{M\times N\times K} \quad \text{and} \quad \mathbf{B}=(b_{mnk})_{M\times N\times K} \quad \text{and} \quad \mathbf{C}=(c_{mnk})_{M\times N\times K} \tag{1}$$

by $M\in\mathbb{N}\backslash\{1\}$, $N\in\mathbb{N}\backslash\{1\}$, $K\in\mathbb{N}\backslash\{1\}$. Let the respective sets of the players' pure strategies be

$$X=\{x_m\}_{m=1}^{M} \quad \text{and} \quad Y=\{y_n\}_{n=1}^{N} \quad \text{and} \quad Z=\{z_k\}_{k=1}^{K}. \tag{2}$$

PMs (1) are presumed to be nonnegative and have identical measurement units. PMs can always be made nonnegative by adding some positive number to them (zeros may be left for convenience in calculations). That will not change NEa, and true payoffs are returned by subtracting the added positive number [1, 2, 5].

Properties of ENEa that make them possible to undergo the refinement

We assume that there are Q ENEa in the 3MG with PMs (1) by $Q>1$. The q-th ENE is

$$e_q=\{x_q^*, y_q^*, z_q^*\} \quad \text{by} \quad x_q^*\in X \quad \text{and} \quad y_q^*\in Y \quad \text{and} \quad z_q^*\in Z, \tag{3}$$

which corresponds to indices $\{m_*^{(q)}, n_*^{(q)}, k_*^{(q)}\}$ by $m_*^{(q)}\in\{\overline{1,M}\}$ and $n_*^{(q)}\in\{\overline{1,N}\}$ and $k_*^{(q)}\in\{\overline{1,K}\}$ in PMs (1). In the q-th ENE (3) the players obtain payoffs

$$\{a_q^*, b_q^*, c_q^*\} \text{ by } a_q^* = a_{m_*^{(q)} n_*^{(q)} k_*^{(q)}} \text{ and } b_q^* = b_{m_*^{(q)} n_*^{(q)} k_*^{(q)}} \text{ and } c_q^* = c_{m_*^{(q)} n_*^{(q)} k_*^{(q)}} . \quad (4)$$

Payoffs $\{a_q^*, b_q^*, c_q^*\}_{q=1}^{Q}$ to be considered only for the refinement in 3MGs have properties of asymmetry and diversity (see the similar properties of PMs in 2MGs in article [5]). The property of diversity is simply that a case

$$\{a_q^*, b_q^*, c_q^*\} = \{\alpha, \beta, \gamma\} \text{ by } [\alpha \;\; \beta \;\; \gamma] \in \mathbb{R}^3 \quad \forall q = \overline{1, Q} \quad (5)$$

is impossible. The property of payoffs' asymmetry in 3MGs looks more complicated that the property of payoffs' asymmetry in 2MGs. For PMs (1) in 3MGs, the membership

$$\{a_q^*, b_q^*, c_q^*\} \in \{\{\alpha, \beta, \gamma\}, \{\beta, \alpha, \gamma\}, \{\beta, \gamma, \alpha\}, \{\alpha, \gamma, \beta\}, \{\gamma, \alpha, \beta\}, \{\gamma, \beta, \alpha\}\} \text{ by } [\alpha \;\; \beta \;\; \gamma] \in \mathbb{R}^3 \quad \forall q = \overline{1, Q} \quad (6)$$

is impossible. The sense of this restriction will be explained in a section below. Factually, impossibility of membership (6) includes impossibility of case (5). Whereas membership (6) does not imply the total nonrefinability of the ENEa (if there are different ENEa, although symmetrical, they may be refined by involving non-equilibrium situations), case (5) has been left apart because it makes the ENEa "more" nonrefinable — they are just identical.

Reasonability, by which players search for a metaequilibrium non-cooperatively

Although the NE is built on a basis of that the player oneself cannot improve one's payoff if springing off an equilibrium strategy in the NE, multiple ENEa generate a non-equilibrium. However, some equilibrium is that what is nonetheless attractive to players. Therefore, the players shall search for an equilibrium (the metaequilibrium) over ENEa. The search is executed non-cooperatively because the players comprehend that, even they cannot guess which ENE is the "right one", they will independently stick to an ENE, at which each of them loses minimally. This is also about a collective loss to be minimized, although some players will not lose at all but win a little bit more. The ENE, at which the sum of the players' payoffs is greater, is reasonably preferable. Eventually, every player comprehends that refusing to lose less in the metaequilibrium leads to lose just more [5]. Of course, this principle of reasonability is feasible only in games whose players or their personifiers are rational.

The four-criteria problem for finding the metaequilibrium in 3MGs along with weights for the criteria

The sum of the players' payoffs at the q-th ENE is a collective utility (CU) of this situation:

$$u(q) = a_q^* + b_q^* + c_q^* \text{ by } q = \overline{1, Q} . \quad (7)$$

Along with CU (7) we have payoff differences for every couple of the players:

$$l_{12}(q) = |a_q^* - b_q^*| , \;\; l_{13}(q) = |a_q^* - c_q^*| , \;\; l_{23}(q) = |b_q^* - c_q^*| , \;\; q = \overline{1, Q} . \quad (8)$$

An ideal case is when those differences are zeros. In this case, there is total parity in payoffs. That is why payoff differences (8) were called payoff parity losses (PPLs) [5].

As PMs (1) are nonnegative and have identical measurement units, then CU and PPLs are normalized [5]:

$$\tilde{u}(q) = u(q) \Big/ \sum_{s=1}^{Q} u(s) \text{ by } q = \overline{1, Q} \quad (9)$$

and

$$\tilde{l}_{12}(q) = l_{12}(q) \Big/ \sum_{s=1}^{Q} u(s) , \;\; \tilde{l}_{13}(q) = l_{13}(q) \Big/ \sum_{s=1}^{Q} u(s) , \;\; \tilde{l}_{23}(q) = l_{23}(q) \Big/ \sum_{s=1}^{Q} u(s) \text{ by } q = \overline{1, Q} . \quad (10)$$

The normalized CU (9) is to be maximized and the normalized PPLs (10) are to be minimized:

$$q_{\text{util}} \in \arg\max_{q=\overline{1,Q}} \tilde{u}(q) , \;\; q_{\text{loss}}^{(12)} \in \arg\min_{q=\overline{1,Q}} \tilde{l}_{12}(q) , \;\; q_{\text{loss}}^{(13)} \in \arg\min_{q=\overline{1,Q}} \tilde{l}_{13}(q) , \;\; q_{\text{loss}}^{(23)} \in \arg\min_{q=\overline{1,Q}} \tilde{l}_{23}(q) . \quad (11)$$

Obviously, four-criteria problem (11) for finding a metaequilibrium in 3MGs will not have a solution, at which $q_{\text{util}} = q_{\text{loss}}^{(12)} = q_{\text{loss}}^{(13)} = q_{\text{loss}}^{(23)}$. Hence, a scalarization is invoked for unifying the criteria in (11) into one. Point

$$\left[\tilde{l}_{12}(q) \;\; \tilde{l}_{13}(q) \;\; \tilde{l}_{23}(q) \;\; \tilde{u}(q)\right] \in \mathbb{R}^4 \quad (12)$$

lies on a hyperplane containing the unreachable minima of normalized PPLs (10) and the unreachable maximum of normalized CU (9). Those three unreachable minima and one unreachable maximum constitute a point, which is

$$[0 \;\; 0 \;\; 0 \;\; 1] \in \mathbb{R}^4 , \quad (13)$$

whence a Euclidean distance between points (12) and (13) is

$$\rho_{\mathbb{R}^4}\left(\left[\tilde{l}_{12}(q) \;\; \tilde{l}_{13}(q) \;\; \tilde{l}_{23}(q) \;\; \tilde{u}(q)\right], [0 \;\; 0 \;\; 0 \;\; 1]\right) = \sqrt{\tilde{l}_{12}^2(q) + \tilde{l}_{13}^2(q) + \tilde{l}_{23}^2(q) + [1 - \tilde{u}(q)]^2} . \quad (14)$$

Nevertheless, criteria of minimizing the PPLs may be significantly less or more important than the criterion of maximizing the CU. Let $\alpha \in (0; 1)$ be a weight of the PPLs criteria. Then distance (14) is re-stated as

$$\rho_{\mathbb{R}^4}\left(\left[\tilde{l}_{12}(q) \;\; \tilde{l}_{13}(q) \;\; \tilde{l}_{23}(q) \;\; \tilde{u}(q)\right], [0 \;\; 0 \;\; 0 \;\; 1]; \alpha\right) = d(q, \alpha) = \sqrt{\alpha\left[\tilde{l}_{12}^2(q) + \tilde{l}_{13}^2(q) + \tilde{l}_{23}^2(q)\right] + (1 - \alpha)[1 - \tilde{u}(q)]^2} . \quad (15)$$

If there is an uncertainty in selecting the weight, then setting $\alpha = 0.5$ is always acceptable. And the solution

$$q_* \in \arg\min_{q=\overline{1,Q}} d(q, \alpha) \quad (16)$$

gives us that metaequilibrium e_{q_*} whose payoffs $\{a_{q_*}^*, b_{q_*}^*, c_{q_*}^*\}$ represent a minimum of PPLs and a CU maximum.

The relative ranking for ordering and scaling ENEa

The approach with distance (15) and minimum (16) does not only imply refining as the metaequilibrium e_{q_*} . Distances $\{d(q, \alpha)\}_{q=1}^{Q}$ are arranged as $\{d(q_v, \alpha)\}_{v=1}^{Q}$, where

$$d(q_v, \alpha) \leqslant d(q_{v+1}, \alpha) \quad v = \overline{1, Q-1} \tag{17}$$

and

$$d(q_*, \alpha) = d(q_1, \alpha) \text{ by } \{q_v\}_{v=1}^{Q} \cap \{\overline{1, Q}\} = \{\overline{1, Q}\}. \tag{18}$$

Then a value

$$r(q_v, \alpha) = d(q_{v+1}, \alpha) / d(q_v, \alpha) \tag{19}$$

is a ratio showing an advantage of the q_v-th ENE over the q_{v+1}-th ENE by $v = \overline{1, Q-1}$. Ratios $\{r(q_v, \alpha)\}_{v=1}^{Q-1}$ showing the relative ranking of ENEa allow ordering and scaling ENEa. This is the refinement in a wide sense. For example, in a $3\times4\times2$ 3MG with PMs

$$\mathbf{A} = \left(\begin{pmatrix} 6 & 8 & 6 & 3 \\ 3 & 5 & 2 & 5 \\ 3 & 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 8 & 2 & 0 & 8 \\ 2 & 7 & 0 & 0 \end{pmatrix} \right) \text{ and } \mathbf{B} = \left(\begin{pmatrix} 9 & 2 & 0 & 0 \\ 1 & 3 & 0 & 5 \\ 6 & 5 & 8 & 3 \end{pmatrix} \begin{pmatrix} 9 & 3 & 0 & 6 \\ 6 & 1 & 9 & 5 \\ 2 & 7 & 5 & 4 \end{pmatrix} \right) \text{ and } \mathbf{C} = \left(\begin{pmatrix} 3 & 6 & 3 & 1 \\ 2 & 7 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 3 & 2 & 4 \\ 6 & 0 & 4 & 3 \\ 7 & 1 & 3 & 1 \end{pmatrix} \right) \tag{20}$$

having four ENEa $e_1 = \{x_1, y_1, z_1\}$, $e_2 = \{x_2, y_3, z_2\}$, $e_3 = \{x_2, y_4, z_1\}$, $e_4 = \{x_3, y_2, z_2\}$, and their respective payoffs $\{6, 9, 3\}$, $\{0, 9, 4\}$, $\{5, 5, 4\}$, $\{7, 7, 1\}$, the distances and the ranking are

$$d(1, 0.5) \approx 0.5025,\ d(2, 0.5) \approx 0.569,\ d(3, 0.5) \approx 0.5424,\ d(4, 0.5) \approx 0.5397,\ d(1, 0.5) < d(4, 0.5) <$$
$$< d(3, 0.5) < d(2, 0.5),\ r(1, 0.5) \approx 1.074,\ r(4, 0.5) = 1.005,\ r(3, 0.5) \approx 1.0491, \tag{21}$$

whence $e_1 \succ e_4 \succ e_3 \succ e_2$. Along with that ENE $e_1 = \{x_1, y_1, z_1\}$ is the metaequilibrium, we learn that payoffs $\{6, 9, 3\}$ in this ENE are 7.4 % better than payoffs $\{7, 7, 1\}$ in the closest ENE $e_4 = \{x_3, y_2, z_2\}$. However, the advantage of this ENE over the next ENE (by its rank) $e_3 = \{x_2, y_4, z_1\}$ is just 0.5 %. That looks pretty weird as much greater non-zero PPLs $l_{13}(4) = l_{23}(4) = 6$ stand against much smaller non-zero PPLs $l_{13}(3) = l_{23}(3) = 1$, while CU $u(4) = 15$ of the higher-ranked ENE exceeds CU $u(3) = 14$ of the lower-ranked ENE just by 1. Subsequently, if to increase the weight of the PPLs criteria, ENE e_3 (having much better factor of PPLs and standing too close by CU) will exceed ENE e_4. Indeed, setting $\alpha > 0.566$ (which is just 13.2 % greater than the initial weight) in 3MG with PMs (20) gives such a result. Thus, 3MG with PMs (20) is an example of that a selection of a weight of the PPLs criteria is not as easy and clear step as it might have seemed before.

Discussion and conclusion

A substantial merit of the proposed method of refining ENEa is that it aligns them in order and also scales them by ratios (19). In particular, that scaling answers a question of "how much is the metaequilibrium better than the other ENEa?" This is very important for 3MGs, where ENEa appear more sophisticated than ENEa in 2MGs, and distinguishing among such 3-payoff ENEa becomes harder. Furthermore, the suggested refinement of ENEa is equivalent to the relative ranking of ENEa, wherein we learn relationships among all ENEa. Nonetheless, this learning potentially depends on what the weight of the PPLs criteria is selected. The example of 3MG with PMs (20) has shown that sometimes relationships among ENEa by ratios (19) are pretty sensitive to the weight. And that is one of the demerits. Another one is that the proposed method does not involve non-equilibrium situations that makes symmetrical ENEa absolutely non-distinguishable here.

The proposed method of refining ENEa is based on the reasonability of "it is better to lose less than risking to lose just more", by which rational players search for a metaequilibrium non-cooperatively. The method can be easily extended to finite noncooperative games of any number of players, and applied for, say, solving queue and priority problems in peer-to-peer computing/networking, where the server acts instead of the players (clients).

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